3

Free-Piston Machines

Hey . . . this engine will work just fine if we simply cut off the rhombic drive . . .

Observation by William T Beale while teaching thermal machinery at the Ohio University, winter 1964

3.1 Background

One of the most novel applications of the Stirling cycle is in free-piston configurations and, indeed, this configuration is the one which holds the most immediate promise.

Free-piston engines operate without physical linkages. They rely only on the gas pressures (and in some cases mechanical springs) to impart the correct motions to the reciprocating elements. Such machines have the advantages of simplicity, low cost, ultra-reliability, and freedom of working gas leakage over conventional Stirling engines. Depending on the particular configuration, these engines may also be designed to operate at constant frequency. They are currently being developed for many diverse applications which include thermally activated heat pumps, solar electric converters, remote area power generators, total energy systems and water pumps (Walker 1980).

The invention of the basic free-piston Stirling engine in the early 1960s is generally attributed to William T Beale (Beale 1969, 1971). Independent inventions of similar types of engines were made by E H Cooke-Yarborough and C West at the Harwell Laboratories of the UK AERE† (Cooke-Yarborough 1967, 1970, Cooke-Yarborough *et al* 1974). G M Benson has also made important early contributions to the field and has patented many novel free-piston Stirling engines (Benson 1973, 1977). Others have since been working on various aspects and modifications of these original ideas (see Martini 1975, Reader 1979, Goldwater and Morrow 1977, Goldberg 1979, Goldberg and Rallis 1979).

An early model free-piston Stirling engine (model M100)‡ produced commercially by Sunpower Inc is shown in figure 3.1. This machine produces a

† Atomic Energy Research Establishment, Harwell, England.

‡ This model is no longer available from the company, having been replaced by a more recently developed one.

nominal 100 W at 50 Hz. From the figure, components typically encountered in a Stirling engine are evident, with the exception of gears, crankshafts and the like, the piston and displacer being mounted on gas springs. This is a common configuration but is certainly not the only one. Many different configurations exist, some of which are analysed later in this chapter.

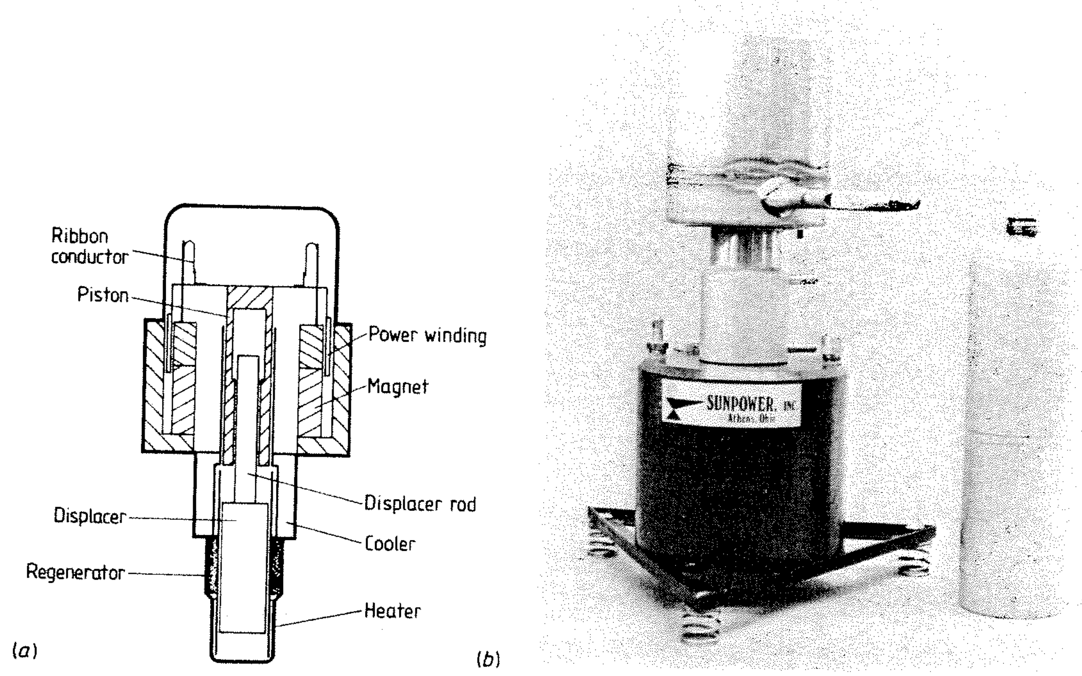

Figure 3.1 Model M100 free-piston Stirling engine (courtesy Sunpower Incorporated).

