# The Volume of a Ball in 4-Space [1]

Let $B(a)$ be the (closed) ball centered at the origin of radius $a$ in $\mathbf{R}^4$. So,

$$B(a) = \left\{(x, y, z, w) \in \mathbf{R}^4 : x^2 + y^2 + z^2 + w^2 \leq a^2\right\}.$$

Let $S(a)$ be the sphere centered at the origin of radius $a$ in $\mathbf{R}^4$. So,

$$S(a) = \left\{(x, y, z, w) \in \mathbf{R}^4 : x^2 + y^2 + z^2 + w^2 = a^2\right\}.$$

Let $V(a)$ be the volume of $B(a)$ and $A(a)$ be the (surface) area of $S(a)$.
Now,

$$V(a) = \iiiint\limits_{B(a)} dV = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} \int_{-\sqrt{a^2-x^2-y^2-z^2}}^{\sqrt{a^2-x^2-y^2-z^2}} dw\,dz\,dy\,dx.$$

Integrating once and using the symmetry we see the iterated integral is equal to

$$16 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \sqrt{a^2 - x^2 - y^2 - z^2}\,dz\,dy\,dx.$$

We can use MATLAB to compute this iterated integral.

1. Enter the following commands:
```
syms x y z
syms a positive ............................................ Makes a > 0.
16*int(int(int(sqrt(a^2-x^2-y^2-z^2),z,0,sqrt(a^2-x^2-y^2)), ...
y,0,sqrt(a^2-x^2)),0,a)
```

2. Given that
$$V(a) = \int_0^a A(r)\,dr$$
find $A(a)$. Hint: Use the Fundamental Theorem of Calculus.

3. Modify the commands above to find the volume of the ball of radius $a$ in $\mathbf{R}^5$. Then find the (surface) area of the sphere of radius $a$ in $\mathbf{R}^5$.

4. Using the formulas that you already know in $\mathbf{R}^2$ and in $\mathbf{R}^3$, try to come up with a general formula for the volume of the ball of radius $a$ and the (surface) area of the sphere of radius $a$ in $\mathbf{R}^n$. (Hint: You will need different formulas for $n$ even and $n$ odd.)

The student uses MATLAB to assist in computing the volume of balls and the (surface) area of spheres in $\mathbf{R}^4$ and $\mathbf{R}^5$.

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