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Welcome to Precalculus, an OpenStax College resource. This textbook has been created with several goals in mind: accessibility, customization, and student engagement—all while encouraging students toward high levels of academic scholarship. Instructors and students alike will find that this textbook offers a strong foundation in precalculus in an accessible format.

About OpenStax College

OpenStax College is a non-profit organization committed to improving student access to quality learning materials. Our free textbooks go through a rigorous editorial publishing process. Our texts are developed and peer-reviewed by educators to ensure they are readable, accurate, and meet the scope and sequence requirements of today’s college courses. Unlike traditional textbooks, OpenStax College resources live online and are owned by the community of educators using them. Through our partnerships with companies and foundations committed to reducing costs for students, OpenStax College is working to improve access to higher education for all. OpenStax College is an initiative of Rice University and is made possible through the generous support of several philanthropic foundations. OpenStax College textbooks are used at many colleges and universities around the world. Please go to https://openstaxcollege.org/pages/adoptions to see our rapidly expanding number of adoptions.

About OpenStax College’s Resources

OpenStax College resources provide quality academic instruction. Three key features set our materials apart from others: they can be customized by instructors for each class, they are a “living” resource that grows online through contributions from educators, and they are available free or for minimal cost.

Customization

OpenStax College learning resources are designed to be customized for each course. Our textbooks provide a solid foundation on which instructors can build, and our resources are conceived and written with flexibility in mind. Instructors can select the sections most relevant to their curricula and create a textbook that speaks directly to the needs of their classes and student body. Teachers are encouraged to expand on existing examples by adding unique context via geographically localized applications and topical connections.

Precalculus can be easily customized using our online platform (http://cnx.org/content/col11667/latest/). Simply select the content most relevant to your current semester and create a textbook that speaks directly to the needs of your class. Precalculus is organized as a collection of sections that can be rearranged, modified, and enhanced through localized examples or to incorporate a specific theme to your course. This customization feature will ensure that your textbook truly reflects the goals of your course.

Curation

To broaden access and encourage community curation, Precalculus is “open source” licensed under a Creative Commons Attribution (CC-BY) license. The mathematics community is invited to submit feedback to enhance and strengthen the material and keep it current and relevant for today’s students. Submit your suggestions to info@openstaxcollege.org, and check in on edition status, alternate versions, errata, and news on the StaxDash at http://openstaxcollege.org.

Cost

Our textbooks are available for free online, and in low-cost print and e-book editions.

About Precalculus

Precalculus is intended for college-level precalculus students. Since precalculus courses vary from one institution to the next, we have attempted to meet the needs of as broad an audience as possible, including all of the content that might be covered in any particular course. The result is a comprehensive book that covers more ground than an instructor could likely cover in a typical one- or two-semester course; but instructors should find, almost without fail, that the topics they wish to include in their syllabus are covered in the text.

Many chapters of Openstax College Precalculus are suitable for other freshman and sophomore math courses such as College Algebra and Trigonometry; however, instructors of those courses might need to supplement or adjust the material. Openstax will also be releasing College Algebra and Algebra and Trigonometry titles tailored to the particular scope, sequence, and pedagogy of those courses.

Coverage and Scope

Precalculus contains twelve chapters, roughly divided into three groups.
Chapters 1-4 discuss various types of functions, providing a foundation for the remainder of the course.

- Chapter 1: Functions
- Chapter 2: Linear Functions
- Chapter 3: Polynomial and Rational Functions
- Chapter 4: Exponential and Logarithmic Functions

Chapters 5-8 focus on Trigonometry. In Precalculus, we approach trigonometry by first introducing angles and the unit circle, as opposed to the right triangle approach more commonly used in College Algebra and Trigonometry courses.

- Chapter 5: Trigonometric Functions
- Chapter 6: Periodic Functions
- Chapter 7: Trigonometric Identities and Equations
- Chapter 8: Further Applications of Trigonometry

Chapters 9-12 present some advanced Precalculus topics that build on topics introduced in chapters 1-8. Most Precalculus syllabi include some of the topics in these chapters, but few include all. Instructors can select material as needed from this group of chapters, since they are not cumulative.

- Chapter 9: Systems of Equations and Inequalities
- Chapter 10: Analytic Geometry
- Chapter 11: Sequences, Probability and Counting Theory
- Chapter 12: Introduction to Calculus

All chapters are broken down into multiple sections, the titles of which can be viewed in the Table of Contents.

Development Overview

OpenStax Precalculus is the product of a collaborative effort by a group of dedicated authors, editors, and instructors whose collective passion for this project has resulted in a text that is remarkably unified in purpose and voice. Special thanks is due to our Lead Author, Jay Abramson of Arizona State University, who provided the overall vision for the book and oversaw the development of each and every chapter, drawing up the initial blueprint, reading numerous drafts, and assimilating field reviews into actionable revision plans for our authors and editors.

The first eight chapters are a derivative work, built on the foundation of Precalculus: An Investigation of Functions, by David Lippman and Melonie Rasmussen. Chapters 9-12 were written and developed from by our expert and highly experienced author team. All twelve chapters follow a new and innovative instructional design, and great care has been taken to maintain a consistent voice from cover to cover. New features have been introduced to flesh out the instruction, all of the graphics have been re-done in a more contemporary style, and much of the content has been revised, replaced, or supplemented to bring the text more in line with mainstream approaches to teaching Precalculus.

Accuracy of the Content

We have taken great pains to ensure the validity and accuracy of this text. Each chapter’s manuscript underwent at least two rounds of review and revision by a panel of active Precalculus instructors. Then, prior to publication, a separate team of experts checked all text, examples, and graphics for mathematical accuracy; multiple reviewers were assigned to each chapter to minimize the chances of any error escaping notice. A third team of experts was responsible for the accuracy of the Answer Key, dutifully re-working every solution to eradicate any lingering errors. Finally, the editorial team conducted a multi-round post-production review to ensure the integrity of the content in its final form. The Solutions Manual, which was written and developed after the Student Edition, has also been rigorously checked for accuracy following a process similar to that described above. Incidentally, the act of writing out solutions step-by-step served as yet another round of validation for the Answer Key in the back of the Student Edition.

In spite of the efforts described above, we acknowledge the possibility that—as with any textbook—some errata have slipped past the guards. We encourage users to report errors via our Errata (http://legacy.cnx.org/content/https://openstaxcollege.org/errata/latest/) page.

Pedagogical Foundations and Features

Learning Objectives

Each chapter is divided into multiple sections (or modules), each of which is organized around a set of learning objectives. The learning objectives are listed explicitly at the beginning of each section, and are the focal point of every instructional element.
Narrative text

Narrative text is used to introduce key concepts, terms, and definitions, to provide real-world context, and to provide transitions between topics and examples. Throughout this book, we rely on a few basic conventions to highlight the most important ideas:

Key terms are boldfaced, typically when first introduced and/or when formally defined

Key concepts and definitions are called out in a blue box for easy reference.

Key equations, formulas, theorems, identities, etc. are assigned a number, which appears near the right margin. Occasionally the text may refer back to an equation or formula by its number.

Example

Each learning objective is supported by one or more worked examples, which demonstrate the problem-solving approaches that students must master. Typically, we include multiple Examples for each learning objective in order to model different approaches to the same type of problem, or to introduce similar problems of increasing complexity. All told, there are more than 650 Examples, or an average of about 55 per chapter.

All Examples follow a simple two- or three-part format. First, we pose a problem or question. Next, we demonstrate the Solution, spelling out the steps along the way. Finally (for select Examples), we conclude with an Analysis reflecting on the broader implications of the Solution just shown.

Figures

Openstax *Precalculus* contains more than 2000 figures and illustrations, the vast majority of which are graphs and diagrams. Art throughout the text adheres to a clear, understated style, drawing the eye to the most important information in each figure while minimizing visual distractions. Color contrast is employed with discretion to distinguish between the different functions or features of a graph.

Supporting Features

Four small but important features, each marked by a distinctive icon, serve to support Examples.

A “How To” is a list of steps necessary to solve a certain type of problem. A How To typically precedes an Example that proceeds to demonstrate the steps in action.
A “Try It” exercise immediately follows an Example or a set of related Examples, providing the student with an immediate opportunity to solve a similar problem. In the Online version of the text, students can click an Answer link directly below the question to check their understanding. In other versions, answers to the Try-It exercises are located in the Answer Key.

A Q&A may appear at any point in the narrative, but most often follows an Example. This feature pre-empt[es misconceptions by posing a commonly asked yes/no question, followed by a detailed answer and explanation.

The “Media” icon appears at the conclusion of each section, just prior to the Section Exercises. This icon marks a list of links to online video tutorials that reinforce the concepts and skills introduced in the section.

Disclaimer: While we have selected tutorials that closely align to our learning objectives, we did not produce these tutorials, nor were they specifically produced or tailored to accompany Openstax Precalculus. We are deeply grateful to James Sousa for compiling his incredibly robust and excellent library of video tutorials, which he has made available to the public under a CC-BY-SA license at http://mathispower4u.yolasite.com/. Most or all of the videos to which we link in our “Media” feature (plus many more) are found in the Algebra 2 and Trigonometry video libraries at the above site.

Section Exercises

Each section of every chapter concludes with a well-rounded set of exercises that can be assigned as homework or used selectively for guided practice. With over 5900 exercises across the 12 chapters, instructors should have plenty to choose from[1].

Section Exercises are organized by question type, and generally appear in the following order:

- **Verbal** questions assess conceptual understanding of key terms and concepts.
- **Algebraic** problems require students to apply algebraic manipulations demonstrated in the section.
- **Graphical** problems assess students’ ability to interpret or produce a graph.
- **Numeric** problems require the student perform calculations or computations.
- **Technology** problems encourage exploration through use of a graphing utility, either to visualize or verify algebraic results or to solve problems via an alternative to the methods demonstrated in the section.
- **Extensions** pose problems more challenging than the Examples demonstrated in the section. They require students to synthesize multiple learning objectives or apply critical thinking to solve complex problems.
- **Real-World Applications** present realistic problem scenarios from fields such as physics, geology, biology, finance, and the social sciences.

Chapter Review Features

Each chapter concludes with a review of the most important takeaways, as well as additional practice problems that students can use to prepare for exams.

- **Key Terms** provides a formal definition for each bold-faced term in the chapter.
- **Key Equations** presents a compilation of formulas, theorems, and standard-form equations.
- **Key Concepts** summarizes the most important ideas introduced in each section, linking back to the relevant Example(s) in case students need to review.
- **Chapter Review Exercises** include 40-80 practice problems that recall the most important concepts from each section.
- **Practice Test** includes 25-50 problems assessing the most important learning objectives from the chapter. Note that the practice test is not organized by section, and may be more heavily weighted toward cumulative objectives as opposed to the foundational objectives covered in the opening sections.
- **Answer Key** includes the answers to all Try It exercises and every other exercise from the Section Exercises, Chapter Review Exercises, and Practice Test.

1. 5,924 total exercises. Includes Chapter Reviews and Practice Tests.
Ancillaries

OpenStax projects offer an array of ancillaries for students and instructors. Currently the following resources are available.

- Instructor’s Solutions Manual
- Student’s Solutions Manual
- PowerPoint Slides

Please visit http://openstaxcollege.org to view an up-to-date list of the Learning Resources for this title and to find information on accessing these resources.

Online Homework

WebAssign

WebAssign is an independent online homework and assessment solution first launched at North Carolina State University in 1997. Today, WebAssign is an employee-owned benefit corporation and participates in the education of over a million students each year. WebAssign empowers faculty to deliver fully customizable assignments and high quality content to their students in an interactive online environment. WebAssign supports Precalculus with hundreds of problems covering every concept in the course, each containing algorithmically-generated values and links directly to the eBook providing a completely integrated online learning experience.

Learningpod is the best place to find high-quality practice and homework questions. Through our partnership with OpenStax College we offer easy-to-use assignment and reporting tools for professors and a beautiful practice experience for students. You can find questions directly from this textbook on Learningpod.com or through the OpenStax mobile app. Look for our links at the end of each chapter!

Practice questions on the Learningpod website: www.learningpod.com
Download the OpenStax Companion Workbooks app (iOS): http://bit.ly/openstaxworkbooks

About Our Team

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Jay Abramson has been teaching Precalculus for 28 years, the last 14 at Arizona State University, where he is the head lecturer in the mathematics department. His accomplishments at ASU include developing the university’s first hybrid and online math courses as well as an extensive library of video lectures and tutorials. In addition, he has served as a contributing author for two of Pearson Education’s math programs, NovaNet Precalculus and Trigonometry. Prior to coming to ASU, Jay taught at Texas Tech University and Amarillo College. He received Teacher of the Year awards at both institutions.

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[2]

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# FUNCTIONS

## 1.1 Introduction to Functions

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## 1.2 Functions and Function Notation

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## 1.3 Domain and Range

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## 1.4 Rates of Change and Behavior of Graphs

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## 1.5 Composition of Functions

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## 1.7 Absolute Value Functions

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## 1.8 Inverse Functions

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2  |  LINEAR FUNCTIONS

Figure 2.1  A bamboo forest in China (credit: “JFXie”/Flickr)

Chapter Outline

2.1 Linear Functions
2.2 Graphs of Linear Functions
2.3 Modeling with Linear Functions
2.4 Fitting Linear Models to Data

Introduction

Imagine placing a plant in the ground one day and finding that it has doubled its height just a few days later. Although it may seem incredible, this can happen with certain types of bamboo species. These members of the grass family are the fastest-growing plants in the world. One species of bamboo has been observed to grow nearly 1.5 inches every hour. In a twenty-four hour period, this bamboo plant grows about 36 inches, or an incredible 3 feet! A constant rate of change, such as the growth cycle of this bamboo plant, is a linear function.

Recall from Section 1.2 that a function is a relation that assigns to every element in the domain exactly one element in the range. Linear functions are a specific type of function that can be used to model many real-world applications, such as plant growth over time. In this chapter, we will explore linear functions, their graphs, and how to relate them to data.

Representing Linear Functions

The function describing the train’s motion is a **linear function**, which is defined as a function with a constant rate of change, that is, a polynomial of degree 1. There are several ways to represent a linear function, including word form, function notation, tabular form, and graphical form. We will describe the train’s motion as a function using each method.

**Representing a Linear Function in Word Form**

Let’s begin by describing the linear function in words. For the train problem we just considered, the following word sentence may be used to describe the function relationship.


Figure 2.2  Shanghai MagLev Train (credit: “kanegen”/Flickr)

Just as with the growth of a bamboo plant, there are many situations that involve constant change over time. Consider, for example, the first commercial maglev train in the world, the Shanghai MagLev Train (Figure 2.2). It carries passengers comfortably for a 30-kilometer trip from the airport to the subway station in only eight minutes.[2]

Suppose a maglev train were to travel a long distance, and that the train maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train’s distance from the station as a function of time? In this section, we will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations such as the train’s distance from the station at a given point in time.
• The train’s distance from the station is a function of the time during which the train moves at a constant speed plus its original distance from the station when it began moving at constant speed.

The speed is the rate of change. Recall that a rate of change is a measure of how quickly the dependent variable changes with respect to the independent variable. The rate of change for this example is constant, which means that it is the same for each input value. As the time (input) increases by 1 second, the corresponding distance (output) increases by 83 meters. The train began moving at this constant speed at a distance of 250 meters from the station.

Representing a Linear Function in Function Notation

Another approach to representing linear functions is by using function notation. One example of function notation is an equation written in the form known as the slope-intercept form of a line, where \( x \) is the input value, \( m \) is the rate of change, and \( b \) is the initial value of the dependent variable.

Equation form \( y = mx + b \)  
Equation notation \( f(x) = mx + b \)

In the example of the train, we might use the notation \( D(t) \) in which the total distance \( D \) is a function of the time \( t \). The rate, \( m \), is 83 meters per second. The initial value of the dependent variable \( b \) is the original distance from the station, 250 meters. We can write a generalized equation to represent the motion of the train.

\[ D(t) = 83t + 250 \]  

(2.2)

Representing a Linear Function in Tabular Form

A third method of representing a linear function is through the use of a table. The relationship between the distance from the station and the time is represented in Figure 2.3. From the table, we can see that the distance changes by 83 meters for every 1 second increase in time.

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D(t) )</td>
<td>250</td>
<td>333</td>
<td>416</td>
<td>499</td>
</tr>
</tbody>
</table>

Figure 2.3 Tabular representation of the function \( D \) showing selected input and output values

Can the input in the previous example be any real number?

No. The input represents time, so while nonnegative rational and irrational numbers are possible, negative real numbers are not possible for this example. The input consists of non-negative real numbers.

Representing a Linear Function in Graphical Form

Another way to represent linear functions is visually, using a graph. We can use the function relationship from above, \( D(t) = 83t + 250 \), to draw a graph, represented in Figure 2.4. Notice the graph is a line. When we plot a linear function, the graph is always a line.

The rate of change, which is constant, determines the slant, or slope of the line. The point at which the input value is zero is the vertical intercept, or \( y \)-intercept, of the line. We can see from the graph in Figure 2.4 that the \( y \)-intercept in the train example we just saw is \( (0, 250) \) and represents the distance of the train from the station when it began moving at a constant speed.
Notice that the graph of the train example is restricted, but this is not always the case. Consider the graph of the line $f(x) = 2x+1$. Ask yourself what numbers can be input to the function, that is, what is the domain of the function? The domain is comprised of all real numbers because any number may be doubled, and then have one added to the product.

**Linear Function**

A **linear function** is a function whose graph is a line. Linear functions can be written in the slope-intercept form of a line

$$f(x) = mx + b \quad (2.3)$$

where $b$ is the initial or starting value of the function (when input, $x = 0$), and $m$ is the constant rate of change, or slope of the function. The **y-intercept** is at $(0, b)$.

**Example 2.1**

**Using a Linear Function to Find the Pressure on a Diver**

The pressure, $P$, in pounds per square inch (PSI) on the diver in **Figure 2.5** depends upon her depth below the water surface, $d$, in feet. This relationship may be modeled by the equation, $P(d) = 0.434d + 14.696$. Restate this function in words.

**Solution**
Solution

Analysis
The initial value, 14.696, is the pressure in PSI on the diver at a depth of 0 feet, which is the surface of the water. The rate of change, or slope, is 0.434 PSI per foot. This tells us that the pressure on the diver increases 0.434 PSI for each foot her depth increases.

Determining whether a Linear Function Is Increasing, Decreasing, or Constant

The linear functions we used in the two previous examples increased over time, but not every linear function does. A linear function may be increasing, decreasing, or constant. For an increasing function, as with the train example, the output values increase as the input values increase. The graph of an increasing function has a positive slope. A line with a positive slope slants upward from left to right as in Figure 2.6(a). For a decreasing function, the slope is negative. The output values decrease as the input values increase. A line with a negative slope slants downward from left to right as in Figure 2.6(b). If the function is constant, the output values are the same for all input values so the slope is zero. A line with a slope of zero is horizontal as in Figure 2.6(c).

Increasing and Decreasing Functions

The slope determines if the function is an increasing linear function, a decreasing linear function, or a constant function.

- \( f(x) = mx + b \) is an increasing function if \( m > 0 \).
- \( f(x) = mx + b \) is a decreasing function if \( m < 0 \).
- \( f(x) = mx + b \) is a constant function if \( m = 0 \).

Example 2.2

Deciding whether a Function Is Increasing, Decreasing, or Constant
Some recent studies suggest that a teenager sends an average of 60 texts per day.\(^3\) For each of the following scenarios, find the linear function that describes the relationship between the input value and the output value. Then, determine whether the graph of the function is increasing, decreasing, or constant.

a. The total number of texts a teen sends is considered a function of time in days. The input is the number of days, and output is the total number of texts sent.

b. A teen has a limit of 500 texts per month in his or her data plan. The input is the number of days, and output is the total number of texts remaining for the month.

c. A teen has an unlimited number of texts in his or her data plan for a cost of $50 per month. The input is the number of days, and output is the total cost of texting each month.

**Solution**

Calculating and Interpreting Slope

In the examples we have seen so far, we have had the slope provided for us. However, we often need to calculate the slope given input and output values. Given two values for the input, \(x_1\) and \(x_2\), and two corresponding values for the output, \(y_1\) and \(y_2\)—which can be represented by a set of points, \((x_1, y_1)\) and \((x_2, y_2)\)—we can calculate the slope \(m\), as follows

\[
m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}
\]

where \(\Delta y\) is the vertical displacement and \(\Delta x\) is the horizontal displacement. Note in function notation two corresponding values for the output \(y_1\) and \(y_2\) for the function \(f\), \(y_1 = f(x_1)\) and \(y_2 = f(x_2)\), so we could equivalently write

\[
m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]

**Figure 2.7** indicates how the slope of the line between the points, \((x_1,y_1)\) and \((x_2,y_2)\), is calculated. Recall that the slope measures steepness. The greater the absolute value of the slope, the steeper the line is.

---

The slope of a function is calculated by the change in \( y \) divided by the change in \( x \). It does not matter which coordinate is used as the \((x_2, y_2)\) and which is the \((x_1, y_1)\), as long as each calculation is started with the elements from the same coordinate pair.

**Are the units for slope always units for the output units for the input?**

Yes. Think of the units as the change of output value for each unit of change in input value. An example of slope could be miles per hour or dollars per day. Notice the units appear as a ratio of units for the output per units for the input.

### Calculate Slope

The slope, or rate of change, of a function \( m \) can be calculated according to the following:

\[
m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}
\]

where \( x_1 \) and \( x_2 \) are input values, \( y_1 \) and \( y_2 \) are output values.

### Given two points from a linear function, calculate and interpret the slope.

1. Determine the units for output and input values.
2. Calculate the change of output values and change of input values.
3. Interpret the slope as the change in output values per unit of the input value.

### Example 2.3

**Finding the Slope of a Linear Function**
If \( f(x) \) is a linear function, and \((3, -2)\) and \((8, 1)\) are points on the line, find the slope. Is this function increasing or decreasing?

### Solution

#### Analysis
As noted earlier, the order in which we write the points does not matter when we compute the slope of the line as long as the first output value, or \( y \)-coordinate, used corresponds with the first input value, or \( x \)-coordinate, used.

### Example 2.4

#### Finding the Population Change from a Linear Function

The population of a city increased from 23,400 to 27,800 between 2008 and 2012. Find the change of population per year if we assume the change was constant from 2008 to 2012.

#### Solution

#### Analysis
Because we are told that the population increased, we would expect the slope to be positive. This positive slope we calculated is therefore reasonable.

### Try It

2.1 If \( f(x) \) is a linear function, and \((2, 3)\) and \((0, 4)\) are points on the line, find the slope. Is this function increasing or decreasing?

2.2 The population of a small town increased from 1,442 to 1,868 between 2009 and 2012. Find the change of population per year if we assume the change was constant from 2009 to 2012.

### Writing the Point-Slope Form of a Linear Equation

Up until now, we have been using the slope-intercept form of a linear equation to describe linear functions. Here, we will learn another way to write a linear function, the **point-slope form**.

\[
y - y_1 = m(x - x_1) \tag{2.7}\]

The point-slope form is derived from the slope formula.

\[
m = \frac{y - y_1}{x - x_1} \quad \text{assuming } x \neq x_1 \tag{2.8}\]

\[
m(x - x_1) = \frac{y - y_1}{x - x_1}(x - x_1) \quad \text{Multiply both sides by } (x - x_1).
\]

\[
m(x - x_1) = y - y_1 \quad \text{Simplify.}
\]

\[
y - y_1 = m(x - x_1) \quad \text{Rearrange.}
\]

Keep in mind that the slope-intercept form and the point-slope form can be used to describe the same function. We can move from one form to another using basic algebra. For example, suppose we are given an equation in point-slope form, \( y - 4 = \frac{-1}{2}(x - 6) \). We can convert it to the slope-intercept form as shown.
Therefore, the same line can be described in slope-intercept form as \( y = -\frac{1}{2}x + 7 \).
2.3 Write the point-slope form of an equation of a line with a slope of \(-2\) that passes through the point \((-2, 2)\). Then rewrite it in the slope-intercept form.

**Writing the Equation of a Line Using Two Points**

The point-slope form of an equation is also useful if we know any two points through which a line passes. Suppose, for example, we know that a line passes through the points \((0, 1)\) and \((3, 2)\). We can use the coordinates of the two points to find the slope.

\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]

\[
= \frac{2 - 1}{3 - 0} = \frac{1}{3}
\]

Now we can use the slope we found and the coordinates of one of the points to find the equation for the line. Let use \((0, 1)\) for our point.

\[
y - y_1 = m(x - x_1)
\]

\[
y - 1 = \frac{1}{3}(x - 0)
\]

As before, we can use algebra to rewrite the equation in the slope-intercept form.

\[
y - 1 = \frac{1}{3}(x - 0)
\]

\[
y - 1 = \frac{1}{3}x
\]

\[
y = \frac{1}{3}x + 1
\]

Add 1 to each side.

Both equations describe the line shown in **Figure 2.9**.
Writing Linear Equations Using Two Points

Write the point-slope form of an equation of a line that passes through the points (5, 1) and (8, 7). Then rewrite it in the slope-intercept form.

Solution

2.4 Write the point-slope form of an equation of a line that passes through the points (–1, 3) and (0, 0). Then rewrite it in the slope-intercept form.

Writing and Interpreting an Equation for a Linear Function

Now that we have written equations for linear functions in both the slope-intercept form and the point-slope form, we can choose which method to use based on the information we are given. That information may be provided in the form of a graph, a point and a slope, two points, and so on. Look at the graph of the function \( f \) in Figure 2.10.

![Figure 2.10](image)

We are not given the slope of the line, but we can choose any two points on the line to find the slope. Let’s choose \((0, 7)\) and \((4, 4)\). We can use these points to calculate the slope.

\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]

\[
= \frac{4 - 7}{4 - 0}
\]

\[
= -\frac{3}{4}
\]

Now we can substitute the slope and the coordinates of one of the points into the point-slope form.

\[
y - y_1 = m(x - x_1)
\]

\[
y - 4 = -\frac{3}{4}(x - 4)
\]

If we want to rewrite the equation in the slope-intercept form, we would find

\[
y - 4 = -\frac{3}{4}(x - 4)
\]

\[
y - 4 = -\frac{3}{4}x + 3
\]

\[
y = -\frac{3}{4}x + 7
\]
If we wanted to find the slope-intercept form without first writing the point-slope form, we could have recognized that the line crosses the $y$-axis when the output value is 7. Therefore, $b = 7$. We now have the initial value $b$ and the slope $m$ so we can substitute $m$ and $b$ into the slope-intercept form of a line.

$$f(x) = mx + b$$

$$f(x) = -\frac{3}{4}x + 7$$

So the function is $f(x) = -\frac{3}{4}x + 7$, and the linear equation would be $y = -\frac{3}{4}x + 7$.

**Example 2.7**

**Writing an Equation for a Linear Function**

Write an equation for a linear function given a graph of $f$ shown in Figure 2.11.

**Solution**

**Analysis**

This makes sense because we can see from Figure 2.11 that the line crosses the $y$-axis at the point $(0, 2)$, which is the $y$-intercept, so $b = 2$. 

**How To**

1. Identify two points on the line.
2. Use the two points to calculate the slope.
3. Determine where the line crosses the $y$-axis to identify the $y$-intercept by visual inspection.
4. Substitute the slope and $y$-intercept into the slope-intercept form of a line equation.
Example 2.8

Writing an Equation for a Linear Cost Function

Suppose Ben starts a company in which he incurs a fixed cost of $1,250 per month for the overhead, which includes his office rent. His production costs are $37.50 per item. Write a linear function \( C \) where \( C(x) \) is the cost for \( x \) items produced in a given month.

Solution

Analysis

If Ben produces 100 items in a month, his monthly cost is represented by

\[
C(100) = 1250 + 37.5(100)
\]

\[
= 5000
\]

So his monthly cost would be $5,000.

Example 2.9

Writing an Equation for a Linear Function Given Two Points

If \( f \) is a linear function, with \( f(3) = -2 \), and \( f(8) = 1 \), find an equation for the function in slope-intercept form.

Solution

Solution

2.5 If \( f(x) \) is a linear function, with \( f(2) = -11 \), and \( f(4) = -25 \), find an equation for the function in slope-intercept form.
Modeling Real-World Problems with Linear Functions

In the real world, problems are not always explicitly stated in terms of a function or represented with a graph. Fortunately, we can analyze the problem by first representing it as a linear function and then interpreting the components of the function. As long as we know, or can figure out, the initial value and the rate of change of a linear function, we can solve many different kinds of real-world problems.

**Given a linear function** \( f \) **and the initial value and rate of change, evaluate** \( f(c) \).

1. Determine the initial value and the rate of change (slope).
2. Substitute the values into \( f(x) = mx + b \).
3. Evaluate the function at \( x = c \).