Consider a free-body diagram (figure 3.2) of the piston and displacer for the model M100. The equations of motion for these two components would be, for the piston:

$$M_P \ddot{x}_p = (p_c - p_b)A_P - C_{pc}(\dot{x}_p + \dot{x}_c) \quad (3.1)$$

where $C_{pc}(\dot{x}_p + \dot{x}_c)$ is the force exerted by the alternator, and for the displacer:

$$M_D \ddot{x}_d = A_D p_e - A_P p_c - A_R p_d \quad (3.2)$$

where $A_P = A_D - A_R$, i.e.

$$M_D \ddot{x}_d = A_D(p_e - p_c) + A_R(p_c - p_d). \quad (3.3)$$

Using p_c as a reference pressure, p_e may be obtained as follows

$$p_e = p_c + \Delta p \quad (3.4)$$

where Δp is the instantaneous pressure difference across the heat exchangers.

Substituting equation (3.4) into equation (3.3) we obtain

$$M_D \ddot{x}_d = A_D \Delta p + A_R(p_c - p_d). \quad (3.5)$$

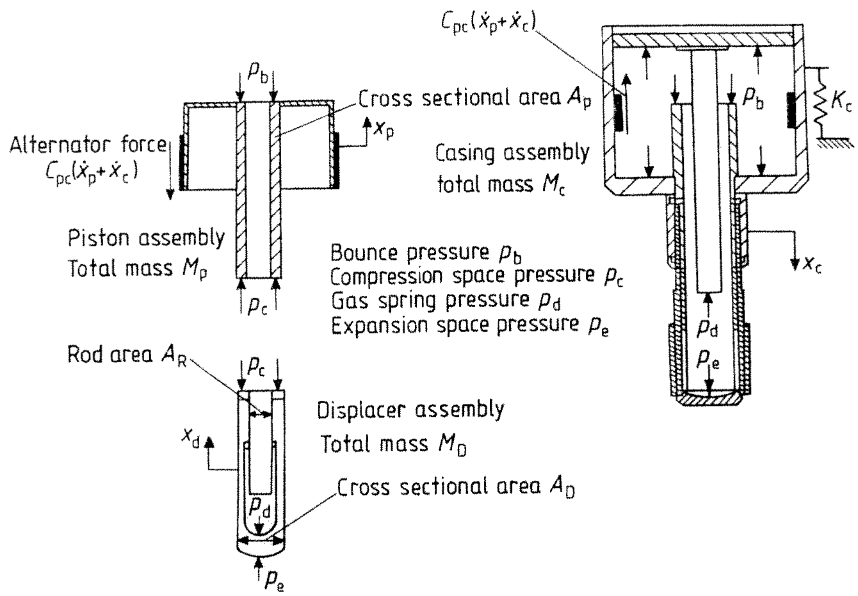

Figure 3.2 Free-body diagram for the Model M100 engine.

Equations (3.1) and (3.5) are then the equations necessary to determine the motions of the piston and displacer respectively. To solve these equations, relationships for p_c and Δp in terms of x_p and x_d are still required. These relationships are obtained from the thermodynamics of the cycle. For the moment we shall assume that the thermodynamic equations are available.

The solution of this set of equations may be attempted by a variety of methods. Possibly the most obvious is by time-stepping integration techniques similar to the simulation method introduced in Chapter 2. This approach has a serious drawback if used for initial sizing, in that the choice of piston and displacer masses, gas springs and other components necessary for the correct dynamic behaviour are not made evident. These parameters would have to be chosen on a more or less arbitrary basis and hence there would be no clear indication, until after the simulation, that a particular engine configuration might work. This trial and error method would clearly prove tedious and expensive, particularly if large scale thermodynamic simulations were used.

The importance of the ability to obtain a preliminary idea of engine configuration in a short space of time cannot be underestimated. It allows the designer to determine at an early stage in the design whether or not the size, performance and operating parameters (such as charge pressure, frequency, etc) are reasonable, and to some extent whether or not a Stirling engine is suitable for the application at hand.

In what follows, a classical dynamic analysis of free-piston Stirling engines is described (Berchowitz and Wyatt-Mair 1979). This analysis is then coupled with the ideal isothermal thermodynamics of Chapter 2 to obtain closed-form solutions for the behaviour, performance and size, as well as the rather narrow criteria under which successful operation may be expected. Furthermore, the solutions also indicate general characteristics such as the effects of changing load, charge pressure, temperature ratio, etc.

3.2 Generalised analysis

For a given geometry, gas type and temperature distribution, the working gas pressure is ideally a function of the volume variations only (cf Chapter 2, table 2.1). Since the volume variations are, in turn, functions of the piston, displacer and casing motions, the working gas, gas spring and bounce space pressures may be written:

$$p = f(x_{\mathrm{p}}, x_{\mathrm{d}}, x_{\mathrm{c}}). \tag{3.6}$$

As shown in Chapter 2, ideal thermodynamics exclude the possibility of pressure gradients. These pressure gradients, however, strongly influence the dynamic behaviour of the machine in that the reciprocating elements are subjected to increased dissipative (damping) forces. The major effects here are the change in phase between the displacer and piston motions, and the change of displacer to piston amplitude ratio. Together, these two effects significantly alter the power output. Thus, for meaningful results the pressure gradient must be accounted for in some way.

