were used, for example the Schmidt isothermal power. Again, the corresponding equations for the casing motion mode are obtained by replacing the subscript p by c and M_P by M_C.

The cyclic work may also be obtained by solving the integral $\oint p\,dv$ for each working space. However, the above results yield good answers when the linear approximations are reasonable and, furthermore, they are easy to apply. Table 3.1 lists all the pertinent equations derived in this section.

Table 3.1 Set of equations for the generalised dynamic analysis piston motion mode.

Frequency	$\omega^2 = (D_{dp}S_{pd} + S_{dp}D_{pd} - D_{dd}S_{pp} - S_{dd}D_{pp})/(D_{dd} + D_{pp})$
Geometric constraint	$\omega^4 + \omega^2(S_{pp} + S_{dd} + D_{dp}D_{pd} - D_{dd}D_{pp}) + S_{dd}S_{pp} - S_{dp}S_{pd} = 0$
Piston–displacer phase angle	$\phi = \tan^{-1}\left(\dfrac{\omega[D_{pp}S_{pd} - D_{pd}(S_{pp} + \omega^2)]}{-[S_{pd}S_{pp} + \omega^2(S_{pd} + D_{pd}D_{pp})]}\right)$
Piston–displacer amplitude ratio	$\dfrac{X_d}{X_p} = r = \dfrac{(\omega^2 + S_{pp})^2 + \omega^2 D_{pp}{}^2}{\{[S_{pd}(\omega^2 + S_{pp}) + \omega^2 D_{pp}D_{pd}]^2 + \omega^2[D_{pd}(\omega^2 + S_{pp}) - D_{pp}S_{pd}]^2\}^{1/2}}$
Irrecoverable work	$W_{ir} = -\pi\omega M_D[(D_{dd} + D_{pd}M_P/M_D)X_d{}^2 + D_{dp}X_p{}^2] + \pi\omega C_{H_{pc}} X_p{}^2$
Useful work	$W_s = -\pi\omega M_P D_{pp} X_p{}^2 - \pi\omega C_{H_{pc}} X_p{}^2$
The above equations are subject to $\omega \gg \sqrt{-S_{cc}}$.	
The casing motion mode equations are obtained by replacing the subscript p by c and M_P by M_C. The resulting set of equations is subject to $\omega \gg \sqrt{-S_{pp}}$.	

3.3 Linearisation

To apply the preceding analysis it is necessary to obtain functions for the pressure variations in the working spaces and the gas springs (also referred to as bounce spaces). The Isothermal analysis lends a convenient closed-form result which is known to be a fairly good approximation to reality. This result, from Chapter 2, table 2.1, is

$$p = MR\left[\frac{V_c}{T_k} + \frac{V_k}{T_k} + \frac{V_r \ln(T_h/T_k)}{(T_h - T_k)} + \frac{V_h}{T_h} + \frac{V_e}{T_h}\right]^{-1}. \quad (3.48)$$

By assuming adiabatic gas springs and a perfect gas, the pressure variations in these spaces may be described by

$$p_b = p_{mean}(V_B/V_b)^{\gamma}, \tag{3.49}$$

where V_B is the average gas spring volume and V_b is the instantaneous value.

Referring to figure 3.6 as a general example of a free-piston engine, the volume variations are as follows:

$$V_c = A_P(C_C + x_p) - (A_D - A_R)x_d \tag{3.50}$$

$$V_e = A_D(E_E + x_d + x_c). \tag{3.51}$$

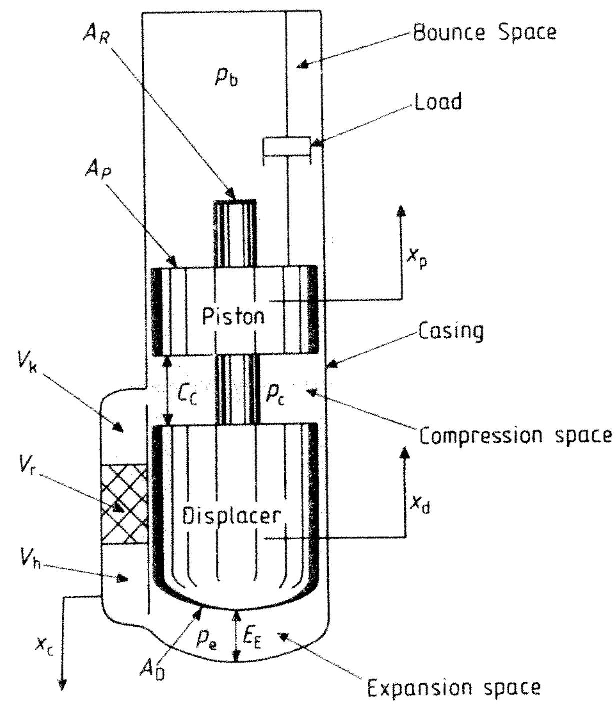

Figure 3.6 Volume variations. In the configuration shown, the displacer rod passes through the piston and is rigidly attached to the displacer. Movement in the positive direction tends to increase working space volumes.

Substituting equations (3.50) and (3.51) into equation (3.48), we obtain

$$p = MR\left(\frac{A_P(C_C + x_p) - (A_D - A_R)x_d}{T_k} + \frac{V_k}{T_k} + \frac{V_r \ln(T_h/T_k)}{(T_h - T_k)} + \frac{V_h}{T_h} + \frac{A_D(E_E + x_d + x_c)}{T_h}\right)^{-1} \tag{3.52}$$

In order to be used in the dynamic analysis, equation (3.52) must be linearised. For the purposes of this analysis the binomial expansion is used.

This method of linearisation has been found to work perfectly adequately and well within the tolerance expected from an analysis such as this. Rewriting equation (3.52)

$$p = \frac{MR}{S}\left(1 + \frac{A_P x_p - (A_D - A_R)x_d}{T_k S} + \frac{A_D(x_d + x_c)}{T_h S}\right)^{-1} \tag{3.53}$$

where

$$S = \frac{A_P C_C}{T_k} + \frac{V_k}{T_k} + \frac{V_r \ln(T_h/T_k)}{(T_h - T_k)} + \frac{V_h}{T_h} + \frac{A_D E_E}{T_h}. \tag{3.54}$$

Using the binomial expansion and neglecting second-order terms we obtain

$$p \simeq \frac{MR}{S}\left(1 - \frac{A_P x_p - (A_D - A_R)x_d}{T_k S} - \frac{A_D(x_d + x_c)}{T_h S} + \ldots\right) \tag{3.55}$$

given the following condition

$$\frac{A_P x_p - (A_D - A_R)x_d}{T_k S} + \frac{A_D(x_d + x_c)}{T_h S} \ll 1,$$

which is usually easily satisfied.

Since the mean pressure p_{mean} in the working spaces is the charge pressure, result (3.55) may be improved further by writing

$$p \simeq p_{mean}\left(1 - \frac{A_D(x_d + x_c)}{T_h S} - \frac{A_P x_p - (A_D - A_R)x_d}{T_k S}\right) \tag{3.56}$$

which implies that all pressure variations occur symmetrically around the charge pressure.

The gas spring pressures are linearised in the same way. These results are not repeated here since they depend on the particular geometry under consideration.

Since the Isothermal analysis does not account for pressure gradients, the pressure variation given by equation (3.56) may be thought of as representing the spatial average pressure at any instant. For the purposes of the dynamic analysis, equation (3.56) is taken to represent the pressure variation in the compression space. Academically, it may be more accurate to assume that this equation describes some intermediate pressure between the compression and expansion spaces. On the other hand, what this pressure variation is referred to does not make any significant difference to the dynamic components and, furthermore, the assumption does simplify the calculations to a great extent. Therefore we assume that

$$p_c \simeq p. \tag{3.57}$$

A result describing the pressure gradient across the heat exchangers is required in addition to the compression space pressure. This result would have to account for the character of the gas flow which typically may be laminar or turbulent at any instant. The exact form of the pressure gradient result, therefore, is quite complex. Fortunately, in most machines the gas flow is predominantly turbulent in the cooler and heater and laminar in the regenerator. Thus it is convenient to assume that the pressure drop is made up as follows:

$$\Delta p = \Delta p_k + \Delta p_r + \Delta p_h \tag{3.58}$$

where Δp_k, Δp_r and Δp_h are the pressure drop over the cooler, regenerator and heater, respectively. For the cooler and heater, then, the pressure drop may be assumed to be of the form

$$\Delta p_{k,h} = \tfrac{1}{2}\rho (f_t + k_h)\bar{u}|\bar{u}| \tag{3.59}$$

where f_t is the turbulent friction factor and is given by $f_t = 4f_f L/d$, where f_f is the Fanning friction factor, L is the heat exchanger length and d is the hydraulic diameter. k_h is the head loss coefficient due to bends and entrance and exit effects. For the regenerator

$$\Delta p_r = f_l \bar{u} \tag{3.60}$$

where f_l is the laminar friction factor.

Equations (3.59) and (3.60) are both approximate but they do describe the pressure gradients quite accurately in the majority of free-piston Stirling engines. Equation (3.59) is non-linear and must therefore be linearised before it can be incorporated into the dynamic analysis. The variation of density with pressure is neglected since in most high pressure free-piston engines the pressure swing is usually a small fraction of the charge pressure (typically less than 15%). It is therefore assumed that density is only a function of local temperature, which in the isothermal case is constant. Thus the major contribution to the non-linearity is the velocity term.

Since the pressure drop constitutes a damping effect on the reciprocating components, it is necessary to find an equivalent linear damper which will dissipate the same energy. The condition for equivalence to be satisfied is: the energy dissipated per quarter cycle by actual damping must equal the energy dissipated per quarter cycle by an equivalent linear damping (Anvoner 1970). From equation (3.59), the energy dissipated per unit flow area is

$$E_1 = \tfrac{1}{2}\frac{\rho}{\omega}(f_t + k_h)\int_0^{\pi/2} (\bar{u})^2 \bar{u}\,\mathrm{d}\omega t. \tag{3.61}$$

For sinusoidal displacements the flow velocity is approximated by

$$\bar{u} = U\cos\omega t. \tag{3.62}$$

Substituting equation (3.62) into equation (3.61) and integrating, we obtain

$$E_1 = \tfrac{1}{3}\rho (f_t + k_h)U^3/\omega. \tag{3.63}$$

In the case of linear damping, the damping force per unit flow area is given by

$$\Delta p = C\bar{u}. \tag{3.64}$$

In this case the energy dissipated per unit flow area is:

$$E_2 = C\int_0^{\pi/2} (\bar{u})^2\,\mathrm{d}\omega t. \tag{3.65}$$

Substituting equation (3.62) into equation (3.65) and integrating, we obtain

$$E_2 = \frac{\pi C}{4\,\omega}U^2. \tag{3.66}$$

Equating (3.63) and (3.66) and solving for C, we obtain

$$C = \tfrac{4}{3}(\rho/\pi)(f_t + k_h)U. \tag{3.67}$$

The turbulent pressure drop may now be represented by

$$\Delta p_{k,h} \simeq \tfrac{4}{3}(\rho/\pi)(f_t + k_h)U. \tag{3.68}$$

Thus the total linear pressure drop across the heat exchangers is given by

$$\Delta p = \tfrac{4}{3}(1/\pi)\{[\rho U(f_t + k_h)\bar{u}]_{\text{cooler}} + [\rho U(f_t + k_h)\bar{u}]_{\text{heater}}\} + f_1\,\bar{u}_{\text{regenerator}}. \tag{3.69}$$

It is now necessary to find the gas flow velocities in terms of the velocities of the reciprocating elements. This is done by obtaining the net volumetric flow rate through the heat exchanger loop and dividing by the relevant flow area to obtain the corresponding gas velocity. Obviously the volumetric flow rate is not uniform throughout the heat exchangers and therefore the velocities so derived are approximate.