**Example 2.10**

**Using a Linear Function to Determine the Number of Songs in a Music Collection**

Marcus currently has 200 songs in his music collection. Every month, he adds 15 new songs. Write a formula for the number of songs, \( N \), in his collection as a function of time, \( t \), the number of months. How many songs will he own in a year?

**Solution**

**Analysis**

Notice that \( N \) is an increasing linear function. As the input (the number of months) increases, the output (number of songs) increases as well.

**Example 2.11**

**Using a Linear Function to Calculate Salary Plus Commission**

Working as an insurance salesperson, Ilya earns a base salary plus a commission on each new policy. Therefore, Ilya’s weekly income, \( I \), depends on the number of new policies, \( n \), he sells during the week. Last week he sold 3 new policies, and earned $760 for the week. The week before, he sold 5 new policies and earned $920. Find an equation for \( I(n) \), and interpret the meaning of the components of the equation.

**Solution**

**Example 2.12**

**Using Tabular Form to Write an Equation for a Linear Function**

Table 2.1 relates the number of rats in a population to time, in weeks. Use the table to write a linear equation.
<table>
<thead>
<tr>
<th>$w$, number of weeks</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(w)$, number of rats</td>
<td>1000</td>
<td>1080</td>
<td>1160</td>
<td>1240</td>
</tr>
</tbody>
</table>

Table 2.1

**Solution**

**Is the initial value always provided in a table of values like Table 2.1?**

No. Sometimes the initial value is provided in a table of values, but sometimes it is not. If you see an input of 0, then the initial value would be the corresponding output. If the initial value is not provided because there is no value of input on the table equal to 0, find the slope, substitute one coordinate pair and the slope into $f(x) = mx + b$, and solve for $b$.

2.6 A new plant food was introduced to a young tree to test its effect on the height of the tree. **Table 2.2** shows the height of the tree, in feet, $x$ months since the measurements began. Write a linear function, $H(x)$, where $x$ is the number of months since the start of the experiment.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H(x)$</td>
<td>12.5</td>
<td>13.5</td>
<td>14.5</td>
<td>16.5</td>
<td>18.5</td>
</tr>
</tbody>
</table>

Table 2.2

Access this online resource for additional instruction and practice with linear functions.

- **Linear Functions** (http://openstaxcollege.org/l/linearfunctions)
2.1 EXERCISES

Verbal

1. Terry is skiing down a steep hill. Terry’s elevation, \( E(t) \), in feet after \( t \) seconds is given by \( E(t) = 3000 - 70t \). Write a complete sentence describing Terry’s starting elevation and how it is changing over time.

2. Maria is climbing a mountain. Maria’s elevation, \( E(t) \), in feet after \( t \) minutes is given by \( E(t) = 1200 + 40t \). Write a complete sentence describing Maria’s starting elevation and how it is changing over time.

3. Jessica is walking home from a friend’s house. After 2 minutes she is 1.4 miles from home. Twelve minutes after leaving, she is 0.9 miles from home. What is her rate in miles per hour?

4. Sonya is currently 10 miles from home and is walking farther away at 2 miles per hour. Write an equation for her distance from home \( t \) hours from now.

5. A boat is 100 miles away from the marina, sailing directly toward it at 10 miles per hour. Write an equation for the distance of the boat from the marina after \( t \) hours.

6. Timmy goes to the fair with $40. Each ride costs $2. How much money will he have left after riding \( n \) rides?

Algebraic

For the following exercises, determine whether the equation of the curve can be written as a linear function.

7. \( y = \frac{1}{4}x + 6 \)

8. \( y = 3x - 5 \)

9. \( y = 3x^2 - 2 \)

10. \( 3x + 5y = 15 \)

11. \( 3x^2 + 5y = 15 \)

12. \( 3x + 5y^2 = 15 \)

13. \( -2x^2 + 3y^2 = 6 \)

14. \( -\frac{x - 3}{5} = 2y \)

For the following exercises, determine whether each function is increasing or decreasing.

15. \( f(x) = 4x + 3 \)

16. \( g(x) = 5x + 6 \)

17. \( a(x) = 5 - 2x \)

18. \( b(x) = 8 - 3x \)

19. \( h(x) = -2x + 4 \)

20. \( k(x) = -4x + 1 \)

21. \( j(x) = \frac{1}{2}x - 3 \)

22.
\[ p(x) = \frac{1}{4}x - 5 \]

23. \[ n(x) = -\frac{1}{3}x - 2 \]

24. \[ m(x) = -\frac{3}{8}x + 3 \]

For the following exercises, find the slope of the line that passes through the two given points.

25. \((2, 4)\) and \((4, 10)\)

26. \((1, 5)\) and \((4, 11)\)

27. \((-1, 4)\) and \((5, 2)\)

28. \((8, -2)\) and \((4, 6)\)

29. \((6, 11)\) and \((-4, 3)\)

For the following exercises, given each set of information, find a linear equation satisfying the conditions, if possible.

30. \(f(-5) = -4\), and \(f(5) = 2\)

31. \(f(-1) = 4\) and \(f(5) = 1\)

32. \((2, 4)\) and \((4, 10)\)

33. Passes through \((1, 5)\) and \((4, 11)\)

34. Passes through \((-1, 4)\) and \((5, 2)\)

35. Passes through \((-2, 8)\) and \((4, 6)\)

36. \(x\) intercept at \((-2, 0)\) and \(y\) intercept at \((0, -3)\)

37. \(x\) intercept at \((-5, 0)\) and \(y\) intercept at \((0, 4)\)

**Graphical**

For the following exercises, find the slope of the lines graphed.

38.
39.

40.
For the following exercises, write an equation for the lines graphed.

41.

42.

43.

44.
45.

46.
**Numeric**

For the following exercises, which of the tables could represent a linear function? For each that could be linear, find a linear equation that models the data.

47.  

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(x)$</td>
<td>5</td>
<td>−10</td>
<td>−25</td>
<td>−40</td>
</tr>
</tbody>
</table>

Table 2.3

48.  

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h(x)$</td>
<td>5</td>
<td>30</td>
<td>105</td>
<td>230</td>
</tr>
</tbody>
</table>

Table 2.4

49.  

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>−5</td>
<td>20</td>
<td>45</td>
<td>70</td>
</tr>
</tbody>
</table>

Table 2.5

50.  

<table>
<thead>
<tr>
<th>$x$</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k(x)$</td>
<td>28</td>
<td>13</td>
<td>58</td>
<td>73</td>
</tr>
</tbody>
</table>

Table 2.6

51.  

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(x)$</td>
<td>6</td>
<td>−19</td>
<td>−44</td>
<td>−69</td>
</tr>
</tbody>
</table>

Table 2.7

52.  

<table>
<thead>
<tr>
<th>$x$</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>−4</td>
<td>16</td>
<td>36</td>
<td>56</td>
</tr>
</tbody>
</table>

Table 2.8

53.
If \( f \) is a linear function, \( f(0.1) = 11.5 \), and \( f(0.4) = -5.9 \), find an equation for the function.

Graph the function \( f \) on a domain of \([-10, 10]\):
\[
f(x) = 0.02x - 0.01.
\]
Enter the function in a graphing utility.

For the viewing window, set the minimum value of \( x \) to be \(-10\) and the maximum value of \( x \) to be \(10\).

Graph the function \( f \) on a domain of \([-10, 10]\):
\[
f(x) = 2,500x + 4,000
\]

Table 2.11 shows the input, \( w \), and output, \( k \), for a linear function \( k \). a. Fill in the missing values of the table. b. Write the linear function \( k \), round to 3 decimal places.

Table 2.12 shows the input, \( p \), and output, \( q \), for a linear function \( q \). a. Fill in the missing values of the table. b. Write the linear function \( q \).

Graph the linear function \( f \) on a domain of \([-10, 10]\) for the function whose slope is \(\frac{1}{8}\) and \(y\)-intercept is \(\frac{31}{16}\). Label the points for the input values of \(-10\) and \(10\).

Graph the linear function \( f \) on a domain of \([-0.1, 0.1]\) for the function whose slope is \(75\) and \(y\)-intercept is \(-22.5\). Label the points for the input values of \(-0.1\) and \(0.1\).
62. Graph the linear function \( f \) where \( f(x) = ax + b \) on the same set of axes on a domain of \([-4, 4]\) for the following values of \( a \) and \( b \).
   i. \( a = 2; \ b = 3 \)
   ii. \( a = 2; \ b = 4 \)
   iii. \( a = 2; \ b = -4 \)
   iv. \( a = 2; \ b = -5 \)

**Extensions**

63. Find the value of \( x \) if a linear function goes through the following points and has the following slope: \((x, 2), (−4, 6), m = 3\)

64. Find the value of \( y \) if a linear function goes through the following points and has the following slope: \((10, y), (25, 100), m = −5\)

65. Find the equation of the line that passes through the following points: \((a, b)\) and \((a, b + 1)\)

66. Find the equation of the line that passes through the following points: \((2a, b)\) and \((a, b + 1)\)

67. Find the equation of the line that passes through the following points: \((a, 0)\) and \((c, d)\)

**Real-World Applications**

68. At noon, a barista notices that she has $20 in her tip jar. If she makes an average of $0.50 from each customer, how much will she have in her tip jar if she serves \( n \) more customers during her shift?

69. A gym membership with two personal training sessions costs $125, while gym membership with five personal training sessions costs $260. What is cost per session?

70. A clothing business finds there is a linear relationship between the number of shirts, \( n \), it can sell and the price, \( p \), it can charge per shirt. In particular, historical data shows that 1,000 shirts can be sold at a price of $30, while 3,000 shirts can be sold at a price of $22. Find a linear equation in the form \( p(n) = mn + b \) that gives the price \( p \) they can charge for \( n \) shirts.

71. A phone company charges for service according to the formula: \( C(n) = 24 + 0.1n \), where \( n \) is the number of minutes talked, and \( C(n) \) is the monthly charge, in dollars. Find and interpret the rate of change and initial value.

72. A farmer finds there is a linear relationship between the number of bean stalks, \( n \), she plants and the yield, \( y \), each plant produces. When she plants 30 stalks, each plant yields 30 oz of beans. When she plants 34 stalks, each plant produces 28 oz of beans. Find a linear equation in the form \( y = mn + b \) that gives the yield when \( n \) stalks are planted.

73. A city’s population in the year 1960 was 287,500. In 1989 the population was 275,900. Compute the rate of growth of the population and make a statement about the population rate of change in people per year.

74. A town’s population has been growing linearly. In 2003, the population was 45,000, and the population has been growing by 1,700 people each year. Write an equation, \( P(t) \), for the population \( t \) years after 2003.

75. Suppose that average annual income (in dollars) for the years 1990 through 1999 is given by the linear function: \( I(x) = 1054x + 23,286 \), where \( x \) is the number of years after 1990. Which of the following interprets the slope in the context of the problem?
   a. As of 1990, average annual income was $23,286.
   b. In the ten-year period from 1990–1999, average annual income increased by a total of $1,054.
   c. Each year in the decade of the 1990s, average annual income increased by $1,054.
   d. Average annual income rose to a level of $23,286 by the end of 1999.
76. When temperature is 0 degrees Celsius, the Fahrenheit temperature is 32. When the Celsius temperature is 100, the corresponding Fahrenheit temperature is 212. Express the Fahrenheit temperature as a linear function of \( C \), the Celsius temperature, \( F(C) \).

a. Find the rate of change of Fahrenheit temperature for each unit change temperature of Celsius.

b. Find and interpret \( F(28) \).

c. Find and interpret \( F(-40) \).
Two competing telephone companies offer different payment plans. The two plans charge the same rate per long distance minute, but charge a different monthly flat fee. A consumer wants to determine whether the two plans will ever cost the same amount for a given number of long distance minutes used. The total cost of each payment plan can be represented by a linear function. To solve the problem, we will need to compare the functions. In this section, we will consider methods of comparing functions using graphs.

Graphing Linear Functions

In Linear Functions (http://legacy.cnx.org/content/m10352/latest/), we saw that that the graph of a linear function is a straight line. We were also able to see the points of the function as well as the initial value from a graph. By graphing two functions, then, we can more easily compare their characteristics.

There are three basic methods of graphing linear functions. The first is by plotting points and then drawing a line through the points. The second is by using the y-intercept and slope. And the third method is by using transformations of the identity function \( f(x) = x \).

### Graphing a Function by Plotting Points

To find points of a function, we can choose input values, evaluate the function at these input values, and calculate output values. The input values and corresponding output values form coordinate pairs. We then plot the coordinate pairs on a grid. In general, we should evaluate the function at a minimum of two inputs in order to find at least two points on the graph. For example, given the function, \( f(x) = 2x \), we might use the input values 1 and 2. Evaluating the function for an input value of 1 yields an output value of 2, which is represented by the point \((1, 2)\). Evaluating the function for an input value of 2 yields an output value of 4, which is represented by the point \((2, 4)\). Choosing three points is often advisable because if all three points do not fall on the same line, we know we made an error.

**How To:** Given a linear function, graph by plotting points.

1. Choose a minimum of two input values.
2. Evaluate the function at each input value.
3. Use the resulting output values to identify coordinate pairs.
4. Plot the coordinate pairs on a grid.
5. Draw a line through the points.

### Example 2.13

**Graphing by Plotting Points**

Graph \( f(x) = -\frac{2}{3}x + 5 \) by plotting points.
Analysis
The graph of the function is a line as expected for a linear function. In addition, the graph has a downward slant, which indicates a negative slope. This is also expected from the negative constant rate of change in the equation for the function.

Graph 2.7  \( f(x) = -\frac{3}{4}x + 6 \) by plotting points.

Graphing a Function Using \( y \)-intercept and Slope
Another way to graph linear functions is by using specific characteristics of the function rather than plotting points. The first characteristic is its \( y \)-intercept, which is the point at which the input value is zero. To find the \( y \)-intercept, we can set \( x = 0 \) in the equation.

The other characteristic of the linear function is its slope \( m \), which is a measure of its steepness. Recall that the slope is the rate of change of the function. The slope of a function is equal to the ratio of the change in outputs to the change in inputs. Another way to think about the slope is by dividing the vertical difference, or rise, by the horizontal difference, or run. We encountered both the \( y \)-intercept and the slope in Linear Functions (http://legacy.cnx.org/content/m10352/latest/).

Let’s consider the following function.

\[
\begin{align*}
f(x) &= \frac{1}{2}x + 1 \\
\end{align*}
\]

(2.19)

The slope is \( \frac{1}{2} \). Because the slope is positive, we know the graph will slant upward from left to right. The \( y \)-intercept is the point on the graph when \( x = 0 \). The graph crosses the \( y \)-axis at \( (0, 1) \). Now we know the slope and the \( y \)-intercept. We can begin graphing by plotting the point \( (0, 1) \) We know that the slope is rise over run, \( m = \frac{\text{rise}}{\text{run}} \). From our example, we have \( m = \frac{1}{2} \), which means that the rise is 1 and the run is 2. So starting from our \( y \)-intercept \( (0, 1) \), we can rise 1 and then run 2, or run 2 and then rise 1. We repeat until we have a few points, and then we draw a line through the points as shown in Figure 2.12.

Figure 2.12

Graphical Interpretation of a Linear Function
In the equation \( f(x) = mx + b \)

- \( b \) is the \( y \)-intercept of the graph and indicates the point \( (0, b) \) at which the graph crosses the \( y \)-axis.
• The slope of the line indicates the vertical displacement (rise) and horizontal displacement (run) between each successive pair of points. Recall the formula for the slope:

\[ m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \] (2.20)

**Do all linear functions have y-intercepts?**

Yes. All linear functions cross the y-axis and therefore have y-intercepts. (Note: A vertical line parallel to the y-axis does not have a y-intercept, but it is not a function.)

**Given the equation for a linear function, graph the function using the y-intercept and slope.**

1. Evaluate the function at an input value of zero to find the y-intercept.
2. Identify the slope as the rate of change of the input value.
3. Plot the point represented by the y-intercept.
4. Use \( \frac{\text{rise}}{\text{run}} \) to determine at least two more points on the line.
5. Sketch the line that passes through the points.

**Example 2.14**

**Graphing by Using the y-intercept and Slope**

Graph \( f(x) = -\frac{2}{3}x + 5 \) using the y-intercept and slope.

**Solution**

**Analysis**

The graph slants downward from left to right, which means it has a negative slope as expected.

**Find a point on the graph we drew in Example 2.14 that has a negative x-value.**

**Graphing a Function Using Transformations**

Another option for graphing is to use transformations of the identity function \( f(x) = x \). A function may be transformed by a shift up, down, left, or right. A function may also be transformed using a reflection, stretch, or compression.

**Vertical Stretch or Compression**

In the equation \( f(x) = mx \), the \( m \) is acting as the vertical stretch or compression of the identity function. When \( m \) is negative, there is also a vertical reflection of the graph. Notice in Figure 2.13 that multiplying the equation of \( f(x) = x \) by \( m \) stretches the graph of \( f \) by a factor of \( m \) units if \( m > 1 \) and compresses the graph of \( f \) by a factor of \( m \) units if \( 0 < m < 1 \). This means the larger the absolute value of \( m \), the steeper the slope.
Figure 2.13  Vertical stretches and compressions and reflections on the function $f(x) = x$.

**Vertical Shift**

In $f(x) = mx + b$, the $b$ acts as the vertical shift, moving the graph up and down without affecting the slope of the line. Notice in Figure 2.14 that adding a value of $b$ to the equation of $f(x) = x$ shifts the graph of $f$ a total of $b$ units up if $b$ is positive and $|b|$ units down if $b$ is negative.
Using vertical stretches or compressions along with vertical shifts is another way to look at identifying different types of linear functions. Although this may not be the easiest way to graph this type of function, it is still important to practice each method.

**How To:** Given the equation of a linear function, use transformations to graph the linear function in the form \( f(x) = mx + b \).

1. Graph \( f(x) = x \).
2. Vertically stretch or compress the graph by a factor \( m \).
3. Shift the graph up or down \( b \) units.

**Example 2.15**

**Graphing by Using Transformations**

Graph \( f(x) = \frac{1}{2}x - 3 \) using transformations.
2.9 Graph \( f(x) = 4 + 2x \), using transformations.

**In Example 2.15, could we have sketched the graph by reversing the order of the transformations?**

No. The order of the transformations follows the order of operations. When the function is evaluated at a given input, the corresponding output is calculated by following the order of operations. This is why we performed the compression first. For example, following the order: Let the input be 2.

\[
f(2) = \frac{1}{2}(2) - 3
\]

\[
= 1 - 3
\]

\[
= -2
\]

**Writing the Equation for a Function from the Graph of a Line**

Recall that in *Linear Functions*, we wrote the equation for a linear function from a graph. Now we can extend what we know about graphing linear functions to analyze graphs a little more closely. Begin by taking a look at Figure 2.15. We can see right away that the graph crosses the \( y \)-axis at the point \((0, 4)\) so this is the \( y \)-intercept.

![Figure 2.15](image)

Then we can calculate the slope by finding the rise and run. We can choose any two points, but let’s look at the point \((-2, 0)\). To get from this point to the \( y \)-intercept, we must move up 4 units (rise) and to the right 2 units (run). So the slope must be

\[
m = \frac{\text{rise}}{\text{run}} = \frac{4}{2} = 2.
\]  

Substituting the slope and \( y \)-intercept into the slope-intercept form of a line gives

\[
y = 2x + 4.
\]

**How to:** **Given a graph of linear function, find the equation to describe the function.**

1. Identify the \( y \)-intercept of an equation.
2. Choose two points to determine the slope.
3. Substitute the \( y \)-intercept and slope into the slope-intercept form of a line.

**Example 2.16**

**Matching Linear Functions to Their Graphs**

Match each equation of the linear functions with one of the lines in Figure 2.16.
Finding the x-intercept of a Line

So far, we have been finding the y-intercepts of a function: the point at which the graph of the function crosses the y-axis. A function may also have an x-intercept, which is the x-coordinate of the point where the graph of the function crosses the x-axis. In other words, it is the input value when the output value is zero.

To find the x-intercept, set a function \( f(x) \) equal to zero and solve for the value of \( x \). For example, consider the function shown.

\[
f(x) = 3x - 6
\]  \hspace{1cm} (2.24)

Set the function equal to 0 and solve for \( x \).

\[
0 = 3x - 6 \\
6 = 3x \\
2 = x \\
x = 2
\]  \hspace{1cm} (2.25)

The graph of the function crosses the x-axis at the point \((2, 0)\).
Do all linear functions have x-intercepts?
No. However, linear functions of the form $y = c$, where $c$ is a nonzero real number are the only examples of linear functions with no x-intercept. For example, $y = 5$ is a horizontal line 5 units above the x-axis. This function has no x-intercepts, as shown in Figure 2.17.

$x$-intercept
The $x$-intercept of the function is value of $x$ when $f(x) = 0$. It can be solved by the equation $0 = mx + b$.

Example 2.17
Finding an x-intercept
Find the x-intercept of $f(x) = \frac{1}{2}x - 3$.

Solution
Analysis
A graph of the function is shown in Figure 2.17. We can see that the x-intercept is $(6, 0)$ as we expected.
2.10 Find the $x$-intercept of $f(x) = \frac{1}{2}x - 3$.

Describing Horizontal and Vertical Lines

There are two special cases of lines on a graph—horizontal and vertical lines. A **horizontal line** indicates a constant output, or $y$-value. In Figure 2.18, we see that the output has a value of 2 for every input value. The change in outputs between any two points, therefore, is 0. In the slope formula, the numerator is 0, so the slope is 0. If we use $m = 0$ in the equation $f(x) = mx + b$, the equation simplifies to $f(x) = b$. In other words, the value of the function is a constant. This graph represents the function $f(x) = 2$.

A **vertical line** indicates a constant input, or $x$-value. We can see that the input value for every point on the line is 2, but the output value varies. Because this input value is mapped to more than one output value, a vertical line does not represent a function. Notice that between any two points, the change in the input values is zero. In the slope formula, the denominator will be zero, so the slope of a vertical line is undefined.
Notice that a vertical line, such as the one in Figure 2.19, has an \( x \)-intercept, but no \( y \)-intercept unless it’s the line \( x = 0 \). This graph represents the line \( x = 2 \).

Horizontal and Vertical Lines

Lines can be horizontal or vertical.

A **horizontal line** is a line defined by an equation in the form \( f(x) = b \).

A **vertical line** is a line defined by an equation in the form \( x = a \).

Example 2.18

**Writing the Equation of a Horizontal Line**

Write the equation of the line graphed in Figure 2.20.
Solution

Solution

Example 2.19

Writing the Equation of a Vertical Line

Write the equation of the line graphed in Figure 2.21.

Solution

Solution

Determining Whether Lines are Parallel or Perpendicular

The two lines in Figure 2.22 are parallel lines: they will never intersect. Notice that they have exactly the same steepness, which means their slopes are identical. The only difference between the two lines is the $y$-intercept. If we shifted one line vertically toward the $y$-intercept of the other, they would become the same line.
We can determine from their equations whether two lines are parallel by comparing their slopes. If the slopes are the same and the y-intercepts are different, the lines are parallel. If the slopes are different, the lines are not parallel.

\[
\begin{align*}
  \frac{f(x)}{g(x)} &= \frac{-2x + 6}{-2x - 4} \quad \text{parallel} \\
  \quad &\quad \frac{f(x)}{g(x)} = \frac{3x + 2}{2x + 2} \quad \text{not parallel}
\end{align*}
\]

Unlike parallel lines, perpendicular lines do intersect. Their intersection forms a right, or 90-degree, angle. The two lines in Figure 2.23 are perpendicular.

Perpendicular lines do not have the same slope. The slopes of perpendicular lines are different from one another in a specific way. The slope of one line is the negative reciprocal of the slope of the other line. The product of a number and its reciprocal is 1. So, if \( m_1 \) and \( m_2 \) are negative reciprocals of one another, they can be multiplied together to yield \(-1\).

\[
m_1 m_2 = -1 \tag{2.26}
\]

To find the reciprocal of a number, divide 1 by the number. So the reciprocal of 8 is \( \frac{1}{8} \) and the reciprocal of \( \frac{1}{8} \) is 8. To find the negative reciprocal, first find the reciprocal and then change the sign.

As with parallel lines, we can determine whether two lines are perpendicular by comparing their slopes, assuming that the lines are neither horizontal nor perpendicular. The slope of each line below is the negative reciprocal of the other so the lines are perpendicular.
negative reciprocal of $\frac{1}{4}$ is $-4$

The product of the slopes is $-1$.

\[
-4 \left( \frac{1}{4} \right) = -1
\]
The graph shows that the lines $f(x) = 2x + 3$ and $j(x) = 2x - 6$ are parallel, and the lines $g(x) = \frac{1}{2}x - 4$ and $h(x) = -2x + 2$ are perpendicular.

Writing the Equation of a Line Parallel or Perpendicular to a Given Line

If we know the equation of a line, we can use what we know about slope to write the equation of a line that is either parallel or perpendicular to the given line.

Writing Equations of Parallel Lines

Suppose for example, we are given the following equation.

$$f(x) = 3x + 1$$ (2.31)

We know that the slope of the line formed by the function is 3. We also know that the y-intercept is $(0, 1)$. Any other line with a slope of 3 will be parallel to $f(x)$. So the lines formed by all of the following functions will be parallel to $f(x)$.

$$g(x) = 3x + 6$$
$$h(x) = 3x + 1$$
$$p(x) = 3x + \frac{2}{3}$$

Suppose then we want to write the equation of a line that is parallel to $f$ and passes through the point $(1, 7)$. We already know that the slope is 3. We just need to determine which value for $b$ will give the correct line. We can begin with the point-slope form of an equation for a line, and then rewrite it in the slope-intercept form.

$$y - y_1 = m(x - x_1)$$ (2.33)

$$y - 7 = 3(x - 1)$$
$$y - 7 = 3x - 3$$
$$y = 3x + 4$$

So $g(x) = 3x + 4$ is parallel to $f(x) = 3x + 1$ and passes through the point $(1, 7)$.

**Given the equation of a function and a point through which its graph passes, write the equation of a line parallel to the given line that passes through the given point.**

1. Find the slope of the function.
2. Substitute the given values into either the general point-slope equation or the slope-intercept equation for a line.

**Example 2.21**

**Finding a Line Parallel to a Given Line**

Find a line parallel to the graph of $f(x) = 3x + 6$ that passes through the point $(3, 0)$.

**Solution**

**Analysis**

We can confirm that the two lines are parallel by graphing them. Figure 2.23 shows that the two lines will never intersect.
Writing Equations of Perpendicular Lines

We can use a very similar process to write the equation for a line perpendicular to a given line. Instead of using the same slope, however, we use the negative reciprocal of the given slope. Suppose we are given the following function:

\[ f(x) = 2x + 4 \]  

(2.34)

The slope of the line is 2, and its negative reciprocal is \(-\frac{1}{2}\). Any function with a slope of \(-\frac{1}{2}\) will be perpendicular to \(f(x)\). So the lines formed by all of the following functions will be perpendicular to \(f(x)\).

\[ g(x) = -\frac{1}{2}x + 4 \]  

(2.35)

\[ h(x) = -\frac{1}{2}x + 2 \]

\[ p(x) = -\frac{1}{2}x - \frac{1}{2} \]

As before, we can narrow down our choices for a particular perpendicular line if we know that it passes through a given point. Suppose then we want to write the equation of a line that is perpendicular to \(f(x)\) and passes through the point \((4, 0)\). We already know that the slope is \(-\frac{1}{2}\). Now we can use the point to find the \(y\)-intercept by substituting the given values into the slope-intercept form of a line and solving for \(b\).

\[ g(x) = mx + b \]

\[ 0 = -\frac{1}{2}(4) + b \]

\[ 0 = -2 + b \]

\[ 2 = b \]

\[ b = 2 \]

(2.36)

The equation for the function with a slope of \(-\frac{1}{2}\) and a \(y\)-intercept of 2 is
\[ g(x) = -\frac{1}{2}x + 2. \quad \text{(2.37)} \]

So \( g(x) = -\frac{1}{2}x + 2 \) is perpendicular to \( f(x) = 2x + 4 \) and passes through the point \((4, 0)\). Be aware that perpendicular lines may not look obviously perpendicular on a graphing calculator unless we use the square zoom feature.

**A horizontal line has a slope of zero and a vertical line has an undefined slope. These two lines are perpendicular, but the product of their slopes is not –1. Doesn’t this fact contradict the definition of perpendicular lines?**

No. For two perpendicular linear functions, the product of their slopes is –1. However, a vertical line is not a function so the definition is not contradicted.

---

**Given the equation of a function and a point through which its graph passes, write the equation of a line perpendicular to the given line.**

1. Find the slope of the function.
2. Determine the negative reciprocal of the slope.
3. Substitute the new slope and the values for \( x \) and \( y \) from the coordinate pair provided into \( g(x) = mx + b \).
4. Solve for \( b \).
5. Write the equation for the line.

---

**Example 2.22**

**Finding the Equation of a Perpendicular Line**

Find the equation of a line perpendicular to \( f(x) = 3x + 3 \) that passes through the point \((3, 0)\).