If one considers that the pressure gradient is predominantly a result of viscous friction in the heat exchangers, it is fairly obvious that work against this pressure gradient is dissipative. Therefore, the force due to the pressure gradient influences the dynamics as if it were a damping load. Equivalent linear dampers have been found to work very well in reproducing the effects of the pressure gradient. For the purposes of this analysis, it will therefore be assumed that the pressure gradient may be represented as a function of piston, displacer and casing velocities:

$$\Delta p = f(\dot{x}_{\mathrm{p}}, \dot{x}_{\mathrm{d}}, \dot{x}_{\mathrm{c}}). \tag{3.7}$$

The details of the Δp relationship will be developed later. For the moment, assume that it is of the following form:

$$A_{\mathrm{D}} \Delta p = C_{\mathrm{p}} \dot{x}_{\mathrm{p}} + C_{\mathrm{d}} \dot{x}_{\mathrm{d}} + C_{\mathrm{c}} \dot{x}_{\mathrm{c}}. \tag{3.8}$$

Another important damping effect is the hysteresis loss associated with gas springs. The development of this loss mechanism is covered in Chapter 7. Dynamically, the effects of gas spring hysteresis may also be accounted for by linear dampers. For the piston gas spring

$$f_{\mathrm{p}} = C_{\mathrm{H_{pc}}} (\dot{x}_{\mathrm{p}} + \dot{x}_{\mathrm{c}}) \tag{3.9}$$

and for the displacer gas spring

$$f_{\mathrm{d}} = C_{\mathrm{H_{dc}}} (\dot{x}_{\mathrm{d}} + \dot{x}_{\mathrm{c}}) \tag{3.10}$$

where $(\dot{x}_p + \dot{x}_c)$ and $(\dot{x}_d + \dot{x}_c)$ are the relative velocities with respect to the casing for the piston and displacer respectively.

The dynamic equations (3.1) and (3.5) should be modified to include gas spring damping as follows

$$M_P\ddot{x}_p = (p_c - p_b)A_P - (C_{pc} + C_{H_{pc}})\dot{x}_p - (C_{H_{pc}} + C_{pc})\dot{x}_c \tag{3.11}$$

$$M_D\ddot{x}_d = A_D\Delta p + A_R(p_c - p_d) - C_{H_{dc}}(\dot{x}_d + \dot{x}_c) \tag{3.12}$$

where the gas spring pressures p_b and p_d now represent only the pure spring component of the gas springs, i.e. the ideal pressure swing, which is a function of displacement only.

Substituting for $A_D\Delta p$ from equation (3.8) into equation (3.12):

$$M_D\ddot{x}_d = C_p\dot{x}_p + (C_d - C_{H_{dc}})\dot{x}_d + (C_c - C_{H_{dc}})\dot{x}_c + A_R(p_c - p_d). \tag{3.13}$$

If it is assumed that p_c, p_b and p_d oscillate around a mean value of pressure given by p_{mean} (the charge pressure), a linear approximation of any of these pressures would have the form

$$p = p_{\text{mean}}(1 + ax_p + bx_d + cx_c). \tag{3.14}$$

Therefore, we may write

$$(p_c - p_b)A_P/M_P = S_{pp}x_p + S_{pd}x_d + S_{pc}x_c \tag{3.15}$$

and

$$(p_c - p_d)A_R/M_D = S_{dp}x_p + S_{dd}x_d + S_{dc}x_c \tag{3.16}$$

where the linear coefficients (i.e. the S) will be developed later.

Substituting equations (3.15) and (3.16) into equations (3.11) and (3.13) respectively, we obtain

$$\ddot{x}_p = S_{pp}x_p + S_{pd}x_d + S_{pc}x_c - (C_{pc} + C_{H_{pc}})\dot{x}_p/M_P - (C_{pc} + C_{H_{pc}})\dot{x}_c/M_P \tag{3.17}$$

and

$$\ddot{x}_d = S_{dp}x_p + S_{dd}x_d + S_{dc}x_c + C_p\dot{x}_p/M_D + (C_d - C_{H_{dc}})\dot{x}_d/M_D + (C_c - C_{H_{dc}})\dot{x}_c/M_D. \tag{3.18}$$

Grouping the damping terms together, the final forms of the piston and displacer dynamic equations are generalised as follows:

$$\ddot{x}_p = S_{pp}x_p + S_{pd}x_d + S_{pc}x_c + D_{pp}\dot{x}_p + D_{pd}\dot{x}_d + D_{pc}\dot{x}_c \tag{3.19}$$

and

$$\ddot{x}_d = S_{dp}x_p + S_{dd}x_d + S_{dc}x_c + D_{dp}\dot{x}_p + D_{dd}\dot{x}_p + D_{dc}\dot{x}_c \tag{3.20}$$

where the D are the composite damping coefficients. Note that in the particular case of the model M100, D_{pd} is zero.