The volumetric flow out of the compression space is given by

$$\dot{V}_c = \mathrm{d}V_c/\mathrm{d}t \tag{3.70}$$

and the volumetric flow out of the expansion space by

$$\dot{V}_e = \mathrm{d}V_e/\mathrm{d}t. \tag{3.71}$$

Therefore, from equations (3.50) and (3.51):

$$\dot{V}_c = A_P\dot{x}_p - (A_D - A_R)\dot{x}_d \tag{3.72}$$

$$\dot{V}_e = A_D(\dot{x}_d + \dot{x}_c). \tag{3.73}$$

Positive values for $\dot{V}_c$ and $\dot{V}_e$ indicate increasing volumes, and therefore the net volumetric flow rate through the heat exchangers is given by

$$\bar{\dot{V}} = \dot{V}_c - \dot{V}_e, \tag{3.74}$$

which from equations (3.72) and (3.73) becomes

$$\bar{\dot{V}} = A_P\dot{x}_p - (2A_D - A_R)\dot{x}_d - A_D\dot{x}_c. \tag{3.75}$$

The approximate velocity through each heat exchanger component is then given by

$$\bar{u}_k = \bar{V}/A_k \qquad \text{(cooler velocity)} \tag{3.76}$$
$$\bar{u}_r = \bar{V}/A_r \qquad \text{(regenerator velocity)} \tag{3.77}$$
$$\bar{u}_h = \bar{V}/A_h \qquad \text{(heater velocity)} \tag{3.78}$$

where A_k, A_r and A_h are the flow areas of the cooler, the regenerator and the heater, respectively.

The pressure drop given by equation (3.69) may now be written

$$\Delta p = \{\tfrac{4}{3}(1/\pi)[\rho_k U_k (f_t + k_h)_k / A_k + \rho_h U_h (f_t + k_h)_h / A_h] + f_l / A_r\} \bar{V} \tag{3.79}$$

where ρ_k and ρ_h are given by

$$\rho_k = p_{mean}/(RT_k) \tag{3.80}$$
$$\rho_h = p_{mean}/(RT_h) \tag{3.81}$$

and U_k and U_h are the velocity amplitudes (or peak velocities) in the cooler and the heater and are given by

$$U_k = V_A/A_k \tag{3.82}$$
$$U_h = V_A/A_h \tag{3.83}$$

where V_A is the net volumetric flow rate amplitude and is evaluated from equation (3.75) by assuming sinusoidal displacements. The results for the two operating modes are:

(i) piston motion mode

$$V_A = \omega[(A_P X_p)^2 - 2(2A_D - A_R)A_P X_p X_d \sin\phi + (2A_D - A_R)^2 X_d^2]^{1/2} \tag{3.84}$$

(ii) casing motion mode

$$V_A = \omega[(A_D X_c)^2 + 2(2A_D - A_R)A_D X_c X_d \sin\phi + (2A_D - A_R)^2 X_d^2]^{1/2}. \tag{3.85}$$

From equations (3.75) and (3.79) it can be seen that the pressure drop is now a linear function of the velocities of the reciprocating elements. Recalling the assumed Δp relationship given by equation (3.8), we have

$$A_D \Delta_p = C_p \dot{x}_p + C_d \dot{x}_d + C_c \dot{x}_c. \tag{3.86}$$

Comparing this result with equation (3.79) and equating the respective coefficients of $\dot{x}_p$, $\dot{x}_d$ and $\dot{x}_c$, we have

$$C_p = A_P A_D P_C \tag{3.87}$$
$$C_d = -(2A_D - A_R)A_D P_C \tag{3.88}$$
$$C_c = -A_D{}^2 P_C \tag{3.89}$$

where P_C is given by

$$P_C = \tfrac{4}{3}(1/\pi)[\rho_k U_k (f_t + k_h)_k / A_k + \rho_h U_h (f_t + k_h)_h / A_h] + f_l / A_r, \quad (3.90)$$

which is the final form of the linear damping due to the heat exchanger pressure drop.

A similar pressure drop analysis was done by G Wood at Sunpower Incorporated (Wood 1980). Table 3.2 lists a comparison of his linear results for Δ_p with those predicted by a more complete simulation routine such as that developed in Chapter 5. The agreement between the linear analysis and the simulation can be seen to be perfectly adequate for the purposes set out for the linear analysis.