**Solution**

A graph of the two lines is shown in Figure 2.23 below.
2.11 Given the function \( h(x) = 2x - 4 \), write an equation for the line passing through \((0, 0)\) that is

- a. parallel to \( h(x) \)
- b. perpendicular to \( h(x) \)

**How To:** Given two points on a line and a third point, write the equation of the perpendicular line that passes through the point.

1. Determine the slope of the line passing through the points.
2. Find the negative reciprocal of the slope.
3. Use the slope-intercept form or point-slope form to write the equation by substituting the known values.
4. Simplify.

### Example 2.23

**Finding the Equation of a Line Perpendicular to a Given Line Passing through a Point**

A line passes through the points \((-2, 6)\) and \((4, 5)\). Find the equation of a perpendicular line that passes through the point \((4, 5)\).

#### Solution

#### Example 2.24

**Finding a Point of Intersection Algebraically**

Find the point of intersection of the lines \( h(t) = 3t - 4 \) and \( j(t) = 5 - t \).

#### Solution

#### Analysis

Looking at **Figure 2.23**, this result seems reasonable.

Solving a System of Linear Equations Using a Graph

A system of linear equations includes two or more linear equations. The graphs of two lines will intersect at a single point if they are not parallel. Two parallel lines can also intersect if they are coincident, which means they are the same line and they intersect at every point. For two lines that are not parallel, the single point of intersection will satisfy both equations and therefore represent the solution to the system.

To find this point when the equations are given as functions, we can solve for an input value so that \( f(x) = g(x) \). In other words, we can set the formulas for the lines equal to one another, and solve for the input that satisfies the equation.
If we were asked to find the point of intersection of two distinct parallel lines, should something in the solution process alert us to the fact that there are no solutions?

Yes. After setting the two equations equal to one another, the result would be the contradiction “0 = non-zero real number”.

Look at the graph in Figure 2.23 and identify the following for the function $j(t)$:

a. $y$-intercept
b. $x$-intercept(s)
c. slope
d. Is $j(t)$ parallel or perpendicular to $h(t)$ (or neither)?
e. Is $j(t)$ an increasing or decreasing function (or neither)?
f. Write a transformation description for $j(t)$ from the identity toolkit function $f(x) = x$.

Example 2.25

Finding a Break-Even Point

A company sells sports helmets. The company incurs a one-time fixed cost for $250,000. Each helmet costs $120 to produce, and sells for $140.

a. Find the cost function, $C$, to produce $x$ helmets, in dollars.

b. Find the revenue function, $R$, from the sales of $x$ helmets, in dollars.

c. Find the break-even point, the point of intersection of the two graphs $C$ and $R$.

Solution

Analysis
This means if the company sells 12,500 helmets, they break even; both the sales and cost incurred equaled 1.75 million dollars. See Figure 2.23.
2.2 EXERCISES

Verbal

77. If the graphs of two linear functions are parallel, describe the relationship between the slopes and the $y$-intercepts.

78. If the graphs of two linear functions are perpendicular, describe the relationship between the slopes and the $y$-intercepts.

79. If a horizontal line has the equation $f(x) = a$ and a vertical line has the equation $x = a$, what is the point of intersection? Explain why what you found is the point of intersection.

80. Explain how to find a line parallel to a linear function that passes through a given point.

81. Explain how to find a line perpendicular to a linear function that passes through a given point.

Algebraic

For the following exercises, determine whether the lines given by the equations below are parallel, perpendicular, or neither parallel nor perpendicular:

82. $4x - 7y = 10$
   $7x + 4y = 1$

83. $3y + x = 12$
   $-y = 8x + 1$

84. $3y + 4x = 12$
   $-6y = 8x + 1$

85. $6x - 9y = 10$
   $3x + 2y = 1$

86. $y = \frac{2}{3}x + 1$
   $3x + 2y = 1$

87. $y = \frac{3}{4}x + 1$
   $-3x + 4y = 1$

For the following exercises, find the $x$- and $y$-intercepts of each equation

88. $f(x) = -x + 2$

89. $g(x) = 2x + 4$

90. $h(x) = 3x - 5$

91. $k(x) = -5x + 1$

92. $-2x + 5y = 20$

93. $7x + 2y = 56$

For the following exercises, use the descriptions of each pair of lines given below to find the slopes of Line 1 and Line 2. Is each pair of lines parallel, perpendicular, or neither?

94. Line 1: Passes through $(0, 6)$ and $(3, -24)$
   Line 2: Passes through $(-1, 19)$ and $(8, -71)$
95. Line 1: Passes through (−8, −55) and (10, 89)
   Line 2: Passes through (9, −44) and (4, −14)

96. Line 1: Passes through (2, 3) and (4, −1)
   Line 2: Passes through (6, 3) and (8, 5)

97. Line 1: Passes through (1, 7) and (5, 5)
   Line 2: Passes through (−1, −3) and (1, 1)

98. Line 1: Passes through (0, 5) and (3, 3)
   Line 2: Passes through (1, −5) and (3, −2)

99. Line 1: Passes through (2, 5) and (5, −1)
   Line 2: Passes through (−3, 7) and (3, −5)

100. Write an equation for a line parallel to \( f(x) = −5x − 3 \) and passing through the point (2, −12).

101. Write an equation for a line parallel to \( g(x) = 3x − 1 \) and passing through the point (4, 9).

102. Write an equation for a line perpendicular to \( h(t) = −2t + 4 \) and passing through the point (−4, −1).

103. Write an equation for a line perpendicular to \( p(t) = 3t + 4 \) and passing through the point (3, 1).

104. Find the point at which the line \( f(x) = −2x − 1 \) intersects the line \( g(x) = −x \).

105. Find the point at which the line \( f(x) = 2x + 5 \) intersects the line \( g(x) = −3x − 5 \).

106. Use algebra to find the point at which the line \( f(x) = −\frac{4}{5}x + \frac{274}{25} \) intersects the line \( h(x) = \frac{9}{4}x + \frac{73}{10} \).

107. Use algebra to find the point at which the line \( f(x) = \frac{7}{4}x + \frac{457}{60} \) intersects the line \( g(x) = \frac{4}{3}x + \frac{31}{5} \).

**Graphical**

For the following exercises, the given linear equation with its graph in Figure 2.24.

108. \( f(x) = −x − 1 \)
109. \( f(x) = -2x - 1 \)
110. \( f(x) = -\frac{1}{2}x - 1 \)
111. \( f(x) = 2 \)
112. \( f(x) = 2 + x \)
113. \( f(x) = 3x + 2 \)

For the following exercises, sketch a line with the given features.

114. An \( x \)-intercept of \((-4, 0)\) and \( y \)-intercept of \((0, -2)\)
115. An \( x \)-intercept of \((-2, 0)\) and \( y \)-intercept of \((0, 4)\)
116. A \( y \)-intercept of \((0, 7)\) and slope \(-\frac{3}{2}\)
117. A \( y \)-intercept of \((0, 3)\) and slope \(\frac{2}{5}\)
118. Passing through the points \((-6, -2)\) and \((6, -6)\)
119. Passing through the points \((-3, -4)\) and \((3, 0)\)

For the following exercises, sketch the graph of each equation.

120. \( f(x) = -2x - 1 \)
121. \( g(x) = -3x + 2 \)
122. \( h(x) = \frac{1}{3}x + 2 \)
123. \( k(x) = \frac{2}{3}x - 3 \)
124. \( f(t) = 3 + 2t \)
125. \( p(t) = -2 + 3t \)
126. \( x = 3 \)
127. \( x = -2 \)
128. \( r(x) = 4 \)
129. \( q(x) = 3 \)
130. \( 4x = -9y + 36 \)
131. \( \frac{x}{3} - \frac{y}{4} = 1 \)
132. \( 3x - 5y = 15 \)
133. \( 3x = 15 \)
134. \( 3y = 12 \)
135. If \( g(x) \) is the transformation of \( f(x) = x \) after a vertical compression by \( \frac{3}{4} \), a shift right by 2, and a shift down by 4
a. Write an equation for \( g(x) \).
b. What is the slope of this line?
c. Find the \( y \)-intercept of this line.

136. If \( g(x) \) is the transformation of \( f(x) = x \) after a vertical compression by \( \frac{1}{3} \), a shift left by 1, and a shift up by 3
a. Write an equation for \( g(x) \).
b. What is the slope of this line?
c. Find the \( y \)-intercept of this line.

For the following exercises, write the equation of the line shown in the graph.

137.

138.

139.
For the following exercises, find the point of intersection of each pair of lines if it exists. If it does not exist, indicate that there is no point of intersection.

140. \( y = \frac{3}{4}x + 1 \)
\(-3x + 4y = 12\)

141. \(2x - 3y = 12\)
\(5y + x = 30\)

142. \(2x = y - 3\)
\(y + 4x = 15\)

143. \(x - 2y + 2 = 3\)
\(x - y = 3\)

144. \(5x + 3y = -65\)
\(x - y = -5\)

**Extensions**

146. Find the equation of the line parallel to the line \( g(x) = -0.01x + 2.01 \) through the point \((1, 2)\).

147. Find the equation of the line perpendicular to the line \( g(x) = -0.01x + 2.01 \) through the point \((1, 2)\).
For the following exercises, use the functions $f(x) = -0.1x+200$ and $g(x) = 20x + 0.1$.

148. Find the point of intersection of the lines $f$ and $g$.

149. Where is $f(x)$ greater than $g(x)$? Where is $g(x)$ greater than $f(x)$?

**Real-World Applications**

150. A car rental company offers two plans for renting a car.
- Plan A: $30 per day and $0.18 per mile
- Plan B: $50 per day with free unlimited mileage

How many miles would you need to drive for plan B to save you money?

151. A cell phone company offers two plans for minutes.
- Plan A: $20 per month and $1 for every one hundred texts.
- Plan B: $50 per month with free unlimited texts.

How many texts would you need to send per month for plan B to save you money?

152. A cell phone company offers two plans for minutes.
- Plan A: $15 per month and $2 for every 300 texts.
- Plan B: $25 per month and $0.50 for every 100 texts.

How many texts would you need to send per month for plan B to save you money?
2.3 | Modeling with Linear Functions

Learning Objectives

In this section, you will:

- **2.3.1** Identify steps for modeling and solving.
- **2.3.2** Build linear models from verbal descriptions.
- **2.3.3** Build systems of linear models.

Emily is a college student who plans to spend a summer in Seattle. She has saved $3,500 for her trip and anticipates spending $400 each week on rent, food, and activities. How can we write a linear model to represent her situation? What would be the x-intercept, and what can she learn from it? To answer these and related questions, we can create a model using a linear function. Models such as this one can be extremely useful for analyzing relationships and making predictions based on those relationships. In this section, we will explore examples of linear function models.

**Identifying Steps to Model and Solve Problems**

When modeling scenarios with linear functions and solving problems involving quantities with a constant rate of change, we typically follow the same problem strategies that we would use for any type of function. Let’s briefly review them:

1. Identify changing quantities, and then define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
2. Carefully read the problem to identify important information. Look for information that provides values for the variables or values for parts of the functional model, such as slope and initial value.
3. Carefully read the problem to determine what we are trying to find, identify, solve, or interpret.
4. Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table, or even finding a formula for the function being used to model the problem.
5. When needed, write a formula for the function.
6. Solve or evaluate the function using the formula.
7. Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
8. Clearly convey your result using appropriate units, and answer in full sentences when necessary.

**Building Linear Models**

Now let’s take a look at the student in Seattle. In her situation, there are two changing quantities: time and money. The amount of money she has remaining while on vacation depends on how long she stays. We can use this information to define our variables, including units.

- **Output**: $M$, money remaining, in dollars
- **Input**: $t$, time, in weeks

So, the amount of money remaining depends on the number of weeks: $M(t)$

We can also identify the initial value and the rate of change.

- **Initial Value**: She saved $3,500, so $3,500 is the initial value for $M$.
- **Rate of Change**: She anticipates spending $400 each week, so $-400$ per week is the rate of change, or slope.

Notice that the unit of dollars per week matches the unit of our output variable divided by our input variable. Also, because the slope is negative, the linear function is decreasing. This should make sense because she is spending money each week.

The rate of change is constant, so we can start with the linear model $M(t) = mt + b$. Then we can substitute the intercept and slope provided.

$$M(t) = mt + b$$

To find the $x$- intercept, we set the output to zero, and solve for the input.

$$0 = -400t + 3500$$

$$t = \frac{3500}{400} = 8.75$$

The $x$- intercept is 8.75 weeks. Because this represents the input value when the output will be zero, we could say that Emily will have no money left after 8.75 weeks.

When modeling any real-life scenario with functions, there is typically a limited domain over which that model will be valid—almost no trend continues indefinitely. Here the domain refers to the number of weeks. In this case, it doesn’t make sense to talk about input values less than zero. A negative input value could refer to a number of weeks before she saved $3,500, but the scenario discussed poses the question once she saved $3,500 because this is when her trip and subsequent spending starts. It is also likely that this model is not valid after the $x$- intercept, unless Emily will use a credit card and goes into debt. The domain represents the set of input values, so the reasonable domain for this function is $0 \leq t \leq 8.75$.

In the above example, we were given a written description of the situation. We followed the steps of modeling a problem to analyze the information. However, the information provided may not always be the same. Sometimes we might be provided with an intercept. Other times we might be provided with an output value. We must be careful to analyze the information we are given, and use it appropriately to build a linear model.

**Using a Given Intercept to Build a Model**

Some real-world problems provide the $y$- intercept, which is the constant or initial value. Once the $y$- intercept is known, the $x$- intercept can be calculated. Suppose, for example, that Hannah plans to pay off a no-interest loan from her parents. Her loan balance is $1,000. She plans to pay $250 per month until her balance is $0. The $y$- intercept is the initial amount of her debt, or $1,000. The rate of change, or slope, is $-250$ per month. We can then use the slope-intercept form and the given information to develop a linear model.

$$f(x) = mx + b$$

$$= -250x + 1000$$
Now we can set the function equal to 0, and solve for $x$ to find the $x$-intercept.

\[
0 = -250x + 1000 \\
1000 = 250x \\
4 = x \\
x = 4
\]

The $x$-intercept is the number of months it takes her to reach a balance of $0$. The $x$-intercept is 4 months, so it will take Hannah four months to pay off her loan.

**Using a Given Input and Output to Build a Model**

Many real-world applications are not as direct as the ones we just considered. Instead they require us to identify some aspect of a linear function. We might sometimes instead be asked to evaluate the linear model at a given input or set the equation of the linear model equal to a specified output.

**How To:**

Given a word problem that includes two pairs of input and output values, use the linear function to solve a problem.

1. Identify the input and output values.
2. Convert the data to two coordinate pairs.
3. Find the slope.
4. Write the linear model.
5. Use the model to make a prediction by evaluating the function at a given $x$-value.
6. Use the model to identify an $x$-value that results in a given $y$-value.
7. Answer the question posed.

**Example 2.26**

**Using a Linear Model to Investigate a Town’s Population**

A town’s population has been growing linearly. In 2004 the population was 6,200. By 2009 the population had grown to 8,100. Assume this trend continues.

a. Predict the population in 2013.

b. Identify the year in which the population will reach 15,000.

**Solution**

2.14 A company sells doughnuts. They incur a fixed cost of $25,000 for rent, insurance, and other expenses. It costs $0.25 to produce each doughnut.

a. Write a linear model to represent the cost $C$ of the company as a function of $x$, the number of doughnuts produced.

b. Find and interpret the $y$-intercept.

2.15 A city’s population has been growing linearly. In 2008, the population was 28,200. By 2012, the population was 36,800. Assume this trend continues.

a. Predict the population in 2014.

b. Identify the year in which the population will reach 54,000.
Using a Diagram to Model a Problem

It is useful for many real-world applications to draw a picture to gain a sense of how the variables representing the input and output may be used to answer a question. To draw the picture, first consider what the problem is asking for. Then, determine the input and the output. The diagram should relate the variables. Often, geometrical shapes or figures are drawn. Distances are often traced out. If a right triangle is sketched, the Pythagorean Theorem relates the sides. If a rectangle is sketched, labeling width and height is helpful.

Example 2.27

Using a Diagram to Model Distance Walked

Anna and Emanuel start at the same intersection. Anna walks east at 4 miles per hour while Emanuel walks south at 3 miles per hour. They are communicating with a two-way radio that has a range of 2 miles. How long after they start walking will they fall out of radio contact?

Solution

Should I draw diagrams when given information based on a geometric shape?

Yes. Sketch the figure and label the quantities and unknowns on the sketch.

Example 2.28

Using a Diagram to Model Distance between Cities

There is a straight road leading from the town of Westborough to Agritown 30 miles east and 10 miles north. Partway down this road, it junctions with a second road, perpendicular to the first, leading to the town of Eastborough. If the town of Eastborough is located 20 miles directly east of the town of Westborough, how far is the road junction from Westborough?

Solution

Analysis

One nice use of linear models is to take advantage of the fact that the graphs of these functions are lines. This means real-world applications discussing maps need linear functions to model the distances between reference points.

Building Systems of Linear Models

Real-world situations including two or more linear functions may be modeled with a system of linear equations. Remember, when solving a system of linear equations, we are looking for points the two lines have in common. Typically, there are three types of answers possible, as shown in Figure 2.26.
Given a situation that represents a system of linear equations, write the system of equations and identify the solution.

1. Identify the input and output of each linear model.
2. Identify the slope and y-intercept of each linear model.
3. Find the solution by setting the two linear functions equal to another and solving for $x$, or find the point of intersection on a graph.

Example 2.29

Building a System of Linear Models to Choose a Truck Rental Company

Jamal is choosing between two truck-rental companies. The first, Keep on Trucking, Inc., charges an up-front fee of $20, then 59 cents a mile. The second, Move It Your Way, charges an up-front fee of $16, then 63 cents a mile\(^4\). When will Keep on Trucking, Inc. be the better choice for Jamal?

Solution

Access this online resource for additional instruction and practice with linear function models.

- Interpreting a Linear Function (http://openstaxcollege.org/l/interpretlinear)

---

2.3 EXERCISES

Verbal

153. Explain how to find the input variable in a word problem that uses a linear function.
154. Explain how to find the output variable in a word problem that uses a linear function.
155. Explain how to interpret the initial value in a word problem that uses a linear function.
156. Explain how to determine the slope in a word problem that uses a linear function.

Algebraic

157. Find the area of a parallelogram bounded by the y axis, the line $x = 3$, the line $f(x) = 1 + 2x$, and the line parallel to $f(x)$ passing through $(2, 7)$.

158. Find the area of a triangle bounded by the x-axis, the line $f(x) = 12 - \frac{1}{3}x$, and the line perpendicular to $f(x)$ that passes through the origin.

159. Find the area of a triangle bounded by the y-axis, the line $f(x) = 9 - \frac{6}{7}x$, and the line perpendicular to $f(x)$ that passes through the origin.

160. Find the area of a parallelogram bounded by the x-axis, the line $g(x) = 2$, the line $f(x) = 3x$, and the line parallel to $f(x)$ passing through $(6, 1)$.

For the following exercises, consider this scenario: A town’s population has been decreasing at a constant rate. In 2010 the population was 5,900. By 2012 the population had dropped 4,700. Assume this trend continues.

161. Predict the population in 2016.
162. Identify the year in which the population will reach 0.

For the following exercises, consider this scenario: A town’s population has been increased at a constant rate. In 2010 the population was 46,020. By 2012 the population had increased to 52,070. Assume this trend continues.

163. Predict the population in 2016.
164. Identify the year in which the population will reach 75,000.

For the following exercises, consider this scenario: A town has an initial population of 75,000. It grows at a constant rate of 2,500 per year for 5 years.

165. Find the linear function that models the town’s population $P$ as a function of the year, $t$, where $t$ is the number of years since the model began.
166. Find a reasonable domain and range for the function $P$.
167. If the function $P$ is graphed, find and interpret the x- and y-intercepts.
168. If the function $P$ is graphed, find and interpret the slope of the function.
169. When will the output reached 100,000?
170. What is the output in the year 12 years from the onset of the model?

For the following exercises, consider this scenario: The weight of a newborn is 7.5 pounds. The baby gained one-half pound a month for its first year.

171. Find the linear function that models the baby’s weight $W$ as a function of the age of the baby, in months, $t$.
172. Find a reasonable domain and range for the function $W$.
173. If the function $W$ is graphed, find and interpret the $x$- and $y$-intercepts.

174. If the function $W$ is graphed, find and interpret the slope of the function.

175. When did the baby weight 10.4 pounds?

176. What is the output when the input is 6.2? Interpret your answer.

For the following exercises, consider this scenario: The number of people afflicted with the common cold in the winter months steadily decreased by 205 each year from 2005 until 2010. In 2005, 12,025 people were inflicted.

177. Find the linear function that models the number of people inflicted with the common cold $C$ as a function of the year, $t$.

178. Find a reasonable domain and range for the function $C$.

179. If the function $C$ is graphed, find and interpret the $x$- and $y$-intercepts.

180. If the function $C$ is graphed, find and interpret the slope of the function.

181. When will the output reach 0?

182. In what year will the number of people be 9,700?

**Graphical**

For the following exercises, use the graph in Figure 2.27, which shows the profit, $y$, in thousands of dollars, of a company in a given year, $t$, where $t$ represents the number of years since 1980.

![Figure 2.27](graph)

183. Find the linear function $y$, where $y$ depends on $t$, the number of years since 1980.

184. Find and interpret the $y$-intercept.

185. Find and interpret the $x$-intercept.

186. Find and interpret the slope.

For the following exercises, use the graph in Figure 2.28, which shows the profit, $y$, in thousands of dollars, of a company in a given year, $t$, where $t$ represents the number of years since 1980.
187. Find the linear function $y$, where $y$ depends on $t$, the number of years since 1980.

188. Find and interpret the $y$-intercept.

189. Find and interpret the $x$-intercept.

190. Find and interpret the slope.

**Numeric**

For the following exercises, use the median home values in Mississippi and Hawaii (adjusted for inflation) shown in Table 2.13. Assume that the house values are changing linearly.

<table>
<thead>
<tr>
<th>Year</th>
<th>Mississippi</th>
<th>Hawaii</th>
</tr>
</thead>
<tbody>
<tr>
<td>1950</td>
<td>$25,200</td>
<td>$74,400</td>
</tr>
<tr>
<td>2000</td>
<td>$71,400</td>
<td>$272,700</td>
</tr>
</tbody>
</table>

Table 2.13

191. In which state have home values increased at a higher rate?

192. If these trends were to continue, what would be the median home value in Mississippi in 2010?

193. If we assume the linear trend existed before 1950 and continues after 2000, the two states’ median house values will be (or were) equal in what year? (The answer might be absurd.)

For the following exercises, use the median home values in Indiana and Alabama (adjusted for inflation) shown in Table 2.14. Assume that the house values are changing linearly.

<table>
<thead>
<tr>
<th>Year</th>
<th>Indiana</th>
<th>Alabama</th>
</tr>
</thead>
<tbody>
<tr>
<td>1950</td>
<td>$37,700</td>
<td>$27,100</td>
</tr>
<tr>
<td>2000</td>
<td>$94,300</td>
<td>$85,100</td>
</tr>
</tbody>
</table>

Table 2.14

194. In which state have home values increased at a higher rate?

195. If these trends were to continue, what would be the median home value in Indiana in 2010?
196. If we assume the linear trend existed before 1950 and continues after 2000, the two states’ median house values will be (or were) equal in what year? (The answer might be absurd.)

**Real-World Applications**

197. In 2004, a school population was 1001. By 2008 the population had grown to 1697. Assume the population is changing linearly.
   a. How much did the population grow between the year 2004 and 2008?
   b. How long did it take the population to grow from 1001 students to 1697 students?
   c. What is the average population growth per year?
   d. What was the population in the year 2000?
   e. Find an equation for the population, \( P \) of the school \( t \) years after 2000.
   f. Using your equation, predict the population of the school in 2011.

198. In 2003, a town’s population was 1431. By 2007 the population had grown to 2134. Assume the population is changing linearly.
   a. How much did the population grow between the year 2003 and 2007?
   b. How long did it take the population to grow from 1431 people to 2134 people?
   c. What is the average population growth per year?
   d. What was the population in the year 2000?
   e. Find an equation for the population, \( P \) of the town \( t \) years after 2000.
   f. Using your equation, predict the population of the town in 2014.

199. A phone company has a monthly cellular plan where a customer pays a flat monthly fee and then a certain amount of money per minute used on the phone. If a customer uses 410 minutes, the monthly cost will be $71.50. If the customer uses 720 minutes, the monthly cost will be $118.
   a. Find a linear equation for the monthly cost of the cell plan as a function of \( x \), the number of monthly minutes used.
   b. Interpret the slope and \( y \)-intercept of the equation.
   c. Use your equation to find the total monthly cost if 687 minutes are used.

200. A phone company has a monthly cellular data plan where a customer pays a flat monthly fee of $10 and then a certain amount of money per megabyte (MB) of data used on the phone. If a customer uses 20 MB, the monthly cost will be $11.20. If the customer uses 130 MB, the monthly cost will be $17.80.
   a. Find a linear equation for the monthly cost of the data plan as a function of \( x \), the number of MB used.
   b. Interpret the slope and \( y \)-intercept of the equation.
   c. Use your equation to find the total monthly cost if 250 MB are used.

201. In 1991, the moose population in a park was measured to be 4,360. By 1999, the population was measured again to be 5,880. Assume the population continues to change linearly.
   a. Find a formula for the moose population, \( P \) since 1990.
   b. What does your model predict the moose population to be in 2003?

202. In 2003, the owl population in a park was measured to be 340. By 2007, the population was measured again to be 285. The population changes linearly. Let the input be years since 1990.
   a. Find a formula for the owl population, \( P \). Let the input be years since 2003.
   b. What does your model predict the owl population to be in 2012?

203. The Federal Helium Reserve held about 16 billion cubic feet of helium in 2010 and is being depleted by about 2.1 billion cubic feet each year.
   a. Give a linear equation for the remaining federal helium reserves, \( R \), in terms of \( t \), the number of years since 2010.
   b. In 2015, what will the helium reserves be?
   c. If the rate of depletion doesn’t change, in what year will the Federal Helium Reserve be depleted?
Suppose the world’s oil reserves in 2014 are 1,820 billion barrels. If, on average, the total reserves are decreasing by 25 billion barrels of oil each year:

a. Give a linear equation for the remaining oil reserves, \( R \), in terms of \( t \), the number of years since now.

b. Seven years from now, what will the oil reserves be?

c. If the rate at which the reserves are decreasing is constant, when will the world’s oil reserves be depleted?

You are choosing between two different prepaid cell phone plans. The first plan charges a rate of 26 cents per minute. The second plan charges a monthly fee of $19.95 plus 11 cents per minute. How many minutes would you have to use in a month in order for the second plan to be preferable?

You are choosing between two different window washing companies. The first charges $5 per window. The second charges a base fee of $40 plus $3 per window. How many windows would you need to have for the second company to be preferable?

When hired at a new job selling jewelry, you are given two pay options:

- Option A: Base salary of $17,000 a year with a commission of 12% of your sales
- Option B: Base salary of $20,000 a year with a commission of 5% of your sales

How much jewelry would you need to sell for option A to produce a larger income?

When hired at a new job selling electronics, you are given two pay options:

- Option A: Base salary of $14,000 a year with a commission of 10% of your sales
- Option B: Base salary of $19,000 a year with a commission of 4% of your sales

How much electronics would you need to sell for option A to produce a larger income?

When hired at a new job selling electronics, you are given two pay options:

- Option A: Base salary of $20,000 a year with a commission of 12% of your sales
- Option B: Base salary of $26,000 a year with a commission of 3% of your sales

How much electronics would you need to sell for option A to produce a larger income?

When hired at a new job selling electronics, you are given two pay options:

- Option A: Base salary of $10,000 a year with a commission of 9% of your sales
- Option B: Base salary of $20,000 a year with a commission of 4% of your sales

How much electronics would you need to sell for option A to produce a larger income?
A professor is attempting to identify trends among final exam scores. His class has a mixture of students, so he wonders if there is any relationship between age and final exam scores. One way for him to analyze the scores is by creating a diagram that relates the age of each student to the exam score received. In this section, we will examine one such diagram known as a scatter plot.

Drawing and Interpreting Scatter Plots

A scatter plot is a graph of plotted points that may show a relationship between two sets of data. If the relationship is from a linear model, or a model that is nearly linear, the professor can draw conclusions using his knowledge of linear functions. Figure 2.29 shows a sample scatter plot.

Example 2.30

Using a Scatter Plot to Investigate Cricket Chirps

Table 2.15 shows the number of cricket chirps in 15 seconds, for several different air temperatures, in degrees Fahrenheit[5]. Plot this data, and determine whether the data appears to be linearly related.
Finding the Line of Best Fit

Once we recognize a need for a linear function to model that data, the natural follow-up question is “what is that linear function?” One way to approximate our linear function is to sketch the line that seems to best fit the data. Then we can extend the line until we can verify the $y$-intercept. We can approximate the slope of the line by extending it until we can estimate the rise over run.