Since there are three unknown quantities (x_p, x_d, x_c), a third equation is required before a solution can be attempted. This equation is the dynamic equation for the casing of the engine. By inspection it will be of the same form as equations (3.19) and (3.20):

$$\ddot{x}_c = S_{cp}x_p + S_{cd}x_d + S_{cc}x_c + D_{cp}\dot{x}_p + D_{cd}\dot{x}_d + D_{cc}\dot{x}_c. \tag{3.21}$$

Thus, in the general case, the dynamic equations (also referred to as the equations of motion) may be conveniently represented as

$$\begin{bmatrix} \ddot{x}_p \\ \ddot{x}_d \\ \ddot{x}_c \end{bmatrix} = \begin{bmatrix} S_{pp} & S_{pd} & S_{pc} \\ S_{dp} & S_{dd} & S_{dc} \\ S_{cp} & S_{cd} & S_{cc} \end{bmatrix} \begin{bmatrix} x_p \\ x_d \\ x_c \end{bmatrix} + \begin{bmatrix} D_{pp} & D_{pd} & D_{pc} \\ D_{dp} & D_{dd} & D_{dc} \\ D_{cp} & D_{cd} & D_{cc} \end{bmatrix} \begin{bmatrix} \dot{x}_p \\ \dot{x}_d \\ \dot{x}_c \end{bmatrix} \tag{3.22}$$

or in shorthand:

$$[\ddot{x}] = [\mathscr{S}][x] + [\mathscr{D}][\dot{x}] \tag{3.23}$$

where $[\mathscr{S}]$ and $[\mathscr{D}]$ are the influence coefficients (per unit element mass) of the springs and dashpots, respectively. For example, coefficient S_{pd} accounts for the influence on the piston motions by virtue of a spring coupling to the motions of the displacer, whereas S_{pp} is a spring effect unique to the piston.

In deriving equation (3.22), it is assumed that the pressure terms in the dynamic equations can be linearised without too great a penalty in accuracy. Typically, the non-linearity associated with the working gas pressure is small and can be safely neglected. However, the non-linearity associated with gas springs and pressure gradient terms can be important. A careful check should always be made on the second and higher-order terms. For acceptable results these should never exceed 10% of the linear terms.

The description of the system's characteristic behaviour may be deduced by using standard control theory methods (Gupta and Hasdorff 1970). This behaviour is the way that the three elements (piston, displacer and casing) move for different values of the damping and spring coefficients, namely divergence, convergence or steady oscillation. Since we are only interested in steady oscillation, it is possible to adopt a more straightforward approach by simply assuming that an oscillatory solution exists. By this method, the frequency and conditions for oscillation are obtained directly. However, it should be noted that this approach presupposes that the non-linear elements or effects will be such that a stabilising influence is generated by their presence. A perfectly linear machine would be impossible to operate in reality, since it would be so finely balanced at its operating point that any variation in any parameter, including the load, would either cause the machine to stop or cause its oscillations to grow until collisions between its reciprocating parts occurred. The effects of the non-linearities need, therefore, to be addressed in any load-matching study.

The following solution is assumed for each element

$$x_i = X_i \exp[j(\omega t + \phi_i] \qquad i = p, d, c \tag{3.24}$$

which may be represented in vector form as shown in figure 3.3, where X_i is the amplitude and ϕ_i is the phase.

From equation (3.24), $\dot{x}_i$ and $\ddot{x}_i$ are obtained by successive differentiation:

$$\dot{x}_i = j\omega X_i \exp[j(\omega t + \phi_i)] \tag{3.25}$$

$$\ddot{x}_i = -\omega^2 X_i \exp[j(\omega t + \phi_i)]. \tag{3.26}$$

Before substituting the above solutions into the relevant equations in (3.22), the system may be simplified further by noting that there are two distinct preferred modes of operation: large piston motions relative to casing motions, when power is mainly removed from the piston, or large casing motions relative to piston motions, when power is mainly removed from the casing. If each element is considered as a separate second-order force damped system, we can represent the behaviour of that element as in figure 3.4 (Gupta and Hasdorff 1970).

Before considering figure 3.4 further, it is necessary to introduce the concept of natural frequency. For a hypothetical single degree of freedom system without damping, as shown in figure 3.5, the equation of motion is

$$M\ddot{x} = -kx \tag{3.27}$$

where M is the mass and k is the spring constant.

