High School Algebra and Fractions

August 24, 2014

Mathematics is a beautiful and vibrant subject. Mathematics is profound because it is simultaneously the most artistic and the most utilitarian accomplishment of humankind. There are several fundamental facts about (real) numbers that you will need to fully understand if you want to be able to continue to improve your comprehension of the world around you. If you struggle with algebra and/or fractions then hopefully this short discussion of

1 Equations

An equality between expressions, such as $5 = 5$, is preserved if we perform the same operation on the left hand side of the equation as we perform on the right hand side. An equation is like a balance beam. If you add the same weight on the left as to the right then it must remain in balance. What may change is the appearance of the expression, but not its truth. For instance, if we multiply both sides of the equation $5 = 5$, by 3, we get the equality $5 \times 3 = 5 \times 3$. It may look different, but whatever truth it carries remains unchanged. Often, especially in applications, it is critical to change the appearance of an expression to simplify or reveal some aspect of the relationship that is not necessarily apparent but may be very useful. A great example of a not obvious but true relationship is $e^{i\pi} = -1$. Did you see that one coming? If this relationship makes no sense today don't worry someday soon it will.

2 Definitions of Addition and Multiplication

You already know that you can add any two real numbers. Lets make sure that you really understand the properties of addition. Often we are going to use letters to stand for numbers. Why? Let me give an example using people. For instance there are a lot of humans in the world. It is a true statement to say that every human who ever lived had a mother. I am assuming that you understand the content of this statement. Did I need to list each person who ever lived to get my point across? No, a single statement covered it all. For this reason we use letters to represent numbers. So that we can write single statements that say it all.

A1: $(a+b)+c = a+(b+c)$. Addition is associative.

A2: $a+b = b+a$. Addition is commutative.

A3: $a+0 = 0+a = a$. Zero is the additive identity.

A4: For every number a there exists a number b such that $a+b = 0$.

Property **A4** says that each real number has an additive inverse. You may not have thought about negative numbers in this way. The number b is usually written as $-a$. The additive inverse of -5 is $-(-5) = 5$ because $-5+5 = 0$. If you add any two numbers and their sum is zero then the two numbers must be the negatives of each other. This is true because negatives are unique. How can we see what this statement means? Unique means that there exists only one. So suppose that there is more than one negative of the number a. Lets suppose that $a+b = 0$ and also that $a+c = 0$. But then if we use properties **A1-A3** we can conclude that $b = c$. Can you see how to fit the puzzle pieces together? We just write $a+b+c$ two ways. $b = 0+b = (a+c)+b = (a+b)+c = 0+c = c$. There are a lot of concepts crammed into a this very short paragraph. Read it a few more times till it makes sense.

Perhaps one of the things that is confusing about expressions like $-(-5)$ is that this expression involves only one number and not two. Addition takes two numbers to work. If we were to write $1+$, this would have no meaning, and you would probably ask, "one plus what"? The origin of this confusion may be that you see expressions like $2-1 = 1$. This is really just a shorthand for $2+(-1) = 1$. So you mistakenly imagine that the negative sign behaves like the plus sign. But logically it does not. And this can be confusing until you either get used to it, or you figure out the real logic of the additive inverse. The geometry of the additive inverse relationship is that on the number line the additive inverse is located at exactly the same place on the *other* side of zero. Additive inverses balance the number line see-saw. See Figure 1. So if you are given any number, to find its additive inverse, go to the number that is at exactly

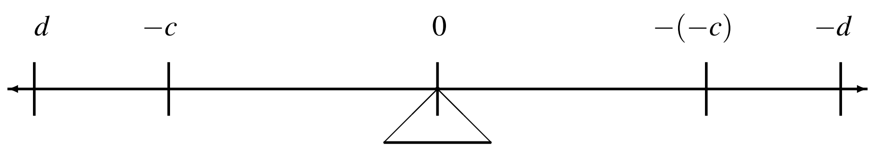

Figure 1: The number line. Additive inverses are located at equal distances on opposite sides of zero.

the same place on the number line but on the opposite side of zero. So if we start on the left of zero, we end up on the right of zero. You might have trouble understanding expressions like $-(-5)$. This is very important. The negative sign is simply a notation that means *the additive inverse of*. So to find $-(-5)$ start at 5 and find its inverse on the see saw at -5. Now to find the additive inverse of -5 go to the other side of zero and you will find 5. We could play this game all day. To find $-(-(-5))$ you make three trips back and forth starting at 5 and ending up at -5. In general an even number of trips ends you back where you started and an odd number lands you on the opposite side of zero from where you began.