**Example 2.31**

**Finding a Line of Best Fit**

Find a linear function that fits the data in Table 2.15 by “eyeballing” a line that seems to fit.

**Solution**

**Analysis**

This linear equation can then be used to approximate answers to various questions we might ask about the trend.

**Recognizing Interpolation or Extrapolation**

While the data for most examples does not fall perfectly on the line, the equation is our best guess as to how the relationship will behave outside of the values for which we have data. We use a process known as interpolation when we predict a value inside the domain and range of the data. The process of extrapolation is used when we predict a value outside the domain and range of the data.

*Figure 2.30* compares the two processes for the cricket-chirp data addressed in Example 2.31. We can see that interpolation would occur if we used our model to predict temperature when the values for chirps are between 18.5 and 44. Extrapolation would occur if we used our model to predict temperature when the values for chirps are less than 18.5 or greater than 44.

There is a difference between making predictions inside the domain and range of values for which we have data and outside that domain and range. Predicting a value outside of the domain and range has its limitations. When our model no longer applies after a certain point, it is sometimes called model breakdown. For example, predicting a cost function for a period of two years may involve examining the data where the input is the time in years and the output is the cost. But if we try to extrapolate a cost when $x = 50$, that is in 50 years, the model would not apply because we could not account for factors fifty years in the future.
Interpolation and Extrapolation

Different methods of making predictions are used to analyze data.

- The method of **interpolation** involves predicting a value inside the domain and/or range of the data.
- The method of **extrapolation** involves predicting a value outside the domain and/or range of the data.
- **Model breakdown** occurs at the point when the model no longer applies.

Example 2.32

**Understanding Interpolation and Extrapolation**

Use the cricket data from Table 2.15 to answer the following questions:

a. Would predicting the temperature when crickets are chirping 30 times in 15 seconds be interpolation or extrapolation? Make the prediction, and discuss whether it is reasonable.

b. Would predicting the number of chirps crickets will make at 40 degrees be interpolation or extrapolation? Make the prediction, and discuss whether it is reasonable.

**Solution**

**Analysis**

Our model predicts the crickets would chirp 8.33 times in 15 seconds. While this might be possible, we have no reason to believe our model is valid outside the domain and range. In fact, generally crickets stop chirping altogether below around 50 degrees.

**Finding the Line of Best Fit Using a Graphing Utility**

While eyeballing a line works reasonably well, there are statistical techniques for fitting a line to data that minimize the differences between the line and data values. One such technique is called **least squares regression** and can be computed...
by many graphing calculators, spreadsheet software, statistical software, and many web-based calculators\(^7\). Least squares regression is one means to determine the line that best fits the data, and here we will refer to this method as linear regression.

**How To:** Given data of input and corresponding outputs from a linear function, find the best fit line using linear regression.

1. Enter the input in List 1 (L1).
2. Enter the output in List 2 (L2).
3. On a graphing utility, select Linear Regression (LinReg).

**Example 2.33**

**Finding a Least Squares Regression Line**

Find the least squares regression line using the cricket-chirp data in Table 2.15.

**Solution**

**Analysis**

Notice that this line is quite similar to the equation we “eyeballed” but should fit the data better. Notice also that using this equation would change our prediction for the temperature when hearing 30 chirps in 15 seconds from 66 degrees to:

\[
T(30) = 30.281 + 1.143(30) \\
= 64.571 \\
≈ 64.6 \text{ degrees}
\]

The graph of the scatter plot with the least squares regression line is shown in Figure 2.30.

**Will there ever be a case where two different lines will serve as the best fit for the data?**

No. There is only one best fit line.

---

6. Technically, the method minimizes the sum of the squared differences in the vertical direction between the line and the data values.

7. For example, http://www.shodor.org/unchem/math/lls/leastsq.html
Distinguishing Between Linear and Non-Linear Models

As we saw above with the cricket-chirp model, some data exhibit strong linear trends, but other data, like the final exam scores plotted by age, are clearly nonlinear. Most calculators and computer software can also provide us with the correlation coefficient, which is a measure of how closely the line fits the data. Many graphing calculators require the user to turn a “diagnostic on” selection to find the correlation coefficient, which mathematicians label as \( r \). The correlation coefficient provides an easy way to get an idea of how close to a line the data falls.

We should compute the correlation coefficient only for data that follows a linear pattern or to determine the degree to which a data set is linear. If the data exhibits a nonlinear pattern, the correlation coefficient for a linear regression is meaningless.

To get a sense for the relationship between the value of \( r \) and the graph of the data, Figure 2.31 shows some large data sets with their correlation coefficients. Remember, for all plots, the horizontal axis shows the input and the vertical axis shows the output.

![Figure 2.31 Plotted data and related correlation coefficients. (credit: “DenisBoigelot,” Wikimedia Commons)](credit: “DenisBoigelot,” Wikimedia Commons)

**Correlation Coefficient**

The correlation coefficient is a value, \( r \), between –1 and 1.

- \( r > 0 \) suggests a positive (increasing) relationship
- \( r < 0 \) suggests a negative (decreasing) relationship
- The closer the value is to 0, the more scattered the data.
- The closer the value is to 1 or –1, the less scattered the data is.

**Example 2.34**

**Finding a Correlation Coefficient**

Calculate the correlation coefficient for cricket-chirp data in Table 2.15.

**Solution**

This content is available for free at http://legacy.cnx.org/content/col11667/1.2
Predicting with a Regression Line

Once we determine that a set of data is linear using the correlation coefficient, we can use the regression line to make predictions. As we learned above, a regression line is a line that is closest to the data in the scatter plot, which means that only one such line is a best fit for the data.

Example 2.35

Using a Regression Line to Make Predictions

Gasoline consumption in the United States has been steadily increasing. Consumption data from 1994 to 2004 is shown in Table 2.16. Determine whether the trend is linear, and if so, find a model for the data. Use the model to predict the consumption in 2008.

<table>
<thead>
<tr>
<th>Year</th>
<th>'94</th>
<th>'95</th>
<th>'96</th>
<th>'97</th>
<th>'98</th>
<th>'99</th>
<th>'00</th>
<th>'01</th>
<th>'02</th>
<th>'03</th>
<th>'04</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consumption (billions of gallons)</td>
<td>113</td>
<td>116</td>
<td>118</td>
<td>119</td>
<td>123</td>
<td>125</td>
<td>126</td>
<td>128</td>
<td>131</td>
<td>133</td>
<td>136</td>
</tr>
</tbody>
</table>

Table 2.16

The scatter plot of the data, including the least squares regression line, is shown in Figure 2.32.

2.18 Use the model we created using technology in Example 2.35 to predict the gas consumption in 2011. Is this an interpolation or an extrapolation?

Access these online resources for additional instruction and practice with fitting linear models to data.

- Introduction to Regression Analysis (http://openstaxcollege.org/l/introregress)
- Linear Regression (http://openstaxcollege.org/l/linearregress)

2.4 EXERCISES

Verbal

211. Describe what it means if there is a model breakdown when using a linear model.
212. What is interpolation when using a linear model?
213. What is extrapolation when using a linear model?
214. Explain the difference between a positive and a negative correlation coefficient.
215. Explain how to interpret the absolute value of a correlation coefficient.

Algebraic

216. A regression was run to determine whether there is a relationship between hours of TV watched per day \( (x) \) and number of sit-ups a person can do \( (y) \). The results of the regression are given below. Use this to predict the number of sit-ups a person who watches 11 hours of TV can do.

\[
y = ax + b
\]

\[
a = -1.341 \\
b = 32.234 \\
r = -0.896
\]

217. A regression was run to determine whether there is a relationship between the diameter of a tree \( (x, \text{ in inches}) \) and the tree’s age \( (y, \text{ in years}) \). The results of the regression are given below. Use this to predict the age of a tree with diameter 10 inches.

\[
y = ax + b
\]

\[
a = 6.301 \\
b = -1.044 \\
r = -0.970
\]

For the following exercises, draw a scatter plot for the data provided. Does the data appear to be linearly related?

218.

<table>
<thead>
<tr>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>-22</td>
<td>-19</td>
<td>-15</td>
<td>-11</td>
<td>-6</td>
<td>-2</td>
</tr>
</tbody>
</table>

Table 2.17

219.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>46</td>
<td>50</td>
<td>59</td>
<td>75</td>
<td>100</td>
<td>136</td>
</tr>
</tbody>
</table>

Table 2.18

220.

<table>
<thead>
<tr>
<th>100</th>
<th>250</th>
<th>300</th>
<th>450</th>
<th>600</th>
<th>750</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>12.6</td>
<td>13.1</td>
<td>14</td>
<td>14.5</td>
<td>15.2</td>
</tr>
</tbody>
</table>

Table 2.19
221.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>9</td>
<td>28</td>
<td>65</td>
<td>125</td>
<td>216</td>
</tr>
</tbody>
</table>

Table 2.20

222. For the following data, draw a scatter plot. If we wanted to know when the population would reach 15,000, would the answer involve interpolation or extrapolation? Eyeball the line, and estimate the answer.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Population</td>
<td>11,500</td>
<td>12,100</td>
<td>12,700</td>
<td>13,000</td>
<td>13,750</td>
</tr>
</tbody>
</table>

Table 2.21

223. For the following data, draw a scatter plot. If we wanted to know when the temperature would reach 28 °F, would the answer involve interpolation or extrapolation? Eyeball the line and estimate the answer.

<table>
<thead>
<tr>
<th>Temperature, °F</th>
<th>16</th>
<th>18</th>
<th>20</th>
<th>25</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time, seconds</td>
<td>46</td>
<td>50</td>
<td>54</td>
<td>55</td>
<td>62</td>
</tr>
</tbody>
</table>

Table 2.22

**Graphical**

For the following exercises, match each scatterplot with one of the four specified correlations in Figure 2.33 and Figure 2.34.

![Graphical](image)

Figure 2.33
Figure 2.34

224. $r = 0.95$
225. $r = -0.89$
226. $r = 0.26$
227. $r = -0.39$

For the following exercises, draw a best-fit line for the plotted data.

228.

229.
232. The U.S. Census tracks the percentage of persons 25 years or older who are college graduates. That data for several years is given in Table 2.23[9]. Determine whether the trend appears linear. If so, and assuming the trend continues, in what year will the percentage exceed 35%?

233. The U.S. import of wine (in hectoliters) for several years is given in Table 2.24. Determine whether the trend appears linear. If so, and assuming the trend continues, in what year will imports exceed 12,000 hectoliters?

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Imports</td>
<td>2665</td>
<td>2688</td>
<td>3565</td>
<td>4129</td>
<td>4584</td>
<td>5655</td>
<td>6549</td>
<td>7950</td>
<td>8487</td>
<td>9462</td>
</tr>
</tbody>
</table>

Table 2.24

234. Table 2.25 shows the year and the number of people unemployed in a particular city for several years. Determine whether the trend appears linear. If so, and assuming the trend continues, in what year will the number of unemployed reach 5?

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Number Unemployed</td>
<td>750</td>
<td>670</td>
<td>650</td>
<td>605</td>
<td>550</td>
<td>510</td>
<td>460</td>
<td>420</td>
<td>380</td>
<td>320</td>
</tr>
</tbody>
</table>

Table 2.25

Technology

For the following exercises, use each set of data to calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to 3 decimal places of accuracy.

235.

<table>
<thead>
<tr>
<th>x</th>
<th>8</th>
<th>15</th>
<th>26</th>
<th>31</th>
<th>56</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>23</td>
<td>41</td>
<td>53</td>
<td>72</td>
<td>103</td>
</tr>
</tbody>
</table>

Table 2.26

236.

<table>
<thead>
<tr>
<th>x</th>
<th>5</th>
<th>7</th>
<th>10</th>
<th>12</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>4</td>
<td>12</td>
<td>17</td>
<td>22</td>
<td>24</td>
</tr>
</tbody>
</table>

Table 2.27

237.
<table>
<thead>
<tr>
<th>$x$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>21.9</td>
<td>22.22</td>
<td>22.74</td>
<td>22.26</td>
<td>20.78</td>
<td>17.6</td>
<td>16.52</td>
<td>18.54</td>
</tr>
</tbody>
</table>

Table 2.28

<table>
<thead>
<tr>
<th>$x$</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>15.76</td>
<td>13.68</td>
<td>14.1</td>
<td>14.02</td>
<td>11.94</td>
<td>12.76</td>
<td>11.28</td>
<td>9.1</td>
</tr>
</tbody>
</table>

Table 2.28

<table>
<thead>
<tr>
<th>$x$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>44.8</td>
<td>43.1</td>
<td>38.8</td>
<td>39</td>
<td>38</td>
<td>32.7</td>
<td>30.1</td>
<td>29.3</td>
<td>27</td>
<td>25.8</td>
</tr>
</tbody>
</table>

Table 2.29

<table>
<thead>
<tr>
<th>$x$</th>
<th>21</th>
<th>25</th>
<th>30</th>
<th>31</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>17</td>
<td>11</td>
<td>2</td>
<td>$-1$</td>
<td>$-18$</td>
<td>$-40$</td>
</tr>
</tbody>
</table>

Table 2.30

<table>
<thead>
<tr>
<th>$x$</th>
<th>100</th>
<th>80</th>
<th>60</th>
<th>55</th>
<th>40</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>2000</td>
<td>1798</td>
<td>1589</td>
<td>1580</td>
<td>1390</td>
<td>1202</td>
</tr>
</tbody>
</table>

Table 2.31

<table>
<thead>
<tr>
<th>$x$</th>
<th>900</th>
<th>988</th>
<th>1000</th>
<th>1010</th>
<th>1200</th>
<th>1205</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>70</td>
<td>80</td>
<td>82</td>
<td>84</td>
<td>105</td>
<td>108</td>
</tr>
</tbody>
</table>

Table 2.32

Extensions

238.

239.

240.

241.

242. Graph $f(x) = 0.5x + 10$. Pick a set of 5 ordered pairs using inputs $x = -2, 1, 5, 6, 9$ and use linear regression to verify that the function is a good fit for the data.
243. Graph \( f(x) = -2x - 10 \). Pick a set of 5 ordered pairs using inputs \( x = -2, 1, 5, 6, 9 \) and use linear regression to verify the function.

For the following exercises, consider this scenario: The profit of a company decreased steadily over a ten-year span. The following ordered pairs shows dollars and the number of units sold in hundreds and the profit in thousands of over the ten-year span, (number of units sold, profit) for specific recorded years:
\[
(46, 1, 600), \ (48, 1, 550), \ (50, 1, 505), \ (52, 1, 540), \ (54, 1, 495).
\]

244. Use linear regression to determine a function \( P \) where the profit in thousands of dollars depends on the number of units sold in hundreds.

245. Find to the nearest tenth and interpret the \( x \)-intercept.

246. Find to the nearest tenth and interpret the \( y \)-intercept.

**Real-World Applications**

For the following exercises, consider this scenario: The profit of a company decreased steadily over a ten-year span. The following ordered pairs shows dollars and the number of units sold in hundreds and the profit in thousands of over the ten-year span, (number of units sold, profit) for specific recorded years:
\[
(46, 1, 600), \ (48, 1, 550), \ (50, 1, 505), \ (52, 1, 540), \ (54, 1, 495).
\]

247. Use linear regression to determine a function \( y \), where the year depends on the population. Round to three decimal places of accuracy.

248. Predict when the population will hit 8,000.

For the following exercises, consider this scenario: The profit of a company increased steadily over a ten-year span. The following ordered pairs show the number of units sold in hundreds and the profit in thousands of over the ten year span, (number of units sold, profit) for specific recorded years:
\[
(46, 250), \ (48, 305), \ (50, 350), \ (52, 390), \ (54, 410).
\]

249. Use linear regression to determine a function \( y \), where the profit in thousands of dollars depends on the number of units sold in hundreds.

250. Predict when the profit will exceed one million dollars.

For the following exercises, consider this scenario: The profit of a company increased steadily over a ten-year span. The following ordered pairs show the number of units sold in hundreds and the profit in thousands of over the ten year span, (number of units sold, profit) for specific recorded years:
\[
(46, 250), \ (48, 225), \ (50, 205), \ (52, 180), \ (54, 165).
\]

251. Use linear regression to determine a function \( y \), where the profit in thousands of dollars depends on the number of units sold in hundreds.

252. Predict when the profit will dip below the $25,000 threshold.
CHAPTER 2 REVIEW

KEY TERMS

correlation coefficient: a value, $r$, between $-1$ and $1$ that indicates the degree of linear correlation of variables, or how closely a regression line fits a data set.

decreasing linear function: a function with a negative slope: If $f(x) = mx + b$, then $m < 0$.

extrapolation: predicting a value outside the domain and range of the data

horizontal line: a line defined by $f(x) = b$, where $b$ is a real number. The slope of a horizontal line is 0.

increasing linear function: a function with a positive slope: If $f(x) = mx + b$, then $m > 0$.

interpolation: predicting a value inside the domain and range of the data

least squares regression: a statistical technique for fitting a line to data in a way that minimizes the differences between the line and data values

linear function: a function with a constant rate of change that is a polynomial of degree 1, and whose graph is a straight line

model breakdown: when a model no longer applies after a certain point

parallel lines: two or more lines with the same slope

perpendicular lines: two lines that intersect at right angles and have slopes that are negative reciprocals of each other

point-slope form: the equation for a line that represents a linear function of the form $y - y_1 = m(x - x_1)$

slope-intercept form: the equation for a line that represents a linear function in the form $f(x) = mx + b$

slope: the ratio of the change in output values to the change in input values; a measure of the steepness of a line

vertical line: a line defined by $x = a$, where $a$ is a real number. The slope of a vertical line is undefined.

x-intercept: the point on the graph of a linear function when the output value is 0; the point at which the graph crosses the horizontal axis

y-intercept: the value of a function when the input value is zero; also known as initial value

KEY EQUATIONS
Table 2.33

<table>
<thead>
<tr>
<th>slope-intercept form of a line</th>
<th>( f(x) = mx + b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>slope</td>
<td>( m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} )</td>
</tr>
<tr>
<td>point-slope form of a line</td>
<td>( y - y_1 = m(x - x_1) )</td>
</tr>
</tbody>
</table>

**KEY CONCEPTS**

**2.1 Linear Functions**

- The ordered pairs given by a linear function represent points on a line.
- Linear functions can be represented in words, function notation, tabular form, and graphical form. See Example 2.1.
- The rate of change of a linear function is also known as the slope.
- An equation in the slope-intercept form of a line includes the slope and the initial value of the function.
- The initial value, or \( y \)-intercept, is the output value when the input of a linear function is zero. It is the \( y \)-value of the point at which the line crosses the \( y \)-axis.
- An increasing linear function results in a graph that slants upward from left to right and has a positive slope.
- A decreasing linear function results in a graph that slants downward from left to right and has a negative slope.
- A constant linear function results in a graph that is a horizontal line.
- Analyzing the slope within the context of a problem indicates whether a linear function is increasing, decreasing, or constant. See Example 2.2.
- The slope of a linear function can be calculated by dividing the difference between \( y \)-values by the difference in corresponding \( x \)-values of any two points on the line. See Example 2.3 and Example 2.4.
- The slope and initial value can be determined given a graph or any two points on the line.
- One type of function notation is the slope-intercept form of an equation.
- The point-slope form is useful for finding a linear equation when given the slope of a line and one point. See Example 2.5.
- The point-slope form is also convenient for finding a linear equation when given two points through which a line passes. See Example 2.6.
- The equation for a linear function can be written if the slope \( m \) and initial value \( b \) are known. See Example 2.7, Example 2.8, and Example 2.9.
- A linear function can be used to solve real-world problems. See Example 2.10 and Example 2.11.
- A linear function can be written from tabular form. See Example 2.12.

**2.2 Graphs of Linear Functions**

- Linear functions may be graphed by plotting points or by using the \( y \)-intercept and slope. See Example 2.13 and Example 2.14.
- Graphs of linear functions may be transformed by using shifts up, down, left, or right, as well as through stretches, compressions, and reflections. See Example 2.15.
- The \( y \)-intercept and slope of a line may be used to write the equation of a line.
• The x-intercept is the point at which the graph of a linear function crosses the x-axis. See Example 2.16 and Example 2.17.

• Horizontal lines are written in the form, \( f(x) = b \). See Example 2.18.

• Vertical lines are written in the form, \( x = b \). See Example 2.19.

• Parallel lines have the same slope.

• Perpendicular lines have negative reciprocal slopes, assuming neither is vertical. See Example 2.20.

• A line parallel to another line, passing through a given point, may be found by substituting the slope value of the line and the x- and y-values of the given point into the equation. \( f(x) = mx + b \), and using the \( b \) that results. Similarly, the point-slope form of an equation can also be used. See Example 2.21.

• A line perpendicular to another line, passing through a given point, may be found in the same manner, with the exception of using the negative reciprocal slope. See Example 2.22 and Example 2.23.

• A system of linear equations may be solved setting the two equations equal to one another and solving for \( x \). The \( y \)-value may be found by evaluating either one of the original equations using this \( x \)-value.

• A system of linear equations may also be solved by finding the point of intersection on a graph. See Example 2.24 and Example 2.25.

2.3 Modeling with Linear Functions

• We can use the same problem strategies that we would use for any type of function.

• When modeling and solving a problem, identify the variables and look for key values, including the slope and \( y \)-intercept. See Example 2.26.

• Draw a diagram, where appropriate. See Example 2.27 and Example 2.28.

• Check for reasonableness of the answer.

• Linear models may be built by identifying or calculating the slope and using the \( y \)-intercept.

• The x-intercept may be found by setting \( y = 0 \), which is setting the expression \( mx + b \) equal to 0.

• The point of intersection of a system of linear equations is the point where the \( x \)- and \( y \)-values are the same. See Example 2.29.

• A graph of the system may be used to identify the points where one line falls below (or above) the other line.

2.4 Fitting Linear Models to Data

• Scatter plots show the relationship between two sets of data. See Example 2.30.

• Scatter plots may represent linear or non-linear models.

• The line of best fit may be estimated or calculated, using a calculator or statistical software. See Example 2.31.

• Interpolation can be used to predict values inside the domain and range of the data, whereas extrapolation can be used to predict values outside the domain and range of the data. See Example 2.32.

• The correlation coefficient, \( r \), indicates the degree of linear relationship between data. See Example 2.34.

• A regression line best fits the data. See Example 2.35.

• The least squares regression line is found by minimizing the squares of the distances of points from a line passing through the data and may be used to make predictions regarding either of the variables. See Example 2.33.

CHAPTER 2 REVIEW EXERCISES

m10352 (http://legacy.cnx.org/content/m10352/latest/)

271. Determine whether the algebraic equation is linear. \( 2x + 3y = 7 \)

272. Determine whether the algebraic equation is linear. \( 6x^2 - y = 5 \)

273. Determine whether the function is increasing or decreasing.
\( f(x) = 7x - 2 \)

274. Determine whether the function is increasing or decreasing.
\( g(x) = -x + 2 \)

275. Given each set of information, find a linear equation that satisfies the given conditions, if possible.
Passes through \((7, 5)\) and \((3, 17)\)

276. Given each set of information, find a linear equation that satisfies the given conditions, if possible.
\( x\)-intercept at \((6, 0)\) and \(y\)-intercept at \((0, 10)\)

277. Find the slope of the line shown in the line graph.

278. Find the slope of the line graphed.
279. Write an equation in slope-intercept form for the line shown.

![Graph of a line](http://legacy.cnx.org/content/m10355/latest/)

280. Does the following table represent a linear function? If so, find the linear equation that models the data.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-4</th>
<th>0</th>
<th>2</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(x)$</td>
<td>18</td>
<td>-2</td>
<td>-12</td>
<td>-52</td>
</tr>
</tbody>
</table>

Table 2.34

281. Does the following table represent a linear function? If so, find the linear equation that models the data.

<table>
<thead>
<tr>
<th>$x$</th>
<th>6</th>
<th>8</th>
<th>12</th>
<th>26</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(x)$</td>
<td>-8</td>
<td>-12</td>
<td>-18</td>
<td>-46</td>
</tr>
</tbody>
</table>

Table 2.35

282. On June 1st, a company has $4,000,000 profit. If the company then loses 150,000 dollars per day thereafter in the month of June, what is the company’s profit $n^{th}$ day after June 1st?

m10355 (http://legacy.cnx.org/content/m10355/latest/)

For the following exercises, determine whether the lines given by the equations below are parallel, perpendicular, or neither parallel nor perpendicular:

283. $\frac{2x}{6} - 6y = 12$
   $\frac{-x}{3} + 3y = 1$

284. $y = \frac{1}{3}x - 2$
   $3x + y = -9$
For the following exercises, find the x- and y-intercepts of the given equation

285. \(7x + 9y = -63\)

286. \(f(x) = 2x - 1\)

For the following exercises, use the descriptions of the pairs of lines to find the slopes of Line 1 and Line 2. Is each pair of lines parallel, perpendicular, or neither?

287.
Line 1: Passes through (5, 11) and (10, 1)
Line 2: Passes through (−1, 3) and (−5, 11)

288.
Line 1: Passes through (8, −10) and (0, −26)
Line 2: Passes through (2, 5) and (4, 4)

289. Write an equation for a line perpendicular to \(f(x) = 5x - 1\) and passing through the point (5, 20).

290. Find the equation of a line with a y-intercept of (0, 2) and slope \(-\frac{1}{2}\).

291. Sketch a graph of the linear function \(f(t) = 2t - 5\).

292. Find the point of intersection for the 2 linear functions:
\[
x = y + 6 \\
2x - y = 13
\]

293. A car rental company offers two plans for renting a car.
Plan A: 25 dollars per day and 10 cents per mile
Plan B: 50 dollars per day with free unlimited mileage
How many miles would you need to drive for plan B to save you money?

294. Find the area of a triangle bounded by the y-axis, the line \(f(x) = 10 - 2x\), and the line perpendicular to \(f\) that passes through the origin.

295. A town’s population increases at a constant rate. In 2010 the population was 55,000. By 2012 the population had increased to 76,000. If this trend continues, predict the population in 2016.

296. The number of people afflicted with the common cold in the winter months dropped steadily by 50 each year since 2004 until 2010. In 2004, 875 people were inflicted.
Find the linear function that models the number of people afflicted with the common cold \(C\) as a function of the year, \(t\). When will no one be afflicted?

For the following exercises, use the graph in Figure 2.35 showing the profit, \(y\), in thousands of dollars, of a company in a given year, \(x\), where \(x\) represents years since 1980.
Find the linear function \( y \), where \( y \) depends on \( x \), the number of years since 1980.

Find and interpret the \( y \)-intercept.

For the following exercise, consider this scenario: In 2004, a school population was 1,700. By 2012 the population had grown to 2,500.

Assume the population is changing linearly.

a. How much did the population grow between the year 2004 and 2012?
b. What is the average population growth per year?
c. Find an equation for the population, \( P \), of the school \( t \) years after 2004.

For the following exercises, consider this scenario: In 2000, the moose population in a park was measured to be 6,500. By 2010, the population was measured to be 12,500. Assume the population continues to change linearly.

Find a formula for the moose population, \( P \).

What does your model predict the moose population to be in 2020?

For the following exercises, consider this scenario: The median home values in subdivisions Pima Central and East Valley (adjusted for inflation) are shown in Table 2.36. Assume that the house values are changing linearly.

<table>
<thead>
<tr>
<th>Year</th>
<th>Pima Central</th>
<th>East Valley</th>
</tr>
</thead>
<tbody>
<tr>
<td>1970</td>
<td>32,000</td>
<td>120,250</td>
</tr>
<tr>
<td>2010</td>
<td>85,000</td>
<td>150,000</td>
</tr>
</tbody>
</table>

In which subdivision have home values increased at a higher rate?

If these trends were to continue, what would be the median home value in Pima Central in 2015?

Draw a scatter plot for the data in Table 2.37. Then determine whether the data appears to be linearly related.
305. Draw a scatter plot for the data in Table 2.38. If we wanted to know when the population would reach 15,000, would the answer involve interpolation or extrapolation?

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Population</td>
<td>5,600</td>
<td>5,950</td>
<td>6,300</td>
<td>6,600</td>
<td>6,900</td>
</tr>
</tbody>
</table>

Table 2.38

306. Eight students were asked to estimate their score on a 10-point quiz. Their estimated and actual scores are given in Table 2.39. Plot the points, then sketch a line that fits the data.

| Predicted | 6 | 7 | 7 | 8 | 7 | 9 | 10 | 10 |
| Actual    | 6 | 7 | 8 | 8 | 9 | 10| 10 | 9 |

Table 2.39

307. Draw a best-fit line for the plotted data.

For the following exercises, consider the data in Table 2.40, which shows the percent of unemployed in a city of people 25 years or older who are college graduates is given below, by year.