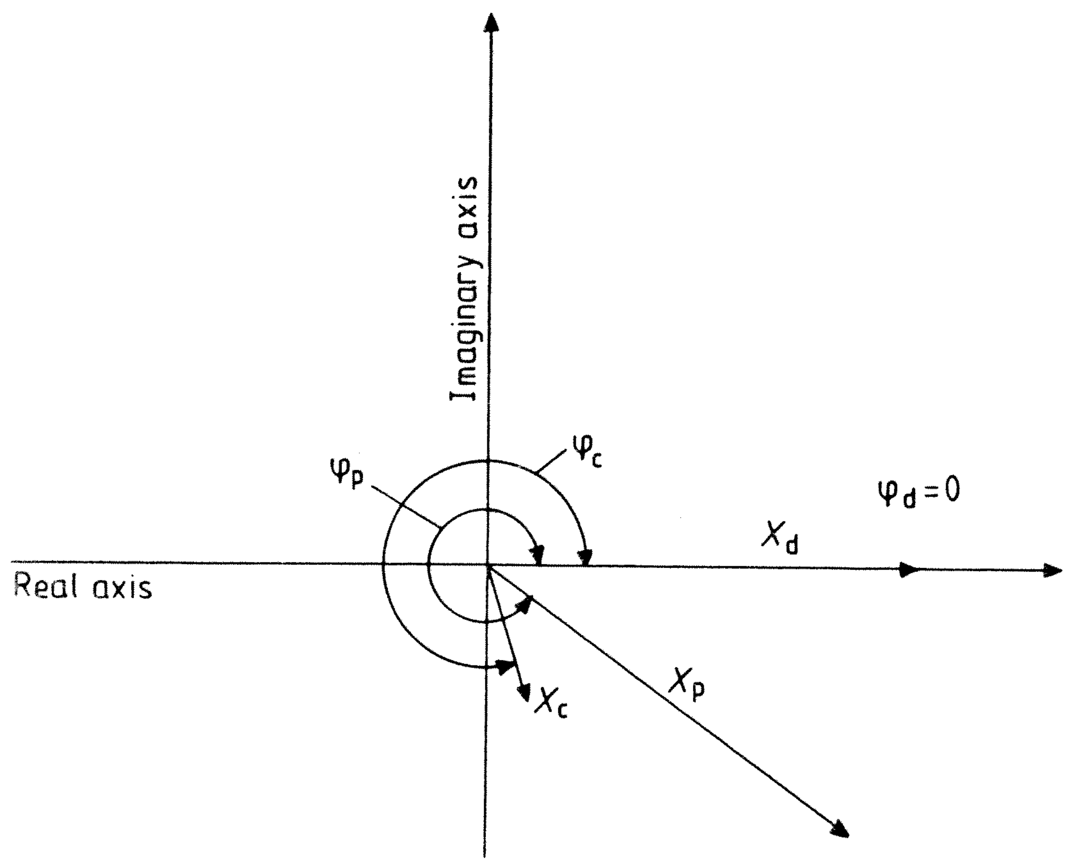

Figure 3.3 Vector notation.