Table 3.2 Comparison of the Linear and Simulation Δp amplitudes (after Wood 1980).

Displacer amplitude (cm)	Displacer–piston phase (degrees)	Piston amplitude (cm)	Peak Reynolds number C	R	H	Δp (bar) Simulation	Linear
1.27	65.00	1.27	19780	6	5136	0.410	0.345
0.98	21.94	1.12	6311	2	3734	0.126	0.118
0.87	25.00	1.12	7543	2	3351	0.108	0.097
0.45	65.00	1.12	15790	3	1917	0.178	0.146
0.40	90.00	1.12	18440	4	1693	0.230	0.193

The above results were evaluated for a machine operating at 60 Hz: C, cooler; R, regenerator; H, heater.

Finally, we require to calculate the damping effect due to gas spring hysteresis loss. The details of this loss mechanism are outlined in Chapter 7. Therefore, referring to Chapter 7, equation (7.31), the hysteresis loss is given by

$$\bar{\dot{W}} = \frac{k}{4}\sqrt{\frac{\omega}{2\alpha_0}}\gamma(\gamma-1)T_w\left(\frac{\Delta V}{V_B}\right)^2. \quad (3.91)$$

The work done against damping is given by (equation 3.45)

$$W = \pi C_H \omega X^2 \quad (3.92)$$

where C_H is the gas spring damping coefficient and X is the amplitude of the damped motions.

Expressing (3.92) as a power

$$\dot{W} = \tfrac{1}{2} C_H (\omega X)^2. \quad (3.93)$$

Equating (3.91) and (3.93) and noting that $\Delta V = AX$

$$C_H = \frac{k}{2}\sqrt{\frac{\omega}{2\alpha_0}}\gamma(\gamma-1)T_w A_w\left(\frac{A}{\omega V_B}\right)^2 \quad (3.94)$$

where A is the relevant presented area of the reciprocating element. Equation (3.94) thus gives the damping due to gas spring hysteresis. All the relevant linear equations are listed in table 3.3.

Table 3.3 Set of linearised equations for pressure, pressure drop and gas spring hysteresis damping.

Compression space pressure	$p_c \simeq p_{mean}\left(1 - \frac{A_D(x_d + x_c)}{T_h S} - \frac{A_P x_p - (A_D - A_R)x_d}{T_k S}\right)$
	where
	$S = \frac{A_P C_C}{T_k} + \frac{V_k}{T_k} + \frac{V_r \ln(T_h/T_k)}{(T_h - T_k)} + \frac{V_h}{T_h} + \frac{A_D E_E}{T_h}$
Pressure drop	$A_D \Delta_p = C_p \dot{x}_p + C_d \dot{x}_d + C_c \dot{x}_c$
	where
	$C_p = A_P A_D P_C \quad C_d = -(2A_D - A_R)A_D P_C \quad C_c = -A_D^2 P_C$
	and
	$P_C = \frac{4}{3}(1/\pi)[\rho_k U_k (f_t + k_h)_k / A_k + \rho_h U_h (f_t + k_h)_h / A_h] + f_l / A_r$
Gas spring hysteresis damping	$C_H = \frac{k}{2}\sqrt{\frac{\omega}{2\alpha_0}\gamma(\gamma - 1)T_w A_w}\left(\frac{A}{\omega V_B}\right)^2$

Note that other methods may be used to generate the linear functions. For example, if a sophisticated computer simulation is used to analyse the thermodynamic cycle, the pressure variations will only be known implicitly. In this case, volume variations would be assumed, the resulting pressure variations analysed by Fourier techniques and then, using only the linear terms, the dynamic analysis would be performed to obtain new volume variations. This process is therefore iterative unless one is solving for the reciprocating masses, in which case the solution is obtained directly. The second-order terms neglected should be of the order of 10% or less of the linear terms (Gedeon 1978). Particular examples are now considered.

3.4 The Sunpower RE-1000 engine

The RE-1000 was developed at Sunpower to investigate free-piston Stirling engine applications. The engine is primarily a research machine and is the seventh in a series which began with an engine being built for the American Gas Association (Beale *et al* 1975). The power to load is of the order of 1 kW (figure 3.7).