<table>
<thead>
<tr>
<th>Year</th>
<th>2000</th>
<th>2002</th>
<th>2005</th>
<th>2007</th>
<th>2010</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent Graduates</td>
<td>6.5</td>
<td>7.0</td>
<td>7.4</td>
<td>8.2</td>
<td>9.0</td>
</tr>
</tbody>
</table>

Table 2.40

308. Determine whether the trend appears to be linear. If so, and assuming the trend continues, find a linear regression model to predict the percent of unemployed in a given year to three decimal places.

309. In what year will the percentage exceed 12%?

310. Based on the set of data given in Table 2.41, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to three decimal places.
Table 2.41

<table>
<thead>
<tr>
<th>$x$</th>
<th>17</th>
<th>20</th>
<th>23</th>
<th>26</th>
<th>29</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>15</td>
<td>25</td>
<td>31</td>
<td>37</td>
<td>40</td>
</tr>
</tbody>
</table>

311. Based on the set of data given in Table 2.42, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to three decimal places.

Table 2.42

<table>
<thead>
<tr>
<th>$x$</th>
<th>10</th>
<th>12</th>
<th>15</th>
<th>18</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>36</td>
<td>34</td>
<td>30</td>
<td>28</td>
<td>22</td>
</tr>
</tbody>
</table>

For the following exercises, consider this scenario: The population of a city increased steadily over a ten-year span. The following ordered pairs show the population and the year over the ten-year span (population, year) for specific recorded years:

(3,600, 2000); (4,000, 2001); (4,700, 2003); (6,000, 2006)

312. Use linear regression to determine a function $y$, where the year depends on the population, to three decimal places of accuracy.

313. Predict when the population will hit 12,000.

314. What is the correlation coefficient for this model to three decimal places of accuracy?

315. According to the model, what is the population in 2014?

CHAPTER 2 PRACTICE TEST

298. Determine whether the following algebraic equation can be written as a linear function. $2x + 3y = 7$

299. Determine whether the following function is increasing or decreasing. $f(x) = -2x + 5$

300. Determine whether the following function is increasing or decreasing. $f(x) = 7x + 9$

301. Given the following set of information, find a linear equation satisfying the conditions, if possible.
   Passes through (5, 1) and (3, −9)

302. Given the following set of information, find a linear equation satisfying the conditions, if possible.
   $x$ intercept at (−4, 0) and $y$-intercept at (0, −6)

303. Find the slope of the line in Figure 2.36.
304. Write an equation for the line in Figure 2.37.

305. Does Table 2.43 represent a linear function? If so, find a linear equation that models the data.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-6</th>
<th>0</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(x)$</td>
<td>14</td>
<td>32</td>
<td>38</td>
<td>44</td>
</tr>
</tbody>
</table>

Table 2.43

306. Does Table 2.44 represent a linear function? If so, find a linear equation that models the data.
307. At 6 am, an online company has sold 120 items that day. If the company sells an average of 30 items per hour for the remainder of the day, write an expression to represent the number of items that were sold $n$ after 6 am.

For the following exercises, determine whether the lines given by the equations below are parallel, perpendicular, or neither parallel nor perpendicular:

$$y = \frac{3}{4}x - 9$$

308. $-4x - 3y = 8$

309. $-2x + y = 3$

$$3x + \frac{3}{2}y = 5$$

310. Find the $x$- and $y$-intercepts of the equation $2x + 7y = -14$.

311. Given below are descriptions of two lines. Find the slopes of Line 1 and Line 2. Is the pair of lines parallel, perpendicular, or neither?

Line 1: Passes through $(-2, -6)$ and $(3, 14)$

Line 2: Passes through $(2, 6)$ and $(4, 14)$

312. Write an equation for a line perpendicular to $f(x) = 4x + 3$ and passing through the point $(8, 10)$.

313. Sketch a line with a $y$-intercept of $(0, 5)$ and slope $-\frac{5}{2}$.

314. Graph of the linear function $f(x) = -x + 6$.

315. For the two linear functions, find the point of intersection:

$$x = y + 2$$

$$2x - 3y = -1$$

316. A car rental company offers two plans for renting a car.

Plan A: $25 per day and $0.10 per mile

Plan B: $40 per day with free unlimited mileage

How many miles would you need to drive for plan B to save you money?

317. Find the area of a triangle bounded by the $y$ axis, the line $f(x) = 12 - 4x$, and the line perpendicular to $f$ that passes through the origin.

318. A town’s population increases at a constant rate. In 2010 the population was 65,000. By 2012 the population had increased to 90,000. Assuming this trend continues, predict the population in 2018.

319. The number of people afflicted with the common cold in the winter months dropped steadily by 25 each year since 2002 until 2012. In 2002, 8,040 people were inflicted. Find the linear function that models the number of people afflicted with the common cold $C$ as a function of the year, $t$. When will less than 6,000 people be afflicted?
For the following exercises, use the graph in Figure 2.38, showing the profit, \( y \), in thousands of dollars, of a company in a given year, \( x \), where \( x \) represents years since 1980.

![Figure 2.38](image)

320. Find the linear function \( y \), where \( y \) depends on \( x \), the number of years since 1980.

321. Find and interpret the \( y \)-intercept.

322. In 2004, a school population was 1250. By 2012 the population had dropped to 875. Assume the population is changing linearly.
   a. How much did the population drop between the year 2004 and 2012?
   b. What is the average population decline per year?
   c. Find an equation for the population, \( P \), of the school \( t \) years after 2004.

323. Draw a scatter plot for the data provided in Table 2.45. Then determine whether the data appears to be linearly related.

<table>
<thead>
<tr>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>-450</td>
<td>-200</td>
<td>10</td>
<td>265</td>
<td>500</td>
<td>755</td>
</tr>
</tbody>
</table>

Table 2.45

324. Draw a best-fit line for the plotted data.

For the following exercises, use Table 2.46, which shows the percent of unemployed persons 25 years or older who are college graduates in a particular city, by year.
325. Determine whether the trend appears linear. If so, and assuming the trend continues, find a linear regression model to predict the percent of unemployed in a given year to three decimal places.

326. In what year will the percentage drop below 4%?

327. Based on the set of data given in Table 2.47, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient. Round to three decimal places of accuracy.

<table>
<thead>
<tr>
<th>Year</th>
<th>2000</th>
<th>2002</th>
<th>2005</th>
<th>2007</th>
<th>2010</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent Graduates</td>
<td>8.5</td>
<td>8.0</td>
<td>7.2</td>
<td>6.7</td>
<td>6.4</td>
</tr>
</tbody>
</table>

Table 2.46

328. Use linear regression to determine a function $y$, where the year depends on the population. Round to three decimal places of accuracy.

329. Predict when the population will hit 20,000.

330. What is the correlation coefficient for this model?
Figure 3.1 35-mm film, once the standard for capturing photographic images, has been made largely obsolete by digital photography. (credit “film”: modification of work by Horia Varlan; credit “memory cards”: modification of work by Paul Hudson)

Chapter Outline

3.1 Complex Numbers
3.2 Quadratic Functions
3.3 Power Functions and Polynomial Functions
3.4 Graphs of Polynomial Functions
3.5 Dividing Polynomials
3.6 Zeros of Polynomial Functions
3.7 Rational Functions
3.8 Inverses and Radical Functions
3.9 Modeling Using Variation

Introduction

Digital photography has dramatically changed the nature of photography. No longer is an image etched in the emulsion on a roll of film. Instead, nearly every aspect of recording and manipulating images is now governed by mathematics. An image becomes a series of numbers, representing the characteristics of light striking an image sensor. When we open an image file, software on a camera or computer interprets the numbers and converts them to a visual image. Photo editing software uses complex polynomials to transform images, allowing us to manipulate the image in order to crop details, change the color palette, and add special effects. Inverse functions make it possible to convert from one file format to another. In this chapter, we will learn about these concepts and discover how mathematics can be used in such applications.
3.1 | Complex Numbers

Learning Objectives

In this section, you will:

3.1.1 Express square roots of negative numbers as multiples of i.
3.1.2 Plot complex numbers on the complex plane.
3.1.3 Add and subtract complex numbers.
3.1.4 Multiply and divide complex numbers.

The study of mathematics continuously builds upon itself. Negative integers, for example, fill a void left by the set of positive integers. The set of rational numbers, in turn, fills a void left by the set of integers. The set of real numbers fills a void left by the set of rational numbers. Not surprisingly, the set of real numbers has voids as well. For example, we still have no solution to equations such as

$$x^2 + 4 = 0. \quad (3.1)$$

Our best guesses might be +2 or −2. But if we test +2 in this equation, it does not work. If we test −2, it does not work. If we want to have a solution for this equation, we will have to go farther than we have so far. After all, to this point we have described the square root of a negative number as undefined. Fortunately, there is another system of numbers that provides solutions to problems such as these. In this section, we will explore this number system and how to work within it.

Expressing Square Roots of Negative Numbers as Multiples of $i$

We know how to find the square root of any positive real number. In a similar way, we can find the square root of a negative number. The difference is that the root is not real. If the value in the radicand is negative, the root is said to be an imaginary number. The imaginary number $i$ is defined as the square root of negative 1.

$$\sqrt{-1} = i \quad (3.2)$$

So, using properties of radicals,

$$i^2 = (\sqrt{-1})^2 = -1. \quad (3.3)$$

We can write the square root of any negative number as a multiple of $i$. Consider the square root of −25.

$$\sqrt{-25} = \sqrt{25 \cdot (-1)}$$

$$= \sqrt{25} \sqrt{-1}$$

$$= 5i \quad (3.4)$$

We use $5i$ and not $-5i$ because the principal root of 25 is the positive root.

A complex number is the sum of a real number and an imaginary number. A complex number is expressed in standard form when written $a + bi$ where $a$ is the real part and $bi$ is the imaginary part. For example, $5 + 2i$ is a complex number. So, too, is $3 + 4\sqrt{3}$.

Imaginary numbers are distinguished from real numbers because a squared imaginary number produces a negative real number. Recall, when a positive real number is squared, the result is a positive real number and when a negative real number is squared, again, the result is a positive real number. Complex numbers are a combination of real and imaginary numbers.

Imaginary and Complex Numbers

A complex number is a number of the form $a + bi$ where

- $a$ is the real part of the complex number.
• $bi$ is the imaginary part of the complex number.

If $b = 0$, then $a + bi$ is a real number. If $a = 0$ and $b$ is not equal to 0, the complex number is called an imaginary number. An imaginary number is an even root of a negative number.

**Given an imaginary number, express it in standard form.**

1. Write $\sqrt{-a}$ as $\sqrt{a}i$.
2. Express $\sqrt{-1}$ as $i$.
3. Write $\sqrt{a} \cdot i$ in simplest form.

**Example 3.1**

**Expressing an Imaginary Number in Standard Form**

Express $\sqrt{-9}$ in standard form.

**Solution**

Express $\sqrt{-24}$ in standard form.

**Plotting a Complex Number on the Complex Plane**

We cannot plot complex numbers on a number line as we might real numbers. However, we can still represent them graphically. To represent a complex number we need to address the two components of the number. We use the complex plane, which is a coordinate system in which the horizontal axis represents the real component and the vertical axis represents the imaginary component. Complex numbers are the points on the plane, expressed as ordered pairs $(a, b)$, where $a$ represents the coordinate for the horizontal axis and $b$ represents the coordinate for the vertical axis.

Let’s consider the number $-2 + 3i$. The real part of the complex number is $-2$ and the imaginary part is $3i$. We plot the ordered pair $(-2, 3)$ to represent the complex number $-2 + 3i$ as shown in Figure 3.2.
Complex Plane

In the complex plane, the horizontal axis is the real axis, and the vertical axis is the imaginary axis as shown in Figure 3.3.

Given a complex number, represent its components on the complex plane.

1. Determine the real part and the imaginary part of the complex number.
2. Move along the horizontal axis to show the real part of the number.
3. Move parallel to the vertical axis to show the imaginary part of the number.
4. Plot the point.

Example 3.2

Plotting a Complex Number on the Complex Plane

Plot the complex number $3 - 4i$ on the complex plane.
3.2 Plot the complex number $-4 - i$ on the complex plane.

Adding and Subtracting Complex Numbers

Just as with real numbers, we can perform arithmetic operations on complex numbers. To add or subtract complex numbers, we combine the real parts and combine the imaginary parts.

### Complex Numbers: Addition and Subtraction

Adding complex numbers:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$  \hspace{1cm} (3.5)

Subtracting complex numbers:

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$  \hspace{1cm} (3.6)

**How To:** Given two complex numbers, find the sum or difference.

1. Identify the real and imaginary parts of each number.
2. Add or subtract the real parts.
3. Add or subtract the imaginary parts.

**Example 3.3**

**Adding Complex Numbers**

Add $3 - 4i$ and $2 + 5i$.

**Solution**

**Example 3.3**

**3.3 Subtract $2 + 5i$ from $3 - 4i$.**

**Multiplying Complex Numbers**

Multiplying complex numbers is much like multiplying binomials. The major difference is that we work with the real and imaginary parts separately.

**Multiplying a Complex Numbers by a Real Number**

Let’s begin by multiplying a complex number by a real number. We distribute the real number just as we would with a binomial. So, for example,
3.4

Given a complex number and a real number, multiply to find the product.

1. Use the distributive property.
2. Simplify.

Example 3.4

Multiplying a Complex Number by a Real Number

Find the product $4(2 + 5i)$.

Solution

Solution

Try it

Find the product $-4(2 + 6i)$.

Multiplying Complex Numbers Together

Now, let’s multiply two complex numbers. We can use either the distributive property or the FOIL method. Recall that FOIL is an acronym for multiplying First, Inner, Outer, and Last terms together. Using either the distributive property or the FOIL method, we get

$$(a + bi)(c + di) = ac + adi + bci + bdi^2.$$  \hspace{1cm} (3.7)

Because $i^2 = -1$, we have

$$(a + bi)(c + di) = ac + adi + bci - bd.$$  \hspace{1cm} (3.8)

To simplify, we combine the real parts, and we combine the imaginary parts.

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$  \hspace{1cm} (3.9)

Given two complex numbers, multiply to find the product.

1. Use the distributive property or the FOIL method.
2. Simplify.

Example 3.5

Multiplying a Complex Number by a Complex Number

Multiply $(4 + 3i)(2 - 5i)$.
Dividing Complex Numbers

Division of two complex numbers is more complicated than addition, subtraction, and multiplication because we cannot divide by an imaginary number, meaning that any fraction must have a real-number denominator. We need to find a term by which we can multiply the numerator and the denominator that will eliminate the imaginary portion of the denominator so that we end up with a real number as the denominator. This term is called the complex conjugate of the denominator, which is found by changing the sign of the imaginary part of the complex number. In other words, the complex conjugate of \(a + bi\) is \(a - bi\).

Note that complex conjugates have a reciprocal relationship: The complex conjugate of \(a + bi\) is \(a - bi\), and the complex conjugate of \(a - bi\) is \(a + bi\). Further, when a quadratic equation with real coefficients has complex solutions, the solutions are always complex conjugates of one another.

Suppose we want to divide \(c + di\) by \(a + bi\), where neither \(a\) nor \(b\) equals zero. We first write the division as a fraction, then find the complex conjugate of the denominator, and multiply.

\[
\frac{c + di}{a + bi} \quad \text{where } a \neq 0 \text{ and } b \neq 0
\]

Multiply the numerator and denominator by the complex conjugate of the denominator.

\[
\frac{(c + di)(a - bi)}{(a + bi)(a - bi)} = \frac{(c + di)(a - bi)}{a^2 + b^2}
\]

Apply the distributive property.

\[
\frac{ca - cbi + adi - bdi^2}{a^2 - abi + abi - b^2i^2}
\]

Simplify, remembering that \(i^2 = -1\).

\[
\frac{ca - cbi + adi - bdi^2}{a^2 - abi + abi - b^2i^2} = \frac{ca - cbi + adi - bd(-1)}{a^2 - abi + abi - b^2(-1)} = \frac{(ca + bd) + (ad - cb)i}{a^2 + b^2}
\]

The Complex Conjugate

The complex conjugate of a complex number \(a + bi\) is \(a - bi\). It is found by changing the sign of the imaginary part of the complex number. The real part of the number is left unchanged.

- When a complex number is multiplied by its complex conjugate, the result is a real number.
- When a complex number is added to its complex conjugate, the result is a real number.

Example 3.6

Finding Complex Conjugates

Find the complex conjugate of each number.

a. \(2 + i\sqrt{3}\)

b. \(-\frac{1}{2}i\)

Solution
Solution

Analysis
Although we have seen that we can find the complex conjugate of an imaginary number, in practice we generally find the complex conjugates of only complex numbers with both a real and an imaginary component. To obtain a real number from an imaginary number, we can simply multiply by $i$.

**How To:** Given two complex numbers, divide one by the other.

1. Write the division problem as a fraction.
2. Determine the complex conjugate of the denominator.
3. Multiply the numerator and denominator of the fraction by the complex conjugate of the denominator.
4. Simplify.

**Example 3.7**

**Dividing Complex Numbers**

Divide $(2 + 5i)$ by $(4 - i)$.

**Solution**

**Example 3.8**

**Substituting a Complex Number into a Polynomial Function**

Let $f(x) = x^2 - 5x + 2$. Evaluate $f(3 + i)$.

**Solution**

**Analysis**
We write $f(3 + i) = -5 + i$. Notice that the input is $3 + i$ and the output is $-5 + i$.

**Example 3.9**

**Substituting an Imaginary Number in a Rational Function**

Let $f(x) = \frac{2 + x}{x + 3}$. Evaluate $f(10i)$. 

3.6 Let $f(x) = 2x^2 - 3x$. Evaluate $f(8 - i)$. 

This content is available for free at http://legacy.cnx.org/content/col11667/1.2
Let \( f(x) = \frac{x + 1}{x - 4} \). Evaluate \( f(-i) \).

### Simplifying Powers of \( i \)

The powers of \( i \) are cyclic. Let’s look at what happens when we raise \( i \) to increasing powers.

\[
\begin{align*}
i^1 &= i \\
i^2 &= -1 \\
i^3 &= i^2 \cdot i = -1 \cdot i = -i \\
i^4 &= i^3 \cdot i = -i \cdot i = -i^2 = -(-1) = 1 \\
i^5 &= i^4 \cdot i = 1 \cdot i = i
\end{align*}
\]

We can see that when we get to the fifth power of \( i \), it is equal to the first power. As we continue to multiply \( i \) by itself for increasing powers, we will see a cycle of 4. Let’s examine the next 4 powers of \( i \).

\[
\begin{align*}
i^6 &= i^5 \cdot i = i \cdot i = i^2 = -1 \\
i^7 &= i^6 \cdot i = i^2 \cdot i = i^3 = -i \\
i^8 &= i^7 \cdot i = i^3 \cdot i = i^4 = 1 \\
i^9 &= i^8 \cdot i = i^4 \cdot i = i^5 = i
\end{align*}
\]

### Example 3.10

#### Simplifying Powers of \( i \)

Evaluate \( i^{35} \).
Can we write $i^{35}$ in other helpful ways?

As we saw in Example 3.10, we reduced $i^{35}$ to $i^3$ by dividing the exponent by 4 and using the remainder to find the simplified form. But perhaps another factorization of $i^{35}$ may be more useful. Table 3.1 shows some other possible factorizations.

<table>
<thead>
<tr>
<th>Factorization of $i^{35}$</th>
<th>$i^{34} \cdot i$</th>
<th>$i^{33} \cdot i^2$</th>
<th>$i^{31} \cdot i^4$</th>
<th>$i^{19} \cdot i^{16}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reduced form</td>
<td>$(i^2)^{17} \cdot i$</td>
<td>$i^{33} \cdot (-1)$</td>
<td>$i^{31} \cdot 1$</td>
<td>$i^{19} \cdot (i^4)^4$</td>
</tr>
<tr>
<td>Simplified form</td>
<td>$(-1)^{17} \cdot i$</td>
<td>$-i^{33}$</td>
<td>$i^{31}$</td>
<td>$i^{19}$</td>
</tr>
</tbody>
</table>

Table 3.1

Each of these will eventually result in the answer we obtained above but may require several more steps than our earlier method.

Access these online resources for additional instruction and practice with complex numbers.

- Adding and Subtracting Complex Numbers (http://openstaxcollege.org/l/addsubcomplex)
- Multiply Complex Numbers (http://openstaxcollege.org/l/multiplycomplex)
- Multiplying Complex Conjugates (http://openstaxcollege.org/l/multcompconj)
- Raising $i$ to Powers (http://openstaxcollege.org/l/raisingi)
3.1 EXERCISES

Verbal
1. Explain how to add complex numbers.
2. What is the basic principle in multiplication of complex numbers?
3. Give an example to show the product of two imaginary numbers is not always imaginary.
4. What is a characteristic of the plot of a real number in the complex plane?

Algebraic
For the following exercises, evaluate the algebraic expressions.

5. If \( f(x) = x^2 + x - 4 \), evaluate \( f(2i) \).
6. If \( f(x) = x^3 - 2 \), evaluate \( f(i) \).
7. If \( f(x) = x^2 + 3x + 5 \), evaluate \( f(2 + i) \).
8. If \( f(x) = 2x^2 + x - 3 \), evaluate \( f(2 - 3i) \).
9. If \( f(x) = \frac{x + 1}{2 - x} \), evaluate \( f(5i) \).
10. If \( f(x) = \frac{1 + 2x}{x + 3} \), evaluate \( f(4i) \).

Graphical
For the following exercises, determine the number of real and nonreal solutions for each quadratic function shown.

11.

12.

For the following exercises, plot the complex numbers on the complex plane.

13.
1 - 2i

14. -2 + 3i

15. i

16. -3 - 4i

**Numeric**

For the following exercises, perform the indicated operation and express the result as a simplified complex number.

17. (3 + 2i) + (5 - 3i)

18. (-2 - 4i) + (1 + 6i)

19. (-5 + 3i) - (6 - i)

20. (2 - 3i) - (3 + 2i)

21. (-4 + 4i) - (-6 + 9i)

22. (2 + 3i)(4i)

23. (5 - 2i)(3i)

24. (6 - 2i)(5)

25. (-2 + 4i)(8)

26. (2 + 3i)(4 - i)

27. (-1 + 2i)(-2 + 3i)

28. (4 - 2i)(4 + 2i)

29. (3 + 4i)(3 - 4i)

30. \( \frac{3 + 4i}{2} \)

31. \( \frac{6 - 2i}{3} \)

32. \( \frac{-5 + 3i}{2i} \)

33. \( \frac{6 + 4i}{i} \)

34. \( \frac{2 - 3i}{4 + 3i} \)

35. \( \frac{3 + 4i}{2 - i} \)

36. \( \frac{2 + 3i}{2 - 3i} \)

37. \( \sqrt{-9} + 3\sqrt{-16} \)

38. \( -\sqrt{-4} - 4\sqrt{-25} \)
39. \( \frac{2 + \sqrt{-12}}{2} \)

40. \( \frac{4 + \sqrt{-20}}{2} \)

41. \( i^8 \)

42. \( i^{15} \)

43. \( i^{22} \)

**Technology**

For the following exercises, use a calculator to help answer the questions.

44. Evaluate \((1 + i)^k\) for \(k = 4, 8,\) and \(12\). Predict the value if \(k = 16\).

45. Evaluate \((1 - i)^k\) for \(k = 2, 6,\) and \(10\). Predict the value if \(k = 14\).

46. Evaluate \((1 + i) - (1 - i)\) for \(k = 4, 8,\) and \(12\). Predict the value for \(k = 16\).

47. Show that a solution of \(x^6 + 1 = 0\) is \(\frac{\sqrt{3}}{2} + \frac{1}{2}i\).

48. Show that a solution of \(x^8 - 1 = 0\) is \(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}i\).

**Extensions**

For the following exercises, evaluate the expressions, writing the result as a simplified complex number.

49. \( \frac{1}{i} + \frac{4}{i^3} \)

50. \( \frac{1}{i^{11}} - \frac{1}{i^{21}} \)

51. \( i^7(1 + i^2) \)

52. \( i^{-3} + 5i^7 \)

53. \( \frac{(2 + i)(4 - 2i)}{(1 + i)} \)

54. \( \frac{(1 + 3i)(2 - 4i)}{(1 + 2i)} \)

55. \( \frac{(3 + i)^2}{(1 + 2i)^2} \)

56. \( \frac{3 + 2i}{2 + i} + (4 + 3i) \)

57. \( \frac{4 + i}{i} + \frac{3 - 4i}{1 - i} \)

58. \( \frac{3 + 2i}{1 + 2i} - \frac{2 - 3i}{3 + i} \)
3.2 | Quadratic Functions

Learning Objectives

In this section, you will:

- **3.2.1** Recognize characteristics of parabolas.
- **3.2.2** Understand how the graph of a parabola is related to its quadratic function.
- **3.2.3** Determine a quadratic function’s minimum or maximum value.
- **3.2.4** Solve problems involving a quadratic function’s minimum or maximum value.

![Figure 3.4](image) An array of satellite dishes. (credit: Matthew Colvin de Valle, Flickr)

Curved antennas, such as the ones shown in Figure 3.4, are commonly used to focus microwaves and radio waves to transmit television and telephone signals, as well as satellite and spacecraft communication. The cross-section of the antenna is in the shape of a parabola, which can be described by a quadratic function.

In this section, we will investigate quadratic functions, which frequently model problems involving area and projectile motion. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

**Recognizing Characteristics of Parabolas**

The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the minimum value of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the maximum value. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in Figure 3.5.
The $y$-intercept is the point at which the parabola crosses the $y$-axis. The $x$-intercepts are the points at which the parabola crosses the $x$-axis. If they exist, the $x$-intercepts represent the zeros, or roots, of the quadratic function, the values of $x$ at which $y = 0$.

**Example 3.11**

**Identifying the Characteristics of a Parabola**

Determine the vertex, axis of symmetry, zeros, and $y$-intercept of the parabola shown in Figure 3.6.
Understanding How the Graphs of Parabolas are Related to Their Quadratic Functions

The general form of a quadratic function presents the function in the form

\[ f(x) = ax^2 + bx + c \]  \hspace{1cm} (3.16)

where \( a, b, \) and \( c \) are real numbers and \( a \neq 0 \). If \( a > 0 \), the parabola opens upward. If \( a < 0 \), the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry.

The axis of symmetry is defined by \( x = -\frac{b}{2a} \). If we use the quadratic formula, \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \), to solve \( ax^2 + bx + c = 0 \) for the \( x \)-intercepts, or zeros, we find the value of \( x \) halfway between them is always \( x = -\frac{b}{2a} \), the equation for the axis of symmetry.

\( \text{Figure 3.7} \) represents the graph of the quadratic function written in general form as \( y = x^2 + 4x + 3 \). In this form, \( a = 1, b = 4, \) and \( c = 3 \). Because \( a > 0 \), the parabola opens upward. The axis of symmetry is \( x = -\frac{4}{2(1)} = -2 \). This also makes sense because we can see from the graph that the vertical line \( x = -2 \) divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, \((-2, -1)\). The \( x \)-intercepts, those points where the parabola crosses the \( x \)-axis, occur at \((-3, 0)\) and \((-1, 0)\).

The standard form of a quadratic function presents the function in the form

\[ f(x) = a(x - h)^2 + k \]  \hspace{1cm} (3.17)

where \((h, k)\) is the vertex. Because the vertex appears in the standard form of the quadratic function, this form is also known as the vertex form of a quadratic function.

As with the general form, if \( a > 0 \), the parabola opens upward and the vertex is a minimum. If \( a < 0 \), the parabola opens downward, and the vertex is a maximum. \( \text{Figure 3.8} \) represents the graph of the quadratic function written in standard form as \( y = -3(x + 2)^2 + 4 \). Since \( x - h = x + 2 \) in this example, \( h = -2 \). In this form, \( a = -3, h = -2, \) and \( k = 4 \). Because \( a < 0 \), the parabola opens downward. The vertex is at \((-2, 4)\).
The standard form is useful for determining how the graph is transformed from the graph of \( y = x^2 \). Figure 3.9 is the graph of this basic function.

If \( k > 0 \), the graph shifts upward, whereas if \( k < 0 \), the graph shifts downward. In Figure 3.8, \( k > 0 \), so the graph is shifted 4 units upward. If \( h > 0 \), the graph shifts toward the right and if \( h < 0 \), the graph shifts to the left. In Figure 3.8, \( h < 0 \), so the graph is shifted 2 units to the left. The magnitude of \( a \) indicates the stretch of the graph. If \( |a| > 1 \), the point associated with a particular \( x \)-value shifts farther from the \( x \)-axis, so the graph appears to become narrower, and there is a vertical stretch. But if \( |a| < 1 \), the point associated with a particular \( x \)-value shifts closer to the \( x \)-axis, so the graph appears to become wider, but in fact there is a vertical compression. In Figure 3.8, \( |a| > 1 \), so the graph becomes narrower.
The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

\[ a(x - h)^2 + k = ax^2 + bx + c \]  
\[ ax^2 - 2ahx + (ah^2 + k) = ax^2 + bx + c \]

For the linear terms to be equal, the coefficients must be equal.

\[-2ah = b, \text{ so } h = -\frac{b}{2a} \]

This is the axis of symmetry we defined earlier. Setting the constant terms equal:

\[ ah^2 + k = c \]
\[ k = c - ah^2 \]
\[ = c - a - \left( \frac{b}{2a} \right)^2 \]
\[ = c - \frac{b^2}{4a} \]

In practice, though, it is usually easier to remember that \( k \) is the output value of the function when the input is \( h \), so \( f(h) = k \).