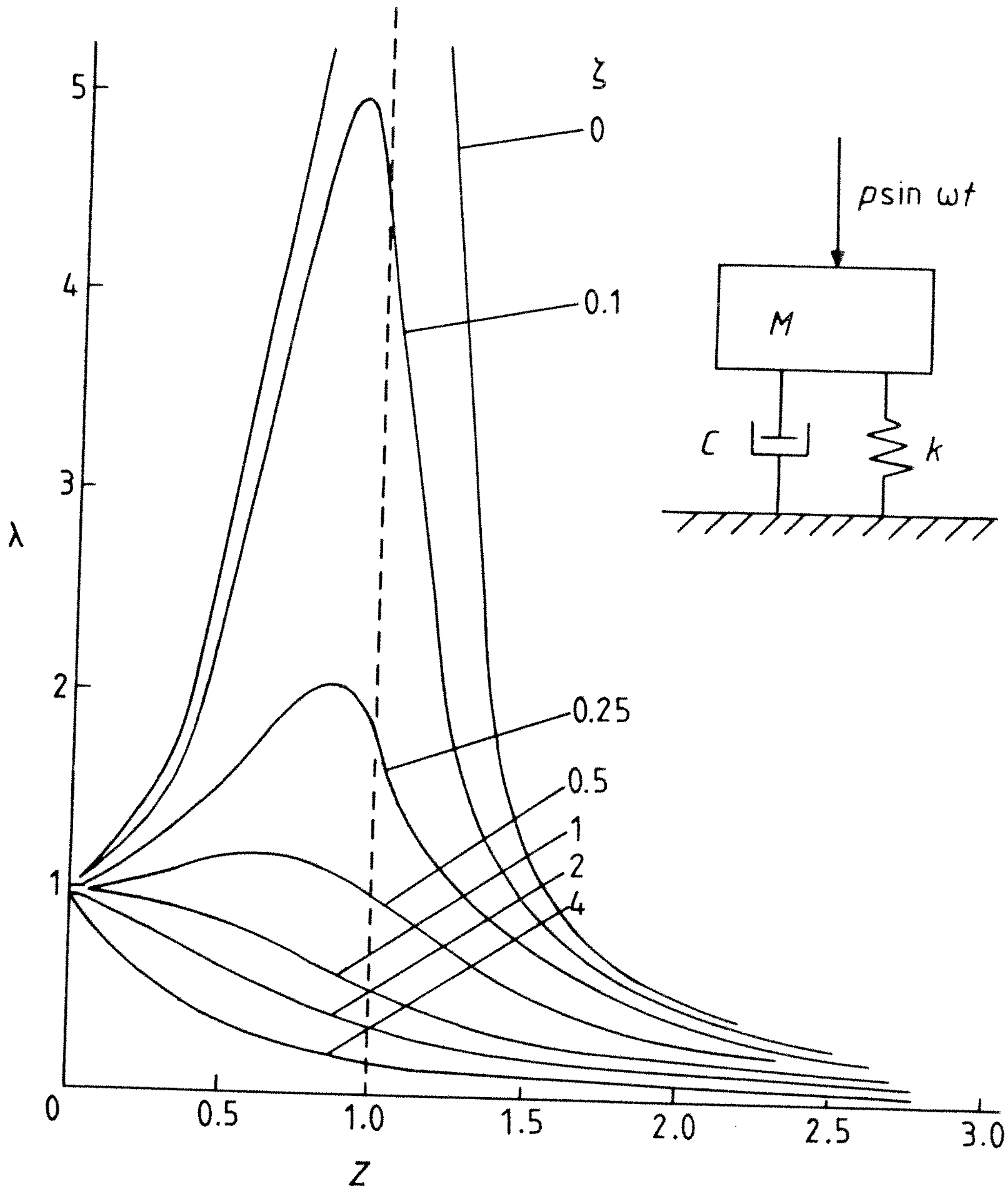

Figure 3.4 Generalised single degree of freedom system:

$$Z = \frac{\omega}{\omega_n} = \frac{\text{applied frequency}}{\text{natural frequency}},$$

$$\lambda = \frac{\text{actual amplitude}}{\text{amplitude due to a static load}},$$

ζ = viscous damping factor $C/(2M\omega_n)$.

Assuming a solution of the form given by equation (3.24), together with equation (3.26), substitution into equation (3.27) yields

$$M\omega^2 = k \tag{3.28}$$

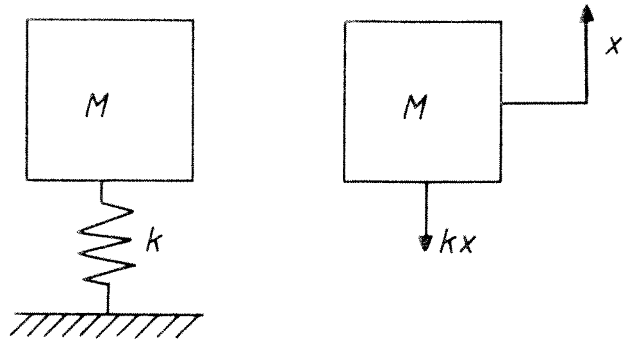

Figure 3.5 Simple spring mass system.

from which the natural frequency follows:

$$\omega_n = \sqrt{k/M}. \tag{3.29}$$

In terms of the notation used in equation set (3.22), this may be written:

$$\omega_n = \sqrt{-S}. \tag{3.30}$$

Thus it is evident that the natural frequencies of the piston, displacer and casing are given by $\sqrt{-S_{pp}}$, $\sqrt{-S_{dd}}$ and $\sqrt{-S_{cc}}$, respectively.

Returning to figure 3.4, it can be seen that operation at a frequency sufficiently higher than the natural frequency of a particular element will cause that element to oscillate at a considerably reduced amplitude.

The conditions for each mode of operation may thus be generalised as follows:

(*a*) predominant piston motion:

natural frequency of casing $\sqrt{-S_{cc}} \ll$ operating frequency ω

(*b*) predominant casing motion:

natural frequency of piston $\sqrt{-S_{pp}} \ll$ operating frequency ω.

It should be sufficient that the unwanted amplitudes be kept to at least one order of magnitude smaller than any of the active amplitudes.