You must internalize these facts if you have not already. You cannot rely on a calculator to understand these concepts for you. If you do not fully understand these facts you will never be able to harness the power that a calculator has to offer. Just as a fine violin requires a skilled musician to produce beautiful music, a calculator will only produce useful results if the operator understands the properties of numbers. Now for multiplication.

The same basic four properties hold for multiplication too. The only difference is that the identity element is not zero. The multiplicative identity is 1.

M1: $(a \times b) \times c = a \times (b \times c)$. Multiplication is associative.

M2: $a \times b = b \times a$. Multiplication is commutative.

M3: $a \times 1 = 1 \times a = a$. One is the multiplicative identity.

M4: If $a \neq 0$ then there exists $b \neq 0$ such that $a \times b = b \times a = 1$.

M5: $a \times (b+c) = a \times b + a \times c$. Multiplication distributes over addition.

The number b, guaranteed by **M4** is called the multiplicative inverse. The multiplicative inverse is often called the reciprocal. Usually people use the notation $\frac{1}{a}$ or a^{-1} to represent the multiplicative inverse or reciprocal. For instance the reciprocal of 4 is $\frac{1}{4}$. How can we know that that is a true statement? Anybody can test if two numbers are reciprocals by multiplying them together. Reciprocals multiplied together must equal 1. And, furthermore any two numbers that multiply together to produce one are reciprocals. That is the meat and potatoes of definition **M4**. You must completely internalize this definition.

Reading **M5** from left to right states how multiplication distributes over addition. Notice that reading **M5** from right to left can be described as factoring out an a. Factoring is just the opposite of distributing. This is a pattern recognition skill that you must develop through practice. For instance $3 \times a + 17 \times a = (3+17) \times a = 20 \times a$. The good news is that the human brain, your brain, is exceptionally good at pattern recognition.

3 Properties of Addition and Multiplication

We now have enough definitions to deduce some important properties of numbers. For instance $0 \times a = 0$ for any real number a. How can we show this? Well $0 \times a = (0+0) \times a$ from property **A3**. Then by **M5** we can continue to say that $0 \times a = (0+0) \times a = 0 \times a + 0 \times a$ and finally by property **A4** we can conclude that $0 = (0 \times a - 0 \times a) = 0 \times a$. Lets call this happy new fact **P1**:

P1: For any real number a, $0 \times a = a \times 0 = 0$.

Some students have told me that a number divided by itself equals zero. Lets think why this cannot be true. For a product of two real numbers to be zero, one or both of them need to be zero. You now have all the tools to demonstrate this fact.

First of all, by property **P1**, if either a or b are zero then $a \times b = 0$. Conversely, suppose that $a \times b = 0$. In any situation either $a = 0$ or $a \neq 0$. So we consider each case one at a time. If $a = 0$ then we have demonstrated what we wanted. What If $a \neq 0$ then by **M4** it has a reciprocal a^{-1}. Multiplying both sides of $a \times b = 0$ by the reciprocal we have $a^{-1} \times (a \times b) = a^{-1} \times 0 = 0$. The equality on the right follows by **P1**. Now lets simplify the left hand side of the equation. By **M1** we can write that $a^{-1} \times (a \times b) = (a^{-1} \times a) \times b = 1 \times b = b$ by **M3** and **M4**. Now putting it all together we see that $b = 0$. Do you see that? Write it all out for yourself, one step at a time. Its like putting together a jigsaw puzzle of ideas.

P2: $a \times b = 0$ if and only if $a = 0$ or $b = 0$.

An important point to understand is why zero has no reciprocal. We often say that we cannot divide by zero. Or we say that it is against the rules. It is far more accurate to say that division by zero is not defined. It has no mathematical meaning. If we try to give it a meaning it leads to a contradiction with the other definitions that we have made.

To help you to better understand the process of division think about the geometric relationship between reciprocals. On the number line reciprocals are always on opposite sides of the number 1, but not an equal distance away. Like 4 and $\frac{1}{4}$. The closer a number gets to zero the farther to the right it's reciprocal is from 1. Like $\frac{1}{100}$ and 100 or one over a million 10^{-6}, and a million 10^{6}. So if we think about it the only candidate for the reciprocal of zero would be ∞. We could add the symbol ∞ to the real numbers and define $0 \times \infty := 1$. What would happen if we made this definition? The real problem is that zero times any number must be zero. We showed that this must be true, **P1**, remember. If $0 \times \infty \neq 0$ then weird things happen. For instance if we run with our new definition we see that

$$1 = 0 \times \infty = (0+0) \times \infty = 0 \times \infty + 0 \times \infty = 1 + 1 = 2$$

Clearly $1 \neq 2$, as you can demonstrate with your fingers. So our new definition leads to a contradiction. So we cannot introduce this definition into our framework. This result has nothing to do with the symbol that we chose to represent the reciprocal of zero. If you don't like the symbol ∞ just use any other symbol you like. You will (must) get the same result. So the conclusion is that we can not define a reciprocal for zero that is consistent with the rest of our framework. Its as simple as that.