**Forms of Quadratic Functions**

A quadratic function is a function of degree two. The graph of a quadratic function is a parabola. The **general form of a quadratic function** is \( f(x) = ax^2 + bx + c \) where \( a, b, \) and \( c \) are real numbers and \( a \neq 0 \).

The **standard form of a quadratic function** is \( f(x) = a(x - h)^2 + k \).

The vertex \((h, k)\) is located at

\[ h = -\frac{b}{2a}, \ k = f(h) = f\left( -\frac{b}{2a} \right) \]

**How To**

Given a graph of a quadratic function, write the equation of the function in general form.

1. Identify the horizontal shift of the parabola; this value is \( h \). Identify the vertical shift of the parabola; this value is \( k \).
2. Substitute the values of the horizontal and vertical shift for \( h \) and \( k \) in the function \( f(x) = a(x - h)^2 + k \).
3. Substitute the values of any point, other than the vertex, on the graph of the parabola for \( x \) and \( f(x) \).
4. Solve for the stretch factor, \(|a|\).
5. If the parabola opens up, \( a > 0 \). If the parabola opens down, \( a < 0 \) since this means the graph was reflected about the \( x \)-axis.
6. Expand and simplify to write in general form.

**Example 3.12**

**Writing the Equation of a Quadratic Function from the Graph**

Write an equation for the quadratic function \( g \) in Figure 3.10 as a transformation of \( f(x) = x^2 \), and then expand the formula, and simplify terms to write the equation in general form.
Solution

Analysis

We can check our work using the table feature on a graphing utility. First enter \( Y1 = \frac{1}{2}(x + 2)^2 - 3 \). Next, select TBLSET, then use TblStart = -6 and \( \Delta Tbl = 2 \), and select TABLE. See Table 3.1.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-6</th>
<th>-4</th>
<th>-2</th>
<th>0</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>5</td>
<td>-1</td>
<td>-3</td>
<td>-1</td>
<td>5</td>
</tr>
</tbody>
</table>

The ordered pairs in the table correspond to points on the graph.
A coordinate grid has been superimposed over the quadratic path of a basketball in Figure 3.11. Find an equation for the path of the ball. Does the shooter make the basket?

Figure 3.11  (credit: modification of work by Dan Meyer)

**Given a quadratic function in general form, find the vertex of the parabola.**

1. Identify $a$, $b$, and $c$.
2. Find $h$, the $x$-coordinate of the vertex, by substituting $a$ and $b$ into $h = -\frac{b}{2a}$.
3. Find $k$, the $y$-coordinate of the vertex, by evaluating $k = f(h) = f\left(-\frac{b}{2a}\right)$.

**Example 3.13**

**Finding the Vertex of a Quadratic Function**

Find the vertex of the quadratic function $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic in standard form (vertex form).

**Solution**

**Analysis**

One reason we may want to identify the vertex of the parabola is that this point will inform us where the maximum or minimum value of the output occurs, $(k)$, and where it occurs, $(x)$.

**Finding the Domain and Range of a Quadratic Function**

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all $y$-values greater than or equal to the $y$-coordinate at the turning point or less than or equal to the $y$-coordinate at the turning point, depending on whether the parabola opens up or down.
Domain and Range of a Quadratic Function

The domain of any quadratic function is all real numbers.

The range of a quadratic function written in general form \( f(x) = ax^2 + bx + c \) with a positive \( a \) value is \( f(x) \geq f\left(-\frac{b}{2a}\right) \), or \( \left[f\left(-\frac{b}{2a}\right), \infty\right) \); the range of a quadratic function written in general form with a negative \( a \) value is \( f(x) \leq f\left(-\frac{b}{2a}\right) \), or \( \left(-\infty, f\left(-\frac{b}{2a}\right)\right] \).

The range of a quadratic function written in standard form \( f(x) = a(x-h)^2 + k \) with a positive \( a \) value is \( f(x) \geq k \); the range of a quadratic function written in standard form with a negative \( a \) value is \( f(x) \leq k \).

Given a quadratic function, find the domain and range.

1. Identify the domain of any quadratic function as all real numbers.
2. Determine whether \( a \) is positive or negative. If \( a \) is positive, the parabola has a minimum. If \( a \) is negative, the parabola has a maximum.
3. Determine the maximum or minimum value of the parabola, \( k \).
4. If the parabola has a minimum, the range is given by \( f(x) \geq k \), or \( [k, \infty) \). If the parabola has a maximum, the range is given by \( f(x) \leq k \), or \( (-\infty, k] \).

Example 3.14

Finding the Domain and Range of a Quadratic Function

Find the domain and range of \( f(x) = -5x^2 + 9x - 1 \).

Solution

Try It

Find the domain and range of \( f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11} \).

Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the parabola. We can see the maximum and minimum values in Figure 3.12.
There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue.

Example 3.15

Finding the Maximum Value of a Quadratic Function

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

a. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length $L$.

b. What dimensions should she make her garden to maximize the enclosed area?

Solution

Analysis

This problem also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function in Figure 3.12.
Given an application involving revenue, use a quadratic equation to find the maximum.

1. Write a quadratic equation for revenue.
2. Find the vertex of the quadratic equation.
3. Determine the $y$-value of the vertex.

**Example 3.16**

**Finding Maximum Revenue**

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of $30. Market research has suggested that if the owners raise the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

**Solution**

**Analysis**

This could also be solved by graphing the quadratic as in Figure 3.12. We can see the maximum revenue on a graph of the quadratic function.
Finding the \( x \)- and \( y \)-Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the \( y \)-intercept of a quadratic by evaluating the function at an input of zero, and we find the \( x \)-intercepts at locations where the output is zero. Notice in Figure 3.13 that the number of \( x \)-intercepts can vary depending upon the location of the graph.

![Figure 3.13](http://legacy.cnx.org/content/col11667/1.2)

**Figure 3.13** Number of \( x \)-intercepts of a parabola.

**Given a quadratic function** \( f(x) \), **find the \( y \)- and \( x \)-intercepts.**

1. Evaluate \( f(0) \) to find the \( y \)-intercept.
2. Solve the quadratic equation \( f(x) = 0 \) to find the \( x \)-intercepts.

**Example 3.17**

**Finding the \( y \)- and \( x \)-Intercepts of a Parabola**

Find the \( y \)- and \( x \)-intercepts of the quadratic \( f(x) = 3x^2 + 5x - 2 \).

**Solution**

**Analysis**
By graphing the function, we can confirm that the graph crosses the $y$-axis at $(0, -2)$. We can also confirm that the graph crosses the $x$-axis at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$. See Figure 3.13

Rewriting Quadratics in Standard Form

In Example 3.17, the quadratic was easily solved by factoring. However, there are many quadratics that cannot be factored. We can solve these quadratics by first rewriting them in standard form.

**How To:** Given a quadratic function, find the $x$-intercepts by rewriting in standard form.

1. Substitute $a$ and $b$ into $h = -\frac{b}{2a}$.
2. Substitute $x = h$ into the general form of the quadratic function to find $k$.
3. Rewrite the quadratic in standard form using $h$ and $k$.
4. Solve for when the output of the function will be zero to find the $x$-intercepts.

**Example 3.18**

**Finding the $x$-Intercepts of a Parabola**

Find the $x$-intercepts of the quadratic function $f(x) = 2x^2 + 4x - 4$.

**Solution**

**Analysis**

We can check our work by graphing the given function on a graphing utility and observing the $x$-intercepts. See Figure 3.13.
In a separate Try It, we found the standard and general form for the function \( g(x) = 13 + x^2 - 6x \). Now find the \( y \)- and \( x \)-intercepts (if any).

**Example 3.19**

**Solving a Quadratic Equation with the Quadratic Formula**

Solve \( x^2 + x + 2 = 0 \).

**Solution**

**Example 3.20**

**Applying the Vertex and \( x \)-Intercepts of a Parabola**

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball’s height above ground can be modeled by the equation \( H(t) = -16t^2 + 80t + 40 \).

a. When does the ball reach the maximum height?

b. What is the maximum height of the ball?

c. When does the ball hit the ground?

**Solution**
A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock’s height above ocean can be modeled by the equation $H(t) = -16t^2 + 96t + 112$.

a. When does the rock reach the maximum height?
b. What is the maximum height of the rock?
c. When does the rock hit the ocean?

Access these online resources for additional instruction and practice with quadratic equations.

- [Graphing Quadratic Functions in General Form](http://openstaxcollege.org/l/graphquadgen)
- [Graphing Quadratic Functions in Standard Form](http://openstaxcollege.org/l/graphquadstan)
- [Quadratic Function Review](http://openstaxcollege.org/l/quadfuncrev)
- [Characteristics of a Quadratic Function](http://openstaxcollege.org/l/characterquad)
3.2 EXERCISES

Verbal

59. Explain the advantage of writing a quadratic function in standard form.

60. How can the vertex of a parabola be used in solving real world problems?

61. Explain why the condition of $a \neq 0$ is imposed in the definition of the quadratic function.

62. What is another name for the standard form of a quadratic function?

63. What two algebraic methods can be used to find the horizontal intercepts of a quadratic function?

Algebraic

For the following exercises, rewrite the quadratic functions in standard form and give the vertex.

64. $f(x) = x^2 - 12x + 32$

65. $g(x) = x^2 + 2x - 3$

66. $f(x) = x^2 - x$

67. $f(x) = x^2 + 5x - 2$

68. $h(x) = 2x^2 + 8x - 10$

69. $k(x) = 3x^2 - 6x - 9$

70. $f(x) = 2x^2 - 10x + 4$

71. $f(x) = 3x^2 - 5x - 1$

For the following exercises, determine whether there is a minimum or maximum value to each quadratic function. Find the value and the axis of symmetry.

72. $y(x) = 2x^2 + 10x + 12$

73. $f(x) = 2x^2 - 10x + 4$

74. $f(x) = -x^2 + 4x + 3$

75. $f(x) = 4x^2 + x - 1$

76. $h(t) = -4t^2 + 6t - 1$

77. $f(x) = \frac{1}{2}x^2 + 3x + 1$

78. $f(x) = -\frac{1}{3}x^2 - 2x + 3$

For the following exercises, determine the domain and range of the quadratic function.

79. $f(x) = (x - 3)^2 + 2$

80. $f(x) = -2(x + 3)^2 - 6$
81. \( f(x) = x^2 + 6x + 4 \)
82. \( f(x) = 2x^2 - 4x + 2 \)
83. \( k(x) = 3x^2 - 6x - 9 \)

For the following exercises, solve the equations over the complex numbers.
84. \( x^2 = -25 \)
85. \( x^2 = -8 \)
86. \( x^2 + 36 = 0 \)
87. \( x^2 + 27 = 0 \)
88. \( x^2 + 2x + 5 = 0 \)
89. \( x^2 - 4x + 5 = 0 \)
90. \( x^2 + 8x + 25 = 0 \)
91. \( x^2 - 4x + 13 = 0 \)
92. \( x^2 + 6x + 25 = 0 \)
93. \( x^2 - 10x + 26 = 0 \)
94. \( x^2 - 6x + 10 = 0 \)
95. \( x(x - 4) = 20 \)
96. \( x(x - 2) = 10 \)
97. \( 2x^2 + 2x + 5 = 0 \)
98. \( 5x^2 - 8x + 5 = 0 \)
99. \( 5x^2 + 6x + 2 = 0 \)
100. \( 2x^2 - 6x + 5 = 0 \)
101. \( x^2 + x + 2 = 0 \)
102. \( x^2 - 2x + 4 = 0 \)

For the following exercises, use the vertex \((h, k)\) and a point on the graph \((x, y)\) to find the general form of the equation of the quadratic function.
103. \((h, k) = (2, 0), (x, y) = (4, 4)\)
104. \((h, k) = (-2, -1), (x, y) = (-4, 3)\)
105. \((h, k) = (0, 1), (x, y) = (2, 5)\)
106. \((h, k) = (2, 3), (x, y) = (5, 12)\)
107. \((h, k) = (-5, 3), (x, y) = (2, 9)\)

108. \((h, k) = (3, 2), (x, y) = (10, 1)\)

109. \((h, k) = (0, 1), (x, y) = (1, 0)\)

110. \((h, k) = (1, 0), (x, y) = (0, 1)\)

**Graphical**

For the following exercises, sketch a graph of the quadratic function and give the vertex, axis of symmetry, and intercepts.

111. \(f(x) = x^2 - 2x\)

112. \(f(x) = x^2 - 6x - 1\)

113. \(f(x) = x^2 - 5x - 6\)

114. \(f(x) = x^2 - 7x + 3\)

115. \(f(x) = -2x^2 + 5x - 8\)

116. \(f(x) = 4x^2 - 12x - 3\)

For the following exercises, write the equation for the graphed function.

117.

![Graph of a quadratic function](image)

118.
119.

120.

121.
For the following exercises, use the table of values that represent points on the graph of a quadratic function. By determining the vertex and axis of symmetry, find the general form of the equation of the quadratic function.

123.

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Table 3.2

124.

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Table 3.3
Table 3.4

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Table 3.5

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<tbody>
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<td>0</td>
<td>2</td>
<td>8</td>
</tr>
</tbody>
</table>

Technology

For the following exercises, use a calculator to find the answer.

128. Graph on the same set of axes the functions \( f(x) = x^2 \), \( f(x) = 2x^2 \), and \( f(x) = \frac{1}{3}x^2 \).

What appears to be the effect of changing the coefficient?

129. Graph on the same set of axes \( f(x) = x^2 \), \( f(x) = x^2 + 2 \) and \( f(x) = x^2 \), \( f(x) = x^2 + 5 \) and \( f(x) = x^2 - 3 \). What appears to be the effect of adding a constant?

130. Graph on the same set of axes \( f(x) = x^2 \), \( f(x) = (x - 2)^2 \), \( f(x - 3)^2 \), and \( f(x) = (x + 4)^2 \).

What appears to be the effect of adding or subtracting those numbers?

131. The path of an object projected at a 45 degree angle with initial velocity of 80 feet per second is given by the function \( h(x) = \frac{-32}{(80)^2}x^2 + x \) where \( x \) is the horizontal distance traveled and \( h(x) \) is the height in feet. Use the TRACE feature of your calculator to determine the height of the object when it has traveled 100 feet away horizontally.

132. A suspension bridge can be modeled by the quadratic function \( h(x) = .0001x^2 \) with \(-2000 \leq x \leq 2000\) where \( |x| \) is the number of feet from the center and \( h(x) \) is height in feet. Use the TRACE feature of your calculator to estimate how far from the center does the bridge have a height of 100 feet.

Extensions

For the following exercises, use the vertex of the graph of the quadratic function and the direction the graph opens to find the domain and range of the function.

133. Vertex \((1, -2)\) opens up.

134. Vertex \((-1, 2)\) opens down.

135.
Vertex $(-5, 11)$, opens down.

136. Vertex $(-100, 100)$, opens up.

For the following exercises, write the equation of the quadratic function that contains the given point and has the same shape as the given function.

137. Contains $(1, 1)$ and has shape of $f(x) = 2x^2$. Vertex is on the $y$-axis.

138. Contains $(-1, 4)$ and has the shape of $f(x) = 2x^2$. Vertex is on the $y$-axis.

139. Contains $(2, 3)$ and has the shape of $f(x) = 3x^2$. Vertex is on the $y$-axis.

140. Contains $(1, -3)$ and has the shape of $f(x) = -x^2$. Vertex is on the $y$-axis.

141. Contains $(4, 3)$ and has the shape of $f(x) = 5x^2$. Vertex is on the $y$-axis.

142. Contains $(1, -6)$ has the shape of $f(x) = 3x^2$. Vertex has $x$-coordinate of $-1$.

**Real-World Applications**

143. Find the dimensions of the rectangular corral producing the greatest enclosed area given 200 feet of fencing.

144. Find the dimensions of the rectangular corral split into 2 pens of the same size producing the greatest possible enclosed area given 300 feet of fencing.

145. Find the dimensions of the rectangular corral producing the greatest enclosed area split into 3 pens of the same size given 500 feet of fencing.

146. Among all of the pairs of numbers whose sum is 6, find the pair with the largest product. What is the product?

147. Among all of the pairs of numbers whose difference is 12, find the pair with the smallest product. What is the product?

148. Suppose that the price per unit in dollars of a cell phone production is modeled by $p = 45 - 0.0125x$, where $x$ is in thousands of phones produced, and the revenue represented by thousands of dollars is $R = x \cdot p$. Find the production level that will maximize revenue.

149. A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 229t + 234$. Find the maximum height the rocket attains.

150. A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 24t + 8$. How long does it take to reach maximum height?

151. A soccer stadium holds 62,000 spectators. With a ticket price of $11, the average attendance has been 26,000. When the price dropped to $9, the average attendance rose to 31,000. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

152. A farmer finds that if she plants 75 trees per acre, each tree will yield 20 bushels of fruit. She estimates that for each additional tree planted per acre, the yield of each tree will decrease by 3 bushels. How many trees should she plant per acre to maximize her harvest?
3.3 | Power Functions and Polynomial Functions

Learning Objectives

In this section, you will:

3.3.1 Identify power functions.
3.3.2 Identify end behavior of power functions.
3.3.3 Identify polynomial functions.
3.3.4 Identify the degree and leading coefficient of polynomial functions.

Figure 3.14 (credit: Jason Bay, Flickr)

Suppose a certain species of bird thrives on a small island. Its population over the last few years is shown in Table 3.7.

<table>
<thead>
<tr>
<th>Year</th>
<th>2009</th>
<th>2010</th>
<th>2011</th>
<th>2012</th>
<th>2013</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bird Population</td>
<td>800</td>
<td>897</td>
<td>992</td>
<td>1,083</td>
<td>1,169</td>
</tr>
</tbody>
</table>

Table 3.7

The population can be estimated using the function \( P(t) = -0.3t^3 + 97t + 800 \), where \( P(t) \) represents the bird population on the island \( t \) years after 2009. We can use this model to estimate the maximum bird population and when it will occur. We can also use this model to predict when the bird population will disappear from the island. In this section, we will examine functions that we can use to estimate and predict these types of changes.

Identifying Power Functions

In order to better understand the bird problem, we need to understand a specific type of function. A power function is a function with a single term that is the product of a real number, a coefficient, and a variable raised to a fixed real number. (A number that multiplies a variable raised to an exponent is known as a coefficient.)

As an example, consider functions for area or volume. The function for the area of a circle with radius \( r \) is

\[
A(r) = \pi r^2
\]

(3.22)

and the function for the volume of a sphere with radius \( r \) is

\[
V(r) = \frac{4}{3}\pi r^3.
\]

(3.23)
Both of these are examples of power functions because they consist of a coefficient, $\pi$ or $\frac{4}{3}\pi$, multiplied by a variable $r$ raised to a power.

**Power Function**

A **power function** is a function that can be represented in the form

$$f(x) = kx^p$$  \hspace{1cm} (3.24)

where $k$ and $p$ are real numbers, and $k$ is known as the **coefficient**.

**Is $f(x) = 2^x$ a power function?**

No. A power function contains a variable base raised to a fixed power. This function has a constant base raised to a variable power. This is called an exponential function, not a power function.

**Example 3.21**

**Identifying Power Functions**

Which of the following functions are power functions?

$f(x) = 1$ Constant function  
$f(x) = x$ Identify function  
$f(x) = x^2$ Quadratic function  
$f(x) = x^3$ Cubic function  
$f(x) = \frac{1}{x}$ Reciprocal function  
$f(x) = \frac{1}{x^2}$ Reciprocal squared function  
$f(x) = \sqrt{x}$ Square root function  
$f(x) = \frac{3}{x}$ Cube root function

**Solution**

3.13 Which functions are power functions?

$$f(x) = 2x^2 \cdot 4x^3$$  
$$g(x) = -x^5 + 5x^3 - 4x$$  
$$h(x) = \frac{2x^5 - 1}{3x^2 + 4}$$

**Identifying End Behavior of Power Functions**

**Figure 3.15** shows the graphs of $f(x) = x^2$, $g(x) = x^4$, and $h(x) = x^6$, which are all power functions with even, whole-number powers. Notice that these graphs have similar shapes, very much like that of the quadratic function in the toolkit. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.
To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol $\infty$ for positive infinity and $-\infty$ for negative infinity. When we say that “$x$ approaches infinity,” which can be symbolically written as $x \to \infty$, we are describing a behavior; we are saying that $x$ is increasing without bound.

With the even-power function, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as $x$ approaches positive or negative infinity, the $f(x)$ values increase without bound. In symbolic form, we could write

$$\text{as } x \to \pm \infty, \quad f(x) \to \infty$$

Figure 3.16 shows the graphs of $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$, which are all power functions with odd, whole-number powers. Notice that these graphs look similar to the cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.

These examples illustrate that functions of the form $f(x) = x^n$ reveal symmetry of one kind or another. First, in Figure 3.15 we see that even functions of the form $f(x) = x^n$, $n$ even, are symmetric about the $y$-axis. In Figure 3.16 we see that odd functions of the form $f(x) = x^n$, $n$ odd, are symmetric about the origin.

For these odd power functions, as $x$ approaches negative infinity, $f(x)$ decreases without bound. As $x$ approaches positive infinity, $f(x)$ increases without bound. In symbolic form we write

$$\text{as } x \to -\infty, \quad f(x) \to -\infty$$
$$\text{as } x \to \infty, \quad f(x) \to \infty$$

The behavior of the graph of a function as the input values get very small ($x \to -\infty$) and get very large ($x \to \infty$) is referred to as the end behavior of the function. We can use words or symbols to describe end behavior.

Figure 3.17 shows the end behavior of power functions in the form $f(x) = kx^n$ where $n$ is a non-negative integer depending on the power and the constant.
Given a power function \( f(x) = kx^n \) where \( n \) is a non-negative integer, identify the end behavior.

1. Determine whether the power is even or odd.
2. Determine whether the constant is positive or negative.
3. Use Figure 3.17 to identify the end behavior.

### Example 3.22
**Identifying the End Behavior of a Power Function**

Describe the end behavior of the graph of \( f(x) = x^8 \).

**Solution**

### Example 3.23
**Identifying the End Behavior of a Power Function**

Describe the end behavior of the graph of \( f(x) = -x^9 \).
Solution
Solution
Analysis
We can check our work by using the table feature on a graphing utility.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td>1,000,000,000</td>
</tr>
<tr>
<td>-5</td>
<td>1,953,125</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>-1,953,125</td>
</tr>
<tr>
<td>10</td>
<td>-1,000,000,000</td>
</tr>
</tbody>
</table>

We can see from Table 3.7 that, when we substitute very small values for $x$, the output is very large, and when we substitute very large values for $x$, the output is very small (meaning that it is a very large negative value).

Describe in words and symbols the end behavior of $f(x) = -5x^4$.

Identifying Polynomial Functions
An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. We want to write a formula for the area covered by the oil slick by combining two functions. The radius $r$ of the spill depends on the number of weeks $w$ that have passed. This relationship is linear.

$$r(w) = 24 + 8w \quad (3.27)$$

We can combine this with the formula for the area $A$ of a circle.

$$A(r) = \pi r^2 \quad (3.28)$$

Composing these functions gives a formula for the area in terms of weeks.

$$A(w) = A(r(w)) = A(24 + 8w) = \pi(24 + 8w)^2 \quad (3.29)$$

Multiplying gives the formula.

$$A(w) = 576\pi + 384\pi w + 64\pi w^2 \quad (3.30)$$

This formula is an example of a polynomial function. A polynomial function consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.
Polynomial Functions

Let \( n \) be a non-negative integer. A polynomial function is a function that can be written in the form

\[
f(x) = a_n x^n + ... + a_2 x^2 + a_1 x + a_0 \tag{3.31}
\]

This is called the general form of a polynomial function. Each \( a_i \) is a coefficient and can be any real number. Each product \( a_i x^i \) is a term of a polynomial function.

Example 3.24

Identifying Polynomial Functions

Which of the following are polynomial functions?

\[
f(x) = 2x^3 \cdot 3x + 4 \\
g(x) = -x(x^2 - 4) \\
h(x) = 5\sqrt{x} + 2
\]

Solution

Identifying the Degree and Leading Coefficient of a Polynomial Function

Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the power of the variable. Although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order of power, or in general form. The degree of the polynomial is the highest power of the variable that occurs in the polynomial; it is the power of the first variable if the function is in general form. The leading term is the term containing the highest power of the variable, or the term with the highest degree. The leading coefficient is the coefficient of the leading term.

Terminology of Polynomial Functions

We often rearrange polynomials so that the powers are descending.

\[
f(x) = a_n x^n + ... + a_2 x^2 + a_1 x + a_0
\]

When a polynomial is written in this way, we say that it is in general form.

How To

Given a polynomial function, identify the degree and leading coefficient.

1. Find the highest power of \( x \) to determine the degree function.
2. Identify the term containing the highest power of \( x \) to find the leading term.
3. Identify the coefficient of the leading term.

Example 3.25
Identifying the Degree and Leading Coefficient of a Polynomial Function

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

\[ f(x) = 3 + 2x^2 - 4x^3 \]
\[ g(t) = 5t^5 - 2t^3 + 7t \]
\[ h(p) = 6p - p^3 - 2 \]

**Solution**

Identify the degree, leading term, and leading coefficient of the polynomial \( f(x) = 4x^2 - x^6 + 2x - 6 \).

Identifying End Behavior of Polynomial Functions

Knowing the degree of a polynomial function is useful in helping us predict its end behavior. To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as \( x \) gets very large or very small, so its behavior will dominate the graph. For any polynomial, the end behavior of the polynomial will match the end behavior of the term of highest degree. See Table 3.8.
<table>
<thead>
<tr>
<th>Polynomial Function</th>
<th>Leading Term</th>
<th>Graph of Polynomial Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = 5x^4 + 2x^3 - x - 4 )</td>
<td>( 5x^4 )</td>
<td><img src="image" alt="Graph of Polynomial Function" /></td>
</tr>
<tr>
<td>( f(x) = -2x^6 - x^5 + 3x^4 + x^3 )</td>
<td>( -2x^6 )</td>
<td><img src="image" alt="Graph of Polynomial Function" /></td>
</tr>
</tbody>
</table>

Table 3.8
<table>
<thead>
<tr>
<th>Polynomial Function</th>
<th>Leading Term</th>
<th>Graph of Polynomial Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = 3x^5 - 4x^4 + 2x^2 + 1$</td>
<td>$3x^5$</td>
<td><img src="image1.png" alt="Graph of Polynomial Function" /></td>
</tr>
<tr>
<td>$f(x) = -6x^3 + 7x^2 + 3x + 1$</td>
<td>$-6x^3$</td>
<td><img src="image2.png" alt="Graph of Polynomial Function" /></td>
</tr>
</tbody>
</table>

Table 3.8

Example 3.26
Identifying End Behavior and Degree of a Polynomial Function

Describe the end behavior and determine a possible degree of the polynomial function in Figure 3.18.

Figure 3.18

Solution

3.16 Describe the end behavior, and determine a possible degree of the polynomial function in Figure 3.19.

Figure 3.19

Example 3.27

Identifying End Behavior and Degree of a Polynomial Function
Given the function \( f(x) = -3x^2(x - 1)(x + 4), \) express the function as a polynomial in general form, and determine the leading term, degree, and end behavior of the function.

**Solution**

3.17 Given the function \( f(x) = 0.2(x - 2)(x + 1)(x - 5), \) express the function as a polynomial in general form and determine the leading term, degree, and end behavior of the function.

**Identifying Local Behavior of Polynomial Functions**

In addition to the end behavior of polynomial functions, we are also interested in what happens in the “middle” of the function. In particular, we are interested in locations where graph behavior changes. A **turning point** is a point at which the function values change from increasing to decreasing or decreasing to increasing.

We are also interested in the intercepts. As with all functions, the **y-intercept** is the point at which the graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one **y-intercept** \((0, a_0)\). The **x-intercepts** occur at the input values that correspond to an output value of zero. It is possible to have more than one x-intercept. See **Figure 3.20**.

**Figure 3.20**

**Intercepts and Turning Points of Polynomial Functions**

A **turning point** of a graph is a point at which the graph changes direction from increasing to decreasing or decreasing to increasing. The **y-intercept** is the point at which the function has an input value of zero. The **x-intercepts** are the points at which the output value is zero.
Given a polynomial function, determine the intercepts.