The equation set in (3.22) may now be considerably simplified by ignoring the terms due to the element with small amplitudes but making sure that the conditions for the required operating mode are satisfied. Therefore, for the predominantly piston motion mode the describing equations are

$$\begin{aligned} \ddot{x}_p &= S_{pp} x_p + S_{pd} x_d + D_{pp} \dot{x}_p + D_{pd} \dot{x}_d \\ \ddot{x}_d &= S_{dp} x_p + S_{dd} x_d + D_{dp} \dot{x}_p + D_{dd} \dot{x}_d, \end{aligned} \tag{3.31}$$

subject to $\sqrt{-S_{cc}} \ll \omega$.

Similarly, for the predominantly casing motion mode, the describing equations are

$$\begin{aligned}\ddot{x}_d &= S_{dd}x_d + S_{dc}x_c + D_{dd}\dot{x}_d + D_{dc}\dot{x}_c \\ \ddot{x}_c &= S_{cd}x_c + S_{cc}x_c + D_{cd}\dot{x}_d + D_{cc}\dot{x}_c,\end{aligned} \tag{3.32}$$

this time subject to $\sqrt{-S_{pp}} \ll \omega$.

By substituting equations (3.24), (3.25) and (3.26) into equation (3.31), the following simultaneous algebraic equations are obtained for the piston motion mode

$$(S_{pd} + j\omega D_{pd})X_d e^{j\phi_d} + (\omega^2 + S_{pp} + j\omega D_{pp})X_p e^{j\phi_p} = 0 \tag{3.33}$$

$$(\omega^2 + S_{dd} + j\omega D_{dd})X_d e^{j\phi_d} + (S_{dp} + j\omega D_{dp})X_p e^{j\phi_p} = 0. \tag{3.34}$$

The corresponding equations for the casing motion mode may be inferred from equations (3.33) and (3.34) by simply replacing the subscript p by c. Since the governing equations are generically similar, only the piston motion mode will be analysed further.

Eliminating the amplitudes by substitution yields the characteristic equation

$$\begin{aligned}&\omega^4 + \omega^2(S_{pp} + S_{dd} + D_{dp}D_{pd} - D_{dd}D_{pp}) + S_{dd}S_{pp} - S_{dp}S_{pd} \\ &+ j\omega[\omega^2(D_{dd} + D_{pp}) + D_{dd}S_{pp} + S_{dd}D_{pp} - D_{pd}S_{pd} - S_{dp}D_{pd}] = 0,\end{aligned} \tag{3.35}$$

where both the real and imaginary parts are identically zero, thus

$$\omega^4 + \omega^2(S_{pp} + S_{dd} + D_{dp}D_{pd} - D_{dd}D_{pp}) + S_{dd}S_{pp} - S_{dp}S_{pd} = 0 \tag{3.36}$$

and

$$\omega^2 = (D_{dp}S_{pd} + S_{dp}D_{pd} - D_{dd}S_{pp} - S_{dd}D_{pp})/(D_{dd} + D_{pp}) \tag{3.37}$$

which are of course, subject to $\omega > \sqrt{-S_{cc}}$ by at least a factor of three. Correspondingly, the equations for the casing motion mode (by replacing the subscript p by c) would be subject to $\omega > \sqrt{-S_{pp}}$ by a similar factor.

The operating frequency is obtained from equation (3.37) whilst the physical constraints for operation are obtained by satisfying both equations (3.36) and (3.37) simultaneously.

Note that D_{pp} (and D_{cc}) are load coefficients. Since they appear in the frequency equation, it can be seen that, in general, free-piston Stirling engines have load-dependent frequencies. However, it will be shown that in some designs, with prudent selection of geometry, it is possible to reduce the variation of frequency with load to a negligible level.

For positive power, the piston and casing motions are required to lag behind those of the displacer. Therefore, it is convenient to measure phase displacements relative to the displacer motions. For the case of predominant piston motions, equation (3.24) becomes

$$x_p = X_p \exp[j(\omega t + \phi)] \tag{3.38}$$

$$x_d = X_d \exp(j\omega t) \tag{3.39}$$

where ϕ is defined as negative for piston motions lagging behind those of the displacer.

This allows equation (3.33) to be rearranged thus

$$X_p \exp(j\phi) = -(S_{pd} + j\omega D_{pd})X_d/(\omega^2 + S_{pp} + j\omega D_{pp}) \tag{3.40}$$

from which the phase and amplitude ratio are obtained:

$$\phi = \tan^{-1}\left(\frac{\omega[D_{pp}S_{pd} - D_{pd}(S_{pp} + \omega^2)]}{-[S_{pd}S_{pp} + \omega^2(S_{pd} + D_{pd}D_{pp})]}\right) \tag{3.41}$$

$$\frac{X_d}{X_p} = r = \frac{(\omega^2 + S_{pp})^2 + \omega^2 D_{pp}^2}{\{[S_{pd}(\omega^2 + S_{pp}) + \omega^2 D_{pp}D_{pd}]^2 + \omega^2[D_{pd}(\omega^2 + S_{pp}) - D_{pp}S_{pd}]^2\}^{1/2}}. \tag{3.42}$$

Typically, ϕ lies in the third or fourth quadrant ($180° < \phi < 360°$) depending on the size of the denominator in equation (3.41).