Exercise: Show that $(-1) \times a = -a$, for any real number a.

4 Working with Fractions

Hopefully it is now getting easier to see relationships, and how they follow logically from the definitions that were set out in **A1-4** and **M1-5**. In preparation for working with fractions lets show that reciprocal of a product is the product of reciprocals. This property will tell us how to multiply fractions. We want to show that $(a \times b)^{-1} = a^{-1} \times b^{-1}$. Remember, from **M4**, all we need to show is that $(a \times b) \times (a^{-1} \times b^{-1}) = 1$. This is true because $(a \times b) \times (a^{-1} \times b^{-1}) = (a \times a^{-1}) \times (b \times b^{-1}) = 1 \times 1 = 1$ because we can multiply in any order and in any association **M1** and **M2**, and the definition of reciprocal **M4**. This is important enough of a property that we should give it a name, so that we can refer to it again later when we need to use it.

P3: $(a \times b)^{-1} = a^{-1} \times b^{-1}$

Now lets talk about fractions. Often adding and subtracting fractions causes confusion. Lets see if we can't clear up the problem. The first thing to understand is that a fraction, $\frac{a}{b} = a \times b^{-1}$, is just a product of two numbers. The first fact that we can establish is that the sum of two fractions with the same denominator can be combined. This follows from **M5**: $\frac{a}{b} + \frac{c}{b} = a \times b^{-1} + c \times b^{-1} = (a+c) \times b^{-1} = \frac{a+c}{b}$. Think about this sentence until it makes sense. Lets give this property a name.

P4: $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$.

Before we can go on to add fractions with different denominators we need to establish how to multiply fractions. But we already know how, it follows from **M1**, **M2** and **M4** and **P3**.

$$\frac{a}{b} \times \frac{c}{d} = (a \times b^{-1}) \times (c \times d^{-1}) = (a \times c) \times (b^{-1} \times d^{-1}) = (a \times c) \times (b \times d)^{-1} = \frac{a \times c}{b \times d}.$$

P5: $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$

For example, lets multiple $\frac{2}{5}$ times $\frac{4}{3}$. The answer is $\frac{2 \times 4}{5 \times 3} = \frac{8}{15}$. Like shooting fish in a barrel. You have done all the hard work establishing the properties now enjoy using them. One more to go.

Ok. Now what if two fractions have different denominators? This is the part that is usually confusing. Is there a way to add $\frac{a}{b} + \frac{c}{d}$? The answer is to bootstrap on what we know from **P4** and to use **M3**. In fact, **A3** and **M3**, are the two most common and useful tools for transforming equations. If we multiply by one in disguise or add zero in disguise we can often radically change the appearance of an equation but not its truth or meaning. Lets think how to do this. $\frac{a}{b}+\frac{c}{d}=(1\times\frac{a}{b})+(\frac{c}{d}\times 1)$. Now on the left hand side lets use $1=\frac{d}{d}$ and on the right hand side lets use $1=\frac{b}{b}$. So we can write

$$\frac{a}{b}+\frac{c}{d}=(1\times\frac{a}{b})+(\frac{c}{d}\times 1)=(\frac{d}{d})\times(\frac{a}{b})+(\frac{c}{d})\times(\frac{b}{b})=\frac{(a\times d)+(c\times b)}{b\times d}$$

P6: $\frac{a}{b}+\frac{c}{d}=\frac{(a\times d)+(c\times b)}{b\times d}$

So lets use it. How about we add $\frac{2}{5}$ plus $\frac{4}{3}$. The answer is $\frac{(2\times 3)+(5\times 4)}{5\times 3}=\frac{26}{15}$.

There is only one hurdle left with fractions. How do we divide two fractions? This seems to cause a lot of confusion. But compared to the other properties that we have established this one is actually the most straightforward application of **M4**. The reciprocal of a fraction $\frac{a}{b}$ is $\frac{b}{a}$. We know this because $\frac{a}{b}\times\frac{b}{a}=\frac{a}{a}\times\frac{b}{b}$ by property **P5**. So by **M4** we can write that $(\frac{a}{b})^{-1}=\frac{b}{a}$. But this finishes our story because if we want to divide **any** number, a fraction or not, its the same as multiplying by the reciprocal. So for instance $\frac{\frac{2}{3}}{\frac{5}{7}}=\frac{2}{3}\times\frac{7}{5}=\frac{14}{15}$.