1. Determine the $y$-intercept by setting $x = 0$ and finding the corresponding output value.
2. Determine the $x$-intercepts by solving for the input values that yield an output value of zero.

Example 3.28

Determining the Intercepts of a Polynomial Function

Given the polynomial function $f(x) = (x - 2)(x + 1)(x - 4)$, written in factored form for your convenience, determine the $y$- and $x$-intercepts.

Solution

Example 3.29

Determining the Intercepts of a Polynomial Function with Factoring

Given the polynomial function $f(x) = x^4 - 4x^2 - 45$, determine the $y$- and $x$-intercepts.

Solution

3.18 Given the polynomial function $f(x) = 2x^3 - 6x^2 - 20x$, determine the $y$- and $x$-intercepts.

Comparing Smooth and Continuous Graphs

The degree of a polynomial function helps us determine the number of $x$-intercepts and the number of turning points. A polynomial function of $n$th degree is the product of $n$ factors, so it will have at most $n$ roots or zeros, or $x$-intercepts. The graph of the polynomial function of degree $n$ must have at most $n - 1$ turning points. This means the graph has at most one fewer turning point than the degree of the polynomial or one fewer than the number of factors.

A **continuous function** has no breaks in its graph: the graph can be drawn without lifting the pen from the paper. A **smooth curve** is a graph that has no sharp corners. The turning points of a smooth graph must always occur at rounded curves. The graphs of polynomial functions are both continuous and smooth.

**Intercepts and Turning Points of Polynomials**

A polynomial of degree $n$ will have, at most, $n$ $x$-intercepts and $n - 1$ turning points.

Example 3.30

Determining the Number of Intercepts and Turning Points of a Polynomial
Without graphing the function, determine the local behavior of the function by finding the maximum number of \( x \)-intercepts and turning points for \( f(x) = -3x^{10} + 4x^7 - x^4 + 2x^3 \).

**Solution**

Example 3.31

**Drawing Conclusions about a Polynomial Function from the Graph**

What can we conclude about the polynomial represented by the graph shown in Figure 3.21 based on its intercepts and turning points?

![Graph of a polynomial function](image)
3.20 What can we conclude about the polynomial represented by the graph shown in Figure 3.22 based on its intercepts and turning points?

![Graph of a polynomial function](image)

Figure 3.22

Example 3.32

**Drawing Conclusions about a Polynomial Function from the Factors**

Given the function \( f(x) = -4x(x + 3)(x - 4) \), determine the local behavior.

**Solution**

Given the function \( f(x) = 0.2(x - 2)(x + 1)(x - 5) \), determine the local behavior.

Access these online resources for additional instruction and practice with power and polynomial functions.

- [Find Key Information about a Given Polynomial Function](http://openstaxcollege.org/l/keyinfopoly)
- [End Behavior of a Polynomial Function](http://openstaxcollege.org/l/endbehavior)
- [Turning Points and x-intercepts of Polynomial Functions](http://openstaxcollege.org/l/turningpoints)
- [Least Possible Degree of a Polynomial Function](http://openstaxcollege.org/l/leastposdegree)
3.3 EXERCISES

Verbal

153. Explain the difference between the coefficient of a power function and its degree.

154. If a polynomial function is in factored form, what would be a good first step in order to determine the degree of the function?

155. In general, explain the end behavior of a power function with odd degree if the leading coefficient is positive.

156. What is the relationship between the degree of a polynomial function and the maximum number of turning points in its graph?

157. What can we conclude if, in general, the graph of a polynomial function exhibits the following end behavior? As $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$.

Algebraic

For the following exercises, identify the function as a power function, a polynomial function, or neither.

158. $f(x) = x^5$

159. $f(x) = (x^2)^3$

160. $f(x) = x - x^4$

161. $f(x) = \frac{x^2}{x^2 - 1}$

162. $f(x) = 2x(x + 2)(x - 1)^2$

163. $f(x) = 3^x + 1$

For the following exercises, find the degree and leading coefficient for the given polynomial.

164. $-3x$

165. $7 - 2x^2$

166. $-2x^2 - 3x^5 + x - 6$

167. $x(4 - x^2)(2x + 1)$

168. $x^2(2x - 3)^2$

For the following exercises, determine the end behavior of the functions.

169. $f(x) = x^4$

170. $f(x) = x^3$

171. $f(x) = -x^4$

172. $f(x) = -x^9$

173. $f(x) = -2x^4 - 3x^2 + x - 1$
174. \( f(x) = 3x^2 + x - 2 \)

175. \( f(x) = x^2(2x^3 - x + 1) \)

176. \( f(x) = (2 - x)^7 \)

For the following exercises, find the intercepts of the functions.

177. \( f(t) = 2(t - 1)(t + 2)(t - 3) \)

178. \( g(n) = -2(3n - 1)(2n + 1) \)

179. \( f(x) = x^4 - 16 \)

180. \( f(x) = x^3 + 27 \)

181. \( f(x) = x(x^2 - 2x - 8) \)

182. \( f(x) = (x + 3)(4x^2 - 1) \)

**Graphical**

For the following exercises, determine the least possible degree of the polynomial function shown.

183.

![Graph 1](image1.png)

184.

![Graph 2](image2.png)

185.
For the following exercises, determine whether the graph of the function provided is a graph of a polynomial function. If so, determine the number of turning points and the least possible degree for the function.

190.

For the following exercises, determine whether the graph of the function provided is a graph of a polynomial function. If so, determine the number of turning points and the least possible degree for the function.

191.

192.
Numeric
For the following exercises, make a table to confirm the end behavior of the function.

198. \( f(x) = -x^3 \)

199. \( f(x) = x^4 - 5x^2 \)

200. \( f(x) = x^2 (1-x)^2 \)

201. \( f(x) = (x-1)(x-2)(3-x) \)

202. \( f(x) = \frac{x^5}{10} - x^4 \)
Technology

For the following exercises, graph the polynomial functions using a calculator. Based on the graph, determine the intercepts and the end behavior.

203. \( f(x) = x^3(x - 2) \)

204. \( f(x) = x(x - 3)(x + 3) \)

205. \( f(x) = x(14 - 2x)(10 - 2x) \)

206. \( f(x) = x(14 - 2x)(10 - 2x)^2 \)

207. \( f(x) = x^3 - 16x \)

208. \( f(x) = x^3 - 27 \)

209. \( f(x) = x^4 - 81 \)

210. \( f(x) = -x^3 + x^2 + 2x \)

211. \( f(x) = x^3 - 2x^2 - 15x \)

212. \( f(x) = x^3 - 0.01x \)

Extensions

For the following exercises, use the information about the graph of a polynomial function to determine the function. Assume the leading coefficient is 1 or –1. There may be more than one correct answer.

213. The y-intercept is \((0, -4)\). The x-intercepts are \((-2, 0), (2, 0)\). Degree is 2.
   End behavior: \( \text{as } x \to -\infty, f(x) \to \infty, \text{ as } x \to \infty, f(x) \to \infty. \)

214. The y-intercept is \((0, 9)\). The x-intercepts are \((-3, 0), (3, 0)\). Degree is 2.
   End behavior: \( \text{as } x \to -\infty, f(x) \to -\infty, \text{ as } x \to \infty, f(x) \to -\infty. \)

215. The y-intercept is \((0, 0)\). The x-intercepts are \((0, 0), (2, 0)\). Degree is 3.
   End behavior: \( \text{as } x \to -\infty, f(x) \to -\infty, \text{ as } x \to \infty, f(x) \to \infty. \)

216. The y-intercept is \((0, 1)\). The x-intercept is \((1, 0)\). Degree is 3.
   End behavior: \( \text{as } x \to -\infty, f(x) \to \infty, \text{ as } x \to \infty, f(x) \to -\infty. \)

217. The y-intercept is \((0, 1)\). There is no x-intercept. Degree is 4.
   End behavior: \( \text{as } x \to -\infty, f(x) \to \infty, \text{ as } x \to \infty, f(x) \to \infty. \)

Real-World Applications

For the following exercises, use the written statements to construct a polynomial function that represents the required information.

218. An oil slick is expanding as a circle. The radius of the circle is increasing at the rate of 20 meters per day. Express the area of the circle as a function of \(d\), the number of days elapsed.

219. A cube has an edge of 3 feet. The edge is increasing at the rate of 2 feet per minute. Express the volume of the cube as a function of \(m\), the number of minutes elapsed.
220. A rectangle has a length of 10 inches and a width of 6 inches. If the length is increased by $x$ inches and the width increased by twice that amount, express the area of the rectangle as a function of $x$.

221. An open box is to be constructed by cutting out square corners of $x$-inch sides from a piece of cardboard 8 inches by 8 inches and then folding up the sides. Express the volume of the box as a function of $x$.

222. A rectangle is twice as long as it is wide. Squares of side 2 feet are cut out from each corner. Then the sides are folded up to make an open box. Express the volume of the box as a function of the width ($x$).
3.4 | Graphs of Polynomial Functions

**Learning Objectives**

In this section, you will:

3.4.1 Recognize characteristics of graphs of polynomial functions.
3.4.2 Use factoring to find zeros of polynomial functions.
3.4.3 Identify zeros and their multiplicities.
3.4.4 Determine end behavior.
3.4.5 Understand the relationship between degree and turning points.
3.4.6 Graph polynomial functions.
3.4.7 Use the Intermediate Value Theorem.

The revenue in millions of dollars for a fictional cable company from 2006 through 2013 is shown in Table 3.9.

<table>
<thead>
<tr>
<th>Year</th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
<th>2011</th>
<th>2012</th>
<th>2013</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revenues</td>
<td>52.4</td>
<td>52.8</td>
<td>51.2</td>
<td>49.5</td>
<td>48.6</td>
<td>48.6</td>
<td>48.7</td>
<td>47.1</td>
</tr>
</tbody>
</table>

*Table 3.9*

The revenue can be modeled by the polynomial function

\[ R(t) = -0.037t^4 + 1.414t^3 - 19.777t^2 + 118.696t - 205.332 \]  

where \( R \) represents the revenue in millions of dollars and \( t \) represents the year, with \( t = 6 \) corresponding to 2006. Over which intervals is the revenue for the company increasing? Over which intervals is the revenue for the company decreasing?

These questions, along with many others, can be answered by examining the graph of the polynomial function. We have already explored the local behavior of quadratics, a special case of polynomials. In this section we will explore the local behavior of polynomials in general.

**Recognizing Characteristics of Graphs of Polynomial Functions**

Polynomial functions of degree 2 or more have graphs that do not have sharp corners; recall that these types of graphs are called smooth curves. Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous. **Figure 3.23** shows a graph that represents a polynomial function and a graph that represents a function that is not a polynomial.
Example 3.33

Recognizing Polynomial Functions

Which of the graphs in Figure 3.24 represents a polynomial function?
Do all polynomial functions have as their domain all real numbers?

Yes. Any real number is a valid input for a polynomial function.

Using Factoring to Find Zeros of Polynomial Functions

Recall that if \( f \) is a polynomial function, the values of \( x \) for which \( f(x) = 0 \) are called zeros of \( f \). If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros.

We can use this method to find \( x \)-intercepts because at the \( x \)-intercepts we find the input values when the output value is zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and fourth-degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases:

1. The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
2. The polynomial is given in factored form.
3. Technology is used to determine the intercepts.
Given a polynomial function \( f \), find the \( x \)-intercepts by factoring.

1. Set \( f(x) = 0 \).
2. If the polynomial function is not given in factored form:
   a. Factor out any common monomial factors.
   b. Factor any factorable binomials or trinomials.
3. Set each factor equal to zero and solve to find the \( x \)-intercepts.

**Example 3.34**

**Finding the \( x \)-Intercepts of a Polynomial Function by Factoring**

Find the \( x \)-intercepts of \( f(x) = x^6 - 3x^4 + 2x^2 \).

**Solution**

**Solution**

**Example 3.35**

**Finding the \( x \)-Intercepts of a Polynomial Function by Factoring**

Find the \( x \)-intercepts of \( f(x) = x^3 - 5x^2 - x + 5 \).

**Solution**

**Solution**

**Example 3.36**

**Finding the \( y \)- and \( x \)-Intercepts of a Polynomial in Factored Form**

Find the \( y \)- and \( x \)-intercepts of \( g(x) = (x - 2)^2(2x + 3) \).

**Solution**

**Solution**

**Analysis**

We can always check that our answers are reasonable by using a graphing calculator to graph the polynomial as shown in **Figure 3.24**.
Example 3.37

Finding the \( x \)-Intercepts of a Polynomial Function Using a Graph

Find the \( x \)-intercepts of \( h(x) = x^3 + 4x^2 + x - 6 \).

Solution

Find the \( y \)- and \( x \)-intercepts of the function \( f(x) = x^4 - 19x^2 + 30x \).

Identifying Zeros and Their Multiplicities

Graphs behave differently at various \( x \)-intercepts. Sometimes, the graph will cross over the horizontal axis at an intercept. Other times, the graph will touch the horizontal axis and bounce off.

Suppose, for example, we graph the function

\[
    f(x) = (x + 3)(x - 2)^2(x + 1)^3. \tag{3.33}
\]

Notice in Figure 3.25 that the behavior of the function at each of the \( x \)-intercepts is different.
The $x$-intercept $x = -3$ is the solution of equation $(x + 3) = 0$. The graph passes directly through the $x$-intercept at $x = -3$. The factor is linear (has a degree of 1), so the behavior near the intercept is like that of a line—it passes directly through the intercept. We call this a single zero because the zero corresponds to a single factor of the function.

The $x$-intercept $x = 2$ is the repeated solution of equation $(x - 2)^2 = 0$. The graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic—it bounces off of the horizontal axis at the intercept.

$$ (x - 2)^2 = (x - 2)(x - 2) \quad (3.34) $$

The factor is repeated, that is, the factor $(x - 2)$ appears twice. The number of times a given factor appears in the factored form of the equation of a polynomial is called the **multiplicity**. The zero associated with this factor, $x = 2$, has multiplicity 2 because the factor $(x - 2)$ occurs twice.

The $x$-intercept $x = -1$ is the repeated solution of factor $(x + 1)^3 = 0$. The graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic—with the same S-shape near the intercept as the toolkit function $f(x) = x^3$. We call this a triple zero, or a zero with multiplicity 3.

For zeros with even multiplicities, the graphs touch or are tangent to the $x$-axis. For zeros with odd multiplicities, the graphs cross or intersect the $x$-axis. See Figure 3.26 for examples of graphs of polynomial functions with multiplicity 1, 2, and 3.

For higher even powers, such as 4, 6, and 8, the graph will still touch and bounce off of the horizontal axis but, for each increasing even power, the graph will appear flatter as it approaches and leaves the $x$-axis.

For higher odd powers, such as 5, 7, and 9, the graph will still cross through the horizontal axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the $x$-axis.
Graphical Behavior of Polynomials at \( x \)-Intercepts

If a polynomial contains a factor of the form \((x - h)^p\), the behavior near the \( x \)-intercept \( h \) is determined by the power \( p \). We say that \( x = h \) is a zero of multiplicity \( p \).

The graph of a polynomial function will touch the \( x \)-axis at zeros with even multiplicities. The graph will cross the \( x \)-axis at zeros with odd multiplicities.

The sum of the multiplicities is the degree of the polynomial function.

**How To:** Given a graph of a polynomial function of degree \( n \), identify the zeros and their multiplicities.

1. If the graph crosses the \( x \)-axis and appears almost linear at the intercept, it is a single zero.
2. If the graph touches the \( x \)-axis and bounces off of the axis, it is a zero with even multiplicity.
3. If the graph crosses the \( x \)-axis at a zero, it is a zero with odd multiplicity.
4. The sum of the multiplicities is \( n \).

**Example 3.38**

**Identifying Zeros and Their Multiplicities**

Use the graph of the function of degree 6 in Figure 3.27 to identify the zeros of the function and their possible multiplicities.
3.23 Use the graph of the function of degree 5 in Figure 3.28 to identify the zeros of the function and their multiplicities.

**Determining End Behavior**

As we have already learned, the behavior of a graph of a polynomial function of the form

\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0
\]

will either ultimately rise or fall as \( x \) increases without bound and will either rise or fall as \( x \) decreases without bound. This is because for very large inputs, say 100 or 1,000, the leading term dominates the size of the output. The same is true for very small inputs, say –100 or –1,000.

Recall that we call this behavior the *end behavior* of a function. As we pointed out when discussing quadratic equations, when the leading term of a polynomial function, \( a_n x^n \), is an even power function, as \( x \) increases or decreases without bound, \( f(x) \) increases without bound. When the leading term is an odd power function, as \( x \) decreases without bound, \( f(x) \) also decreases without bound; as \( x \) increases without bound, \( f(x) \) also increases without bound. If the leading term is negative, it will change the direction of the end behavior. Figure 3.29 summarizes all four cases.
### Understanding the Relationship between Degree and Turning Points

In addition to the end behavior, recall that we can analyze a polynomial function’s local behavior. It may have a turning point where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). Look at the graph of the polynomial function \( f(x) = x^4 - x^3 - 4x^2 + 4x \) in **Figure 3.30**. The graph has three turning points.

<table>
<thead>
<tr>
<th>Even Degree</th>
<th>Odd Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Positive Leading Coefficient, ( a_n &gt; 0 )</strong></td>
<td><strong>Positive Leading Coefficient, ( a_n &gt; 0 )</strong></td>
</tr>
<tr>
<td><img src="image1" alt="Graph" /></td>
<td><img src="image2" alt="Graph" /></td>
</tr>
<tr>
<td><strong>End Behavior:</strong></td>
<td><strong>End Behavior:</strong></td>
</tr>
<tr>
<td>( x \to \infty, f(x) \to \infty )</td>
<td>( x \to \infty, f(x) \to \infty )</td>
</tr>
<tr>
<td>( x \to -\infty, f(x) \to -\infty )</td>
<td>( x \to -\infty, f(x) \to \infty )</td>
</tr>
</tbody>
</table>

**Figure 3.29**

### Understanding the Relationship between Degree and Turning Points

In addition to the end behavior, recall that we can analyze a polynomial function’s local behavior. It may have a turning point where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). Look at the graph of the polynomial function \( f(x) = x^4 - x^3 - 4x^2 + 4x \) in **Figure 3.30**. The graph has three turning points.

**Figure 3.30**
This function $f$ is a 4th degree polynomial function and has 3 turning points. The maximum number of turning points of a polynomial function is always one less than the degree of the function.

**Interpreting Turning Points**

A turning point is a point of the graph where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising).

A polynomial of degree $n$ will have at most $n - 1$ turning points.

**Example 3.39**

**Finding the Maximum Number of Turning Points Using the Degree of a Polynomial Function**

Find the maximum number of turning points of each polynomial function.

a. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$

b. $f(x) = -(x - 1)^2(1 + 2x^2)$

**Solution**

**Graphing Polynomial Functions**

We can use what we have learned about multiplicities, end behavior, and turning points to sketch graphs of polynomial functions. Let us put this all together and look at the steps required to graph polynomial functions.

1. **Given a polynomial function, sketch the graph.**
   1. Find the intercepts.
   2. Check for symmetry. If the function is an even function, its graph is symmetrical about the $y$-axis, that is, $f(-x) = f(x)$. If a function is an odd function, its graph is symmetrical about the origin, that is, $f(-x) = -f(x)$.
   3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the $x$-intercepts.
   4. Determine the end behavior by examining the leading term.
   5. Use the end behavior and the behavior at the intercepts to sketch a graph.
   6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
   7. Optionally, use technology to check the graph.

**Example 3.40**

**Sketching the Graph of a Polynomial Function**

Sketch a graph of $f(x) = -2(x + 3)^2(x - 5)$. 

**Solution**
3.24 Sketch a graph of $f(x) = \frac{1}{4}(x - 1)^4(x + 3)^3$.

Using the Intermediate Value Theorem

In some situations, we may know two points on a graph but not the zeros. If those two points are on opposite sides of the $x$-axis, we can confirm that there is a zero between them. Consider a polynomial function $f$ whose graph is smooth and continuous. The Intermediate Value Theorem states that for two numbers $a$ and $b$ in the domain of $f$, if $a < b$ and $f(a) \neq f(b)$, then the function $f$ takes on every value between $f(a)$ and $f(b)$. We can apply this theorem to a special case that is useful in graphing polynomial functions. If a point on the graph of a continuous function $f$ at $x = a$ lies above the $x$-axis and another point at $x = b$ lies below the $x$-axis, there must exist a third point between $x = a$ and $x = b$ where the graph crosses the $x$-axis. Call this point $(c, f(c))$. This means that we are assured there is a solution $c$ where $f(c) = 0$.

In other words, the Intermediate Value Theorem tells us that when a polynomial function changes from a negative value to a positive value, the function must cross the $x$-axis. Figure 3.31 shows that there is a zero between $a$ and $b$.

Figure 3.31 Using the Intermediate Value Theorem to show there exists a zero.

Intermediate Value Theorem

Let $f$ be a polynomial function. The Intermediate Value Theorem states that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value $c$ between $a$ and $b$ for which $f(c) = 0$.

Example 3.41

Using the Intermediate Value Theorem

Show that the function $f(x) = x^3 - 5x^2 + 3x + 6$ has at least two real zeros between $x = 1$ and $x = 4$.

Solution

Analysis

We can also see on the graph of the function in Figure 3.31 that there are two real zeros between $x = 1$ and $x = 4$. 

3.25 Show that the function \( f(x) = 7x^5 - 9x^4 - x^2 \) has at least one real zero between \( x = 1 \) and \( x = 2 \).

**Writing Formulas for Polynomial Functions**

Now that we know how to find zeros of polynomial functions, we can use them to write formulas based on graphs. Because a polynomial function written in factored form will have an \( x \)-intercept where each factor is equal to zero, we can form a function that will pass through a set of \( x \)-intercepts by introducing a corresponding set of factors.

**Factored Form of Polynomials**

If a polynomial of lowest degree \( p \) has horizontal intercepts at \( x = x_1, x_2, \ldots, x_n \), then the polynomial can be written in the factored form: \( f(x) = a(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n} \) where the powers \( p_j \) on each factor can be determined by the behavior of the graph at the corresponding intercept, and the stretch factor \( a \) can be determined given a value of the function other than the \( x \)-intercept.

**How To:** Given a graph of a polynomial function, write a formula for the function.

1. Identify the \( x \)-intercepts of the graph to find the factors of the polynomial.
2. Examine the behavior of the graph at the \( x \)-intercepts to determine the multiplicity of each factor.
3. Find the polynomial of least degree containing all the factors found in the previous step.
4. Use any other point on the graph (the \( y \)-intercept may be easiest) to determine the stretch factor.

**Example 3.42**
Writing a Formula for a Polynomial Function from the Graph

Write a formula for the polynomial function shown in Figure 3.32.

![Graph of a polynomial function](image)

**Solution**

Given the graph shown in Figure 3.33, write a formula for the function shown.

![Graph of another polynomial function](image)

**3.26** Given the graph shown in Figure 3.33, write a formula for the function shown.

Using Local and Global Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Each turning point represents a local minimum or maximum. Sometimes, a turning point is the highest or lowest point on the entire graph. In these cases, we say that the turning point is a **global maximum** or a **global minimum**. These are also referred to as the absolute maximum and absolute minimum values of the function.

**Local and Global Extrema**

A local maximum or local minimum at \( x = a \) (sometimes called the relative maximum or minimum, respectively) is the output at the highest or lowest point on the graph in an open interval around \( x = a \). If a function has a local
maximum at $a$, then $f(a) \geq f(x)$ for all $x$ in an open interval around $x = a$. If a function has a local minimum at $a$, then $f(a) \leq f(x)$ for all $x$ in an open interval around $x = a$.

A **global maximum** or **global minimum** is the output at the highest or lowest point of the function. If a function has a global maximum at $a$, then $f(a) \geq f(x)$ for all $x$. If a function has a global minimum at $a$, then $f(a) \leq f(x)$ for all $x$.

We can see the difference between local and global extrema in **Figure 3.34**.

![Figure 3.34](image)

**Do all polynomial functions have a global minimum or maximum?**

No. Only polynomial functions of even degree have a global minimum or maximum. For example, $f(x) = x$ has neither a global maximum nor a global minimum.

**Example 3.43**

**Using Local Extrema to Solve Applications**

An open-top box is to be constructed by cutting out squares from each corner of a 14 cm by 20 cm sheet of plastic then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.

**Solution**

Use technology to find the maximum and minimum values on the interval $[-1, 4]$ of the function

$$f(x) = -0.2(x - 2)^3(x + 1)^2(x - 4).$$

Access the following online resource for additional instruction and practice with graphing polynomial functions.

- **Intermediate Value Theorem** (http://openstaxcollege.org/l/ivt)
3.4 EXERCISES

Verbal
223. What is the difference between an $x$-intercept and a zero of a polynomial function $f$?
224. If a polynomial function of degree $n$ has $n$ distinct zeros, what do you know about the graph of the function?
225. Explain how the Intermediate Value Theorem can assist us in finding a zero of a function.
226. Explain how the factored form of the polynomial helps us in graphing it.
227. If the graph of a polynomial just touches the $x$-axis and then changes direction, what can we conclude about the factored form of the polynomial?

Algebraic
For the following exercises, find the $x$- or $t$-intercepts of the polynomial functions.

228. $C(t) = 2(t - 4)(t + 1)(t - 6)$
229. $C(t) = 3(t + 2)(t - 3)(t + 5)$
230. $C(t) = 4t(t - 2)^2(t + 1)$
231. $C(t) = 2(t - 3)(t + 1)^2$
232. $C(t) = 2t^4 - 8t^3 + 6t^2$
233. $C(t) = 4t^4 + 12t^3 - 40t^2$
234. $f(x) = x^4 - x^2$
235. $f(x) = x^3 + x^2 - 20x$
236. $f(x) = x^3 + 6x^2 - 7x$
237. $f(x) = x^3 + x^2 - 4x - 4$
238. $f(x) = x^3 + 2x^2 - 9x - 18$
239. $f(x) = 2x^3 - x^2 - 8x + 4$
240. $f(x) = x^6 - 7x^3 - 8$
241. $f(x) = 2x^4 + 6x^2 - 8$
242. $f(x) = x^3 - 3x^2 - x + 3$
243. $f(x) = x^6 - 2x^4 - 3x^2$
244. $f(x) = x^6 - 3x^4 - 4x^2$
245. $f(x) = x^5 - 5x^3 + 4x$
For the following exercises, use the Intermediate Value Theorem to confirm that the given polynomial has at least one zero within the given interval.

246. \( f(x) = x^3 - 9x \), between \( x = -4 \) and \( x = -2 \).

247. \( f(x) = x^3 - 9x \), between \( x = 2 \) and \( x = 4 \).

248. \( f(x) = x^5 - 2x \), between \( x = 1 \) and \( x = 2 \).

249. \( f(x) = -x^4 + 4 \), between \( x = 1 \) and \( x = 3 \).

250. \( f(x) = -2x^3 - x \), between \( x = -1 \) and \( x = 1 \).

251. \( f(x) = x^3 - 100x + 2 \), between \( x = 0.01 \) and \( x = 0.1 \).

For the following exercises, find the zeros and give the multiplicity of each.

252. \( f(x) = (x + 2)^3(x - 3)^2 \)

253. \( f(x) = x^2(2x + 3)^5(x - 4)^2 \)

254. \( f(x) = x^3(x - 1)^3(x + 2) \)

255. \( f(x) = x^2(x^2 + 4x + 4) \)

256. \( f(x) = (2x + 1)^3(9x^2 - 6x + 1) \)

257. \( f(x) = (3x + 2)^5(x^2 - 10x + 25) \)

258. \( f(x) = x(4x^2 - 12x + 9)(x^2 + 8x + 16) \)

259. \( f(x) = x^6 - x^5 - 2x^4 \)

260. \( f(x) = 3x^4 + 6x^3 + 3x^2 \)

261. \( f(x) = 4x^5 - 12x^4 + 9x^3 \)

262. \( f(x) = 2x^4(x^3 - 4x^2 + 4x) \)

263. \( f(x) = 4x^4(9x^4 - 12x^3 + 4x^2) \)

**Graphical**

For the following exercises, graph the polynomial functions. Note \( x \)- and \( y \)-intercepts, multiplicity, and end behavior.

264. \( f(x) = (x + 3)^2(x - 2) \)

265. \( g(x) = (x + 4)(x - 1)^2 \)

266. \( h(x) = (x - 1)^3(x + 3)^2 \)

267. \( k(x) = (x - 3)^3(x - 2)^2 \)
268. \( m(x) = -2x(x - 1)(x + 3) \)

269. \( n(x) = -3x(x + 2)(x - 4) \)

For the following exercises, use the graphs to write the formula for a polynomial function of least degree.

270.

271.