Note that both amplitudes cannot be obtained simultaneously. One amplitude must be specified, usually estimated from geometric limitations or other non-linear effects.

Once the motions of the reciprocating elements have been determined, the work transfer may be calculated easily. The cyclic work is divided into two parts: irrecoverable work and useful (or recoverable) work, both of which are evaluated from the damping coefficients.

Cyclic work done against damping is given by

$$W = \oint F\dot{x}\,dx \tag{3.43}$$

where F is the damping coefficient in $NS\,m^{-1}$. Assume x is given by

$$x = X \sin \omega t. \tag{3.44}$$

Then equation (3.43) may be integrated over a cycle to give

$$W = \pi F \omega X^2. \tag{3.45}$$

The irrecoverable work is then

$$W_{ir} = -\pi\omega M_D[(D_{dd} + D_{pd}M_P/M_D)X_d^2 + D_{dd}X_p^2] + \pi\omega C_{H_{pc}}X_p^2, \tag{3.46}$$

being the work done against the damping caused by working gas viscous friction and gas spring hysteresis, and the useful work is

$$W_s = -\pi\omega M_P D_{pp} X_p^2 - \pi\omega C_{H_{pc}} X_p^2, \tag{3.47}$$

being the difference between the work done against the damping due to the load and the irrecoverable work due to the piston gas spring. The first term on the right-hand side of equation (3.47) may be equated with the thermodynamic pV power, in order to estimate the piston amplitude. Of course, this would be an iterative procedure unless a simple functional relationship for pV power were used, for example the Schmidt isothermal power. Again, the corresponding equations for the casing motion mode are obtained by replacing the subscript p by c and M_P by M_C.

The cyclic work may also be obtained by solving the integral $\oint p\,dv$ for each working space. However, the above results yield good answers when the linear approximations are reasonable and, furthermore, they are easy to apply. Table 3.1 lists all the pertinent equations derived in this section.

Table 3.1 Set of equations for the generalised dynamic analysis piston motion mode.

Frequency	$\omega^2 = (D_{dp}S_{pd} + S_{dp}D_{pd} - D_{dd}S_{pp} - S_{dd}D_{pp})/(D_{dd} + D_{pp})$
Geometric constraint	$\omega^4 + \omega^2(S_{pp} + S_{dd} + D_{dp}D_{pd} - D_{dd}D_{pp}) + S_{dd}S_{pp} - S_{dp}S_{pd} = 0$
Piston–displacer phase angle	$\phi = \tan^{-1}\left(\dfrac{\omega[D_{pp}S_{pd} - D_{pd}(S_{pp} + \omega^2)]}{-[S_{pd}S_{pp} + \omega^2(S_{pd} + D_{pd}D_{pp})]}\right)$
Piston–displacer amplitude ratio	$\dfrac{X_d}{X_p} = r = \dfrac{(\omega^2 + S_{pp})^2 + \omega^2 D_{pp}{}^2}{\{[S_{pd}(\omega^2 + S_{pp}) + \omega^2 D_{pp}D_{pd}]^2 + \omega^2[D_{pd}(\omega^2 + S_{pp}) - D_{pp}S_{pd}]^2\}^{1/2}}$
Irrecoverable work	$W_{ir} = -\pi\omega M_D[(D_{dd} + D_{pd}M_P/M_D)X_d{}^2 + D_{dp}X_p{}^2] + \pi\omega C_{H_{pc}}X_p{}^2$
Useful work	$W_s = -\pi\omega M_P D_{pp}X_p{}^2 - \pi\omega C_{H_{pc}}X_p{}^2$
The above equations are subject to $\omega \gg \sqrt{-S_{cc}}$.	
The casing motion mode equations are obtained by replacing the subscript p by c and M_P by M_C. The resulting set of equations is subject to $\omega \gg \sqrt{-S_{pp}}$.	

3.3 Linearisation

To apply the preceding analysis it is necessary to obtain functions for the pressure variations in the working spaces and the gas springs (also referred to as bounce spaces). The Isothermal analysis lends a convenient closed-form result which is known to be a fairly good approximation to reality. This result, from Chapter 2, table 2.1, is

$$p = MR\left[\frac{V_c}{T_k} + \frac{V_k}{T_k} + \frac{V_r \ln(T_h/T_k)}{(T_h - T_k)} + \frac{V_h}{T_h} + \frac{V_e}{T_h}\right]^{-1}. \qquad (3.48)$$