272.
For the following exercises, use the graph to identify zeros and multiplicity.
For the following exercises, use the given information about the polynomial graph to write the equation.

277. Degree 3. Zeros at \( x = -2, \) \( x = 1, \) and \( x = 3. \) \( y\)-intercept at \((0, -4)\).

278. Degree 3. Zeros at \( x = -5, \) \( x = -2, \) and \( x = 1. \) \( y\)-intercept at \((0, 6)\).

279. Degree 3. Zeros at \( x = -2, \) \( x = 1, \) and \( x = 3. \) \( y\)-intercept at \((0, -4)\).

280. Degree 3. Zeros at \( x = -5, \) \( x = -2, \) and \( x = 1. \) \( y\)-intercept at \((0, 6)\).

281.
282. Degree 5. Roots of multiplicity 2 at \( x = 3 \) and \( x = 1 \), and a root of multiplicity 1 at \( x = -3 \). \( y \)-intercept at \((0, 9)\).

283. Degree 4. Root of multiplicity 2 at \( x = 4 \), and a root of multiplicity 1 at \( x = 1 \) and \( x = -2 \). \( y \)-intercept at \((0, -3)\).

284. Degree 5. Double zero at \( x = 1 \), and triple zero at \( x = 3 \). Passes through the point \((2, 15)\).

285. Degree 3. Zeros at \( x = 4 \), \( x = 3 \), and \( x = 2 \). \( y \)-intercept at \((0, -24)\).

286. Degree 3. Zeros at \( x = -3 \), \( x = -2 \) and \( x = 1 \). \( y \)-intercept at \((0, 12)\).

287. Degree 4. Roots of multiplicity 2 at \( x = -3 \) and \( x = 2 \) and a root of multiplicity 1 at \( x = -2 \). \( y \)-intercept at \((0, -4)\).

288. Double zero at \( x = -3 \) and triple zero at \( x = 0 \). Passes through the point \((1, 32)\).

**Technology**

For the following exercises, use a calculator to approximate local minima and maxima or the global minimum and maximum.

289. \( f(x) = x^3 - x - 1 \)

290. \( f(x) = 2x^3 - 3x - 1 \)

291. \( f(x) = x^4 + x \)

292. \( f(x) = -x^4 + 3x - 2 \)

293. \( f(x) = x^4 - x^3 + 1 \)

**Extensions**

For the following exercises, use the graphs to write a polynomial function of least degree.

294. 

295. 

[Graph Image]
Real-World Applications

For the following exercises, write the polynomial function that models the given situation.

297. A rectangle has a length of 10 units and a width of 8 units. Squares of \( x \) by \( x \) units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a polynomial function in terms of \( x \).

298. Consider the same rectangle of the preceding problem. Squares of \( 2x \) by \( 2x \) units are cut out of each corner. Express the volume of the box as a polynomial in terms of \( x \).

299. A square has sides of 12 units. Squares \( x + 1 \) by \( x + 1 \) units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a function in terms of \( x \).

300. A cylinder has a radius of \( x + 2 \) units and a height of 3 units greater. Express the volume of the cylinder as a polynomial function.
301. A right circular cone has a radius of $3x + 6$ and a height 3 units less. Express the volume of the cone as a polynomial function. The volume of a cone is $V = \frac{1}{3}\pi r^2h$ for radius $r$ and height $h$. 
3.5 | Dividing Polynomials

Learning Objectives

In this section, you will:

3.5.1 Use long division to divide polynomials.
3.5.2 Use synthetic division to divide polynomials.

Figure 3.35  Lincoln Memorial, Washington, D.C. (credit: Ron Cogswell, Flickr)

The exterior of the Lincoln Memorial in Washington, D.C., is a large rectangular solid with length 61.5 meters (m), width 40 m, and height 30 m. We can easily find the volume using elementary geometry.

\[ V = l \cdot w \cdot h \]
\[ = 61.5 \cdot 40 \cdot 30 \]
\[ = 73,800 \]  \hspace{1cm} (3.36)

So the volume is 73,800 cubic meters (m³). Suppose we knew the volume, length, and width. We could divide to find the height.

\[ h = \frac{V}{l \cdot w} \]
\[ = \frac{73,800}{61.5 \cdot 40} \]
\[ = 30 \]  \hspace{1cm} (3.37)

As we can confirm from the dimensions above, the height is 30 m. We can use similar methods to find any of the missing dimensions. We can also use the same method if any or all of the measurements contain variable expressions. For example, suppose the volume of a rectangular solid is given by the polynomial \(3x^4 - 3x^3 - 33x^2 + 54x\). The length of the solid is given by \(3x\); the width is given by \(x - 2\). To find the height of the solid, we can use polynomial division, which is the focus of this section.

Using Long Division to Divide Polynomials

We are familiar with the long division algorithm for ordinary arithmetic. We begin by dividing into the digits of the dividend that have the greatest place value. We divide, multiply, subtract, include the digit in the next place value position, and repeat. For example, let’s divide 178 by 3 using long division.

---

Another way to look at the solution is as a sum of parts. This should look familiar, since it is the same method used to check division in elementary arithmetic.

\[
\text{dividend} = (\text{divisor} \cdot \text{quotient}) + \text{remainder}
\]

\[
178 = (3 \cdot 59) + 1 = 177 + 1 = 178
\]

We call this the Division Algorithm and will discuss it more formally after looking at an example.

Division of polynomials that contain more than one term has similarities to long division of whole numbers. We can write a polynomial dividend as the product of the divisor and the quotient added to the remainder. The terms of the polynomial division correspond to the digits (and place values) of the whole number division. This method allows us to divide two polynomials. For example, if we were to divide \(2x^3 - 3x^2 + 4x + 5\) by \(x + 2\) using the long division algorithm, it would look like this:

\[
x + 2 | 2x^3 - 3x^2 + 4x + 5
\]

Set up the division problem.

\[
x + 2 \quad 2x^2
\]

\[
x + 2 \quad 2x^2 - 7x
\]

2x³ divided by \(x\) is \(2x^2\).

\[
x + 2 \quad 2x^2 - 7x + 18
\]

\[
-7x^2 + 4x
\]

-7x² divided by \(x\) is -7x.

\[
x + 2 \quad 2x^2 - 7x + 18
\]

\[
-7x^2 + 4x
\]

\[
-7x^2 + 14x
\]

\[
18x + 5
\]

18x divided by \(x\) is 18.

\[
x + 2 \quad 2x^2 - 7x + 18
\]

\[
-7x^2 + 14x
\]

\[
-7x^2 - 14x
\]

\[
18x + 5
\]

\[
-18x + 36
\]

Multiply \(x + 2\) by -7x.

\[
-31
\]

Multiply \(x + 2\) by 18.

We have found

\[
\frac{2x^3 - 3x^2 + 4x + 5}{x + 2} = 2x^2 - 7x + 18 - \frac{31}{x + 2}
\]

or

\[
2x^3 - 3x^2 + 4x + 5 = (x + 2)(2x^2 - 7x + 18) - 31
\]
Writing the result in this manner illustrates the Division Algorithm.

### The Division Algorithm

The **Division Algorithm** states that, given a polynomial dividend \( f(x) \) and a non-zero polynomial divisor \( d(x) \) where the degree of \( d(x) \) is less than or equal to the degree of \( f(x) \), there exist unique polynomials \( q(x) \) and \( r(x) \) such that

\[
f(x) = d(x)q(x) + r(x) \quad (3.42)
\]

\( q(x) \) is the quotient and \( r(x) \) is the remainder. The remainder is either equal to zero or has degree strictly less than \( d(x) \).

If \( r(x) = 0 \), then \( d(x) \) divides evenly into \( f(x) \). This means that, in this case, both \( d(x) \) and \( q(x) \) are factors of \( f(x) \).

---

**How To**: **Given a polynomial and a binomial, use long division to divide the polynomial by the binomial.**

1. Set up the division problem.
2. Determine the first term of the quotient by dividing the leading term of the dividend by the leading term of the divisor.
3. Multiply the answer by the divisor and write it below the like terms of the dividend.
4. Subtract the bottom binomial from the top binomial.
5. Bring down the next term of the dividend.
6. Repeat steps 2–5 until reaching the last term of the dividend.
7. If the remainder is non-zero, express as a fraction using the divisor as the denominator.

---

**Example 3.44**

**Using Long Division to Divide a Second-Degree Polynomial**

Divide \( 5x^2 + 3x - 2 \) by \( x + 1 \).

**Solution**

**Analysis**

This division problem had a remainder of 0. This tells us that the dividend is divided evenly by the divisor, and that the divisor is a factor of the dividend.

---

**Example 3.45**

**Using Long Division to Divide a Third-Degree Polynomial**

Divide \( 6x^3 + 11x^2 - 31x + 15 \) by \( 3x - 2 \).
Solution

Analysis
We can check our work by using the Division Algorithm to rewrite the solution. Then multiply.

$$(3x - 2)(2x^2 + 5x - 7) + 1 = 6x^3 + 11x^2 - 31x + 15$$

Notice, as we write our result,
- the dividend is $6x^3 + 11x^2 - 31x + 15$
- the divisor is $3x - 2$
- the quotient is $2x^2 + 5x - 7$
- the remainder is 1

3.28 Divide $16x^3 - 12x^2 + 20x - 3$ by $4x + 5$.

Using Synthetic Division to Divide Polynomials

As we’ve seen, long division of polynomials can involve many steps and be quite cumbersome. Synthetic division is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1.

To illustrate the process, recall the example at the beginning of the section.

Divide $2x^3 - 3x^2 + 4x + 5$ by $x + 2$ using the long division algorithm.

The final form of the process looked like this:

$$(3.43)$$

$$x + 2 \overline{2x^3 - 3x^2 + 4x + 5}$$
$$2x^2 \quad + \quad x \quad 18$$
$$\underline{- (2x^3 + 4x^2)}$$
$$- 7x^2 + 4x$$
$$\underline{- ( - 7x^2 - 14x)}$$
$$18x + 5$$
$$\underline{- (18x + 36)}$$
$$-31$$

There is a lot of repetition in the table. If we don’t write the variables but, instead, line up their coefficients in columns under the division sign and also eliminate the partial products, we already have a simpler version of the entire problem.

$$2 \overline{2 \quad -3 \quad 4 \quad 5}$$
$$-2 \quad -4$$
$$\underline{-7 \quad 14 \quad -36}$$
$$18 \quad -36$$
$$\underline{-31}$$

Synthetic division carries this simplification even a few more steps. Collapse the table by moving each of the rows up to fill any vacant spots. Also, instead of dividing by 2, as we would in division of whole numbers, then multiplying and subtracting the middle product, we change the sign of the “divisor” to −2, multiply and add. The process starts by bringing down the leading coefficient.

$$-2 \overline{2 \quad -3 \quad 4 \quad 5}$$
$$\underline{-4 \quad 14 \quad -36}$$
$$2 \quad -7 \quad 18 \quad -31$$
We then multiply it by the “divisor” and add, repeating this process column by column, until there are no entries left. The bottom row represents the coefficients of the quotient; the last entry of the bottom row is the remainder. In this case, the quotient is \(2x^2 - 7x + 18\) and the remainder is \(-31\). The process will be made more clear in Example 3.46.

**Synthetic Division**

Synthetic division is a shortcut that can be used when the divisor is a binomial in the form \(x - k\). In *synthetic division*, only the coefficients are used in the division process.

**How To:**

1. Write \(k\) for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by \(k\). Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by \(k\). Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0, the next number from the right has degree 1, the next number from the right has degree 2, and so on.

**Example 3.46**

**Using Synthetic Division to Divide a Second-Degree Polynomial**

Use synthetic division to divide \(5x^2 - 3x - 36\) by \(x - 3\).

**Solution**

**Analysis**

Just as with long division, we can check our work by multiplying the quotient by the divisor and adding the remainder.

\[(x - 3)(5x + 12) + 0 = 5x^2 - 3x - 36\]

**Example 3.47**

**Using Synthetic Division to Divide a Third-Degree Polynomial**

Use synthetic division to divide \(4x^3 + 10x^2 - 6x - 20\) by \(x + 2\).

**Solution**

**Analysis**

The graph of the polynomial function \(f(x) = 4x^3 + 10x^2 - 6x - 20\) in Figure 3.35 shows a zero at \(x = k = -2\). This confirms that \(x + 2\) is a factor of \(4x^3 + 10x^2 - 6x - 20\).
Example 3.48

Using Synthetic Division to Divide a Fourth-Degree Polynomial

Use synthetic division to divide \(-9x^4 + 10x^3 + 7x^2 - 6\) by \(x - 1\).

Solution

Solution

Try it 3.29 Use synthetic division to divide \(3x^4 + 18x^3 - 3x + 40\) by \(x + 7\).

Using Polynomial Division to Solve Application Problems

Polynomial division can be used to solve a variety of application problems involving expressions for area and volume. We looked at an application at the beginning of this section. Now we will solve that problem in the following example.

Example 3.49
Using Polynomial Division in an Application Problem

The volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$ and the width is given by $x - 2$. Find the height of the solid.

Solution

3.30 The area of a rectangle is given by $3x^3 + 14x^2 - 23x + 6$. The width of the rectangle is given by $x + 6$. Find an expression for the length of the rectangle.

Access these online resources for additional instruction and practice with polynomial division.

- Dividing a Trinomial by a Binomial Using Long Division (http://openstaxcollege.org/l/dividetribild)
- Dividing a Polynomial by a Binomial Using Long Division (http://openstaxcollege.org/l/dividepolybild)
- Ex 2: Dividing a Polynomial by a Binomial Using Synthetic Division (http://openstaxcollege.org/l/dividepolybisd2)
- Ex 4: Dividing a Polynomial by a Binomial Using Synthetic Division (http://openstaxcollege.org/l/dividepolybisd4)
3.5 EXERCISES

Verbal

302. If division of a polynomial by a binomial results in a remainder of zero, what can be concluded?

303. If a polynomial of degree \( n \) is divided by a binomial of degree 1, what is the degree of the quotient?

Algebraic

For the following exercises, use long division to divide. Specify the quotient and the remainder.

304. \( (x^2 + 5x - 1) \div (x - 1) \)

305. \( (2x^2 - 9x - 5) \div (x - 5) \)

306. \( (3x^2 + 23x + 14) \div (x + 7) \)

307. \( (4x^2 - 10x + 6) \div (4x + 2) \)

308. \( (6x^2 - 25x - 25) \div (6x + 5) \)

309. \( (-x^2 - 1) \div (x + 1) \)

310. \( (2x^2 - 3x + 2) \div (x + 2) \)

311. \( (x^3 - 126) \div (x - 5) \)

312. \( (3x^2 - 5x + 4) \div (3x + 1) \)

313. \( (x^3 - 3x^2 + 5x - 6) \div (x - 2) \)

314. \( (2x^3 + 3x^2 - 4x + 15) \div (x + 3) \)

For the following exercises, use synthetic division to find the quotient.

315. \( (3x^3 - 2x^2 + x - 4) \div (x + 3) \)

316. \( (2x^3 - 6x^2 - 7x + 6) \div (x - 4) \)

317. \( (6x^3 - 10x^2 - 7x - 15) \div (x + 1) \)

318. \( (4x^3 - 12x^2 - 5x - 1) \div (2x + 1) \)

319. \( (9x^3 - 9x^2 + 18x + 5) \div (3x - 1) \)

320. \( (3x^3 - 2x^2 + x - 4) \div (x + 3) \)

321. \( (-6x^3 + x^2 - 4) \div (2x - 3) \)

322. \( (2x^3 + 7x^2 - 13x - 3) \div (2x - 3) \)
323. \( (3x^3 - 5x^2 + 2x + 3) \div (x + 2) \)
324. \( (4x^3 - 5x^2 + 13) \div (x + 4) \)
325. \( (x^3 - 3x + 2) \div (x + 2) \)
326. \( (x^3 - 21x^2 + 147x - 343) \div (x - 7) \)
327. \( (x^3 - 15x^2 + 75x - 125) \div (x - 5) \)
328. \( (9x^3 - x + 2) \div (3x - 1) \)
329. \( (6x^3 - x^2 + 5x + 2) \div (3x + 1) \)
330. \( (x^4 + x^3 - 3x^2 - 2x + 1) \div (x + 1) \)
331. \( (x^4 - 3x^2 + 1) \div (x - 1) \)
332. \( (x^4 + 2x^3 - 3x^2 + 2x + 6) \div (x + 3) \)
333. \( (x^4 - 10x^3 + 37x^2 - 60x + 36) \div (x - 2) \)
334. \( (x^4 - 8x^3 + 24x^2 - 32x + 16) \div (x - 2) \)
335. \( (x^4 + 5x^3 - 3x^2 - 13x + 10) \div (x + 5) \)
336. \( (x^4 - 12x^3 + 54x^2 - 108x + 81) \div (x - 3) \)
337. \( (4x^4 - 2x^3 - 4x + 2) \div (2x - 1) \)
338. \( (4x^4 + 2x^3 - 4x^2 + 2x + 2) \div (2x + 1) \)

For the following exercises, use synthetic division to determine whether the first expression is a factor of the second. If it is, indicate the factorization.

339. \( x - 2, \ 4x^3 - 3x^2 - 8x + 4 \)
340. \( x - 2, \ 3x^4 - 6x^3 - 5x + 10 \)
341. \( x + 3, \ -4x^3 + 5x^2 + 8 \)
342. \( x - 2, \ 4x^4 - 15x^2 - 4 \)
343. \( x - \frac{1}{2}, \ 2x^4 - x^3 + 2x - 1 \)
344. \( x + \frac{1}{3}, \ 3x^4 + x^3 - 3x + 1 \)
Graphical

For the following exercises, use the graph of the third-degree polynomial and one factor to write the factored form of the polynomial suggested by the graph. The leading coefficient is one.

345. Factor is $x^2 - x + 3$

346. Factor is $(x^2 + 2x + 4)$

347. Factor is $x^2 + 2x + 5$

348. Factor is $x^2 + x + 1$
349. Factor is $x^2 + 2x + 2$

For the following exercises, use synthetic division to find the quotient and remainder.

350. \[ \frac{4x^3 - 33}{x - 2} \]
351. \[ \frac{2x^3 + 25}{x + 3} \]
352. \[ \frac{3x^3 + 2x - 5}{x - 1} \]
353. \[ \frac{-4x^3 - x^2 - 12}{x + 4} \]
354. \[ \frac{x^4 - 2x}{x + 2} \]

**Technology**

For the following exercises, use a calculator with CAS to answer the questions.

355. Consider \[ \frac{x^k - 1}{x - 1} \] with $k = 1, 2, 3$. What do you expect the result to be if $k = 4$?
Consider \( \frac{x^k + 1}{x + 1} \) for \( k = 1, \ 3, \ 5 \). What do you expect the result to be if \( k = 7 \)?

357. Consider \( \frac{x^k - k^4}{x - k} \) for \( k = 1, \ 2, \ 3 \). What do you expect the result to be if \( k = 4 \)?

358. Consider \( \frac{x^k}{x + 1} \) with \( k = 1, \ 2, \ 3 \). What do you expect the result to be if \( k = 4 \)?

359. Consider \( \frac{x^k}{x - 1} \) with \( k = 1, \ 2, \ 3 \). What do you expect the result to be if \( k = 4 \)?

### Extensions

For the following exercises, use synthetic division to determine the quotient involving a complex number.

360. \( \frac{x + 1}{x - i} \)

361. \( \frac{x^2 + 1}{x - i} \)

362. \( \frac{x + 1}{x + i} \)

363. \( \frac{x^2 + 1}{x + i} \)

364. \( \frac{x^3 + 1}{x - i} \)

### Real-World Applications

For the following exercises, use the given length and area of a rectangle to express the width algebraically.

365. Length is \( x + 5 \), area is \( 2x^2 + 9x - 5 \).

366. Length is \( 2x + 5 \), area is \( 4x^3 + 10x^2 + 6x + 15 \).

367. Length is \( 3x - 4 \), area is \( 6x^4 - 8x^3 + 9x^2 - 9x - 4 \).

For the following exercises, use the given volume of a box and its length and width to express the height of the box algebraically.

368. Volume is \( 12x^3 + 20x^2 - 21x - 36 \), length is \( 2x + 3 \), width is \( 3x - 4 \).

369. Volume is \( 18x^3 - 21x^2 - 40x + 48 \), length is \( 3x - 4 \), width is \( 3x - 4 \).

370. Volume is \( 10x^3 + 27x^2 + 2x - 24 \), length is \( 5x - 4 \), width is \( 2x + 3 \).

371. Volume is \( 10x^3 + 30x^2 - 8x - 24 \), length is \( 2 \), width is \( x + 3 \).

For the following exercises, use the given volume and radius of a cylinder to express the height of the cylinder algebraically.

372. Volume is \( \pi(25x^3 - 65x^2 - 29x - 3) \), radius is \( 5x + 1 \).

373. Volume is \( \pi(4x^3 + 12x^2 - 15x - 50) \), radius is \( 2x + 5 \).

374. Volume is \( \pi(3x^4 + 24x^3 + 46x^2 - 16x - 32) \), radius is \( x + 4 \).
A new bakery offers decorated sheet cakes for children’s birthday parties and other special occasions. The bakery wants the volume of a small cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

This problem can be solved by writing a cubic function and solving a cubic equation for the volume of the cake. In this section, we will discuss a variety of tools for writing polynomial functions and solving polynomial equations.

### Evaluating a Polynomial Using the Remainder Theorem

In the last section, we learned how to divide polynomials. We can now use polynomial division to evaluate polynomials using the **Remainder Theorem**. If the polynomial is divided by \( x - k \), the remainder may be found quickly by evaluating the polynomial function at \( k \), that is, \( f(k) \) Let’s walk through the proof of the theorem.

Recall that the Division Algorithm states that, given a polynomial dividend \( f(x) \) and a non-zero polynomial divisor \( d(x) \) where the degree of \( d(x) \) is less than or equal to the degree of \( f(x) \), there exist unique polynomials \( q(x) \) and \( r(x) \) such that

\[
f(x) = d(x)q(x) + r(x).
\]  

(3.44)

If the divisor, \( d(x) \), is \( x - k \), this takes the form

\[
f(x) = (x - k)q(x) + r.
\]  

(3.45)

Since the divisor \( x - k \) is linear, the remainder will be a constant, \( r \). And, if we evaluate this for \( x = k \), we have

\[
f(k) = (k - k)q(k) + r
\]

\[
= 0 \cdot q(k) + r
\]

\[
= r
\]

(3.46)

In other words, \( f(k) \) is the remainder obtained by dividing \( f(x) \) by \( x - k \).

#### The Remainder Theorem

If a polynomial \( f(x) \) is divided by \( x - k \), then the remainder is the value \( f(k) \).

**How To:**

1. Use synthetic division to divide the polynomial by \( x - k \).
2. The remainder is the value \( f(k) \).

#### Example 3.50
Using the Remainder Theorem to Evaluate a Polynomial

Use the Remainder Theorem to evaluate $f(x) = 6x^4 - x^3 - 15x^2 + 2x - 7$ at $x = 2$.

Solution

Analysis

We can check our answer by evaluating $f(2)$.

$$f(x) = 6x^4 - x^3 - 15x^2 + 2x - 7$$

$$f(2) = 6(2)^4 - (2)^3 - 15(2)^2 + 2(2) - 7$$

$$= 25$$

Try It 3.31 Use the Remainder Theorem to evaluate $f(x) = 2x^5 - 3x^4 - 9x^3 + 8x^2 + 2$ at $x = -3$.

Using the Factor Theorem to Solve a Polynomial Equation

The Factor Theorem is another theorem that helps us analyze polynomial equations. It tells us how the zeros of a polynomial are related to the factors. Recall that the Division Algorithm tells us

$$f(x) = (x - k)q(x) + r$$

If $k$ is a zero, then the remainder $r$ is $f(k) = 0$ and $f(x) = (x - k)q(x) + 0$ or $f(x) = (x - k)q(x)$.

Notice, written in this form, $x - k$ is a factor of $f(x)$. We can conclude if $k$ is a zero of $f(x)$, then $x - k$ is a factor of $f(x)$.

Similarly, if $x - k$ is a factor of $f(x)$, then the remainder of the Division Algorithm $f(x) = (x - k)q(x) + r$ is 0. This tells us that $k$ is a zero.

This pair of implications is the Factor Theorem. As we will soon see, a polynomial of degree $n$ in the complex number system will have $n$ zeros. We can use the Factor Theorem to completely factor a polynomial into the product of $n$ factors. Once the polynomial has been completely factored, we can easily determine the zeros of the polynomial.

The Factor Theorem

According to the Factor Theorem, $k$ is a zero of $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$.

How To

Given a factor and a third-degree polynomial, use the Factor Theorem to factor the polynomial.

1. Use synthetic division to divide the polynomial by $(x - k)$.
2. Confirm that the remainder is 0.
3. Write the polynomial as the product of $(x - k)$ and the quadratic quotient.
4. If possible, factor the quadratic.
5. Write the polynomial as the product of factors.

Example 3.51
Using the Factor Theorem to Solve a Polynomial Equation

Show that \((x + 2)\) is a factor of \(x^3 - 6x^2 - x + 30\). Find the remaining factors. Use the factors to determine the zeros of the polynomial.

**Solution**

Use the Factor Theorem to find the zeros of \(f(x) = x^3 + 4x^2 - 4x - 16\) given that \((x - 2)\) is a factor of the polynomial.

Using the Rational Zero Theorem to Find Rational Zeros

Another use for the Remainder Theorem is to test whether a rational number is a zero for a given polynomial. But first we need a pool of rational numbers to test. The **Rational Zero Theorem** helps us to narrow down the number of possible rational zeros using the ratio of the factors of the constant term and factors of the leading coefficient of the polynomial.

Consider a quadratic function with two zeros, \(x = \frac{2}{5}\) and \(x = \frac{3}{4}\). By the Factor Theorem, these zeros have factors associated with them. Let us set each factor equal to 0, and then construct the original quadratic function absent its stretching factor.

\[
\begin{align*}
x - \frac{2}{5} &= 0 \quad \text{or} \quad x - \frac{3}{4} = 0 \\
5x - 2 &= 0 \quad \text{or} \quad 4x - 3 = 0 \\
f(x) &= (5x - 2)(4x - 3) \\
f(x) &= 20x^2 - 23x + 6 \\
f(x) &= (5 \cdot 2)x^2 - 23x + (2 \cdot 3)
\end{align*}
\]

Notice that two of the factors of the constant term, 6, are the two numerators from the original rational roots: 2 and 3. Similarly, two of the factors from the leading coefficient, 20, are the two denominators from the original rational roots: 5 and 4.

We can infer that the numerators of the rational roots will always be factors of the constant term and the denominators will be factors of the leading coefficient. This is the essence of the Rational Zero Theorem; it is a means to give us a pool of possible rational zeros.

**The Rational Zero Theorem**

The **Rational Zero Theorem** states that, if the polynomial \(f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0\) has integer coefficients, then every rational zero of \(f(x)\) has the form \(\frac{p}{q}\) where \(p\) is a factor of the constant term \(a_0\) and \(q\) is a factor of the leading coefficient \(a_n\).

When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.
Given a polynomial function \( f(x) \), use the Rational Zero Theorem to find rational zeros.

1. Determine all factors of the constant term and all factors of the leading coefficient.
2. Determine all possible values of \( \frac{p}{q} \), where \( p \) is a factor of the constant term and \( q \) is a factor of the leading coefficient. Be sure to include both positive and negative candidates.
3. Determine which possible zeros are actual zeros by evaluating each case of \( f\left(\frac{p}{q}\right) \).

Example 3.52

Listing All Possible Rational Zeros

List all possible rational zeros of \( f(x) = 2x^4 - 5x^3 + x^2 - 4 \).

Solution

Example 3.53

Using the Rational Zero Theorem to Find Rational Zeros

Use the Rational Zero Theorem to find the rational zeros of \( f(x) = 2x^3 + x^2 - 4x + 1 \).

Solution

Try it

Use the Rational Zero Theorem to find the rational zeros of \( f(x) = x^3 - 5x^2 + 2x + 1 \).

Finding the Zeros of Polynomial Functions

The Rational Zero Theorem helps us to narrow down the list of possible rational zeros for a polynomial function. Once we have done this, we can use synthetic division repeatedly to determine all of the zeros of a polynomial function.

Given a polynomial function \( f \), use synthetic division to find its zeros.

1. Use the Rational Zero Theorem to list all possible rational zeros of the function.
2. Use synthetic division to evaluate a given possible zero by synthetically dividing the candidate into the polynomial. If the remainder is 0, the candidate is a zero. If the remainder is not zero, discard the candidate.
3. Repeat step two using the quotient found with synthetic division. If possible, continue until the quotient is a quadratic.
4. Find the zeros of the quadratic function. Two possible methods for solving quadratics are factoring and using the quadratic formula.

Example 3.54
Finding the Zeros of a Polynomial Function with Repeated Real Zeros

Find the zeros of \( f(x) = 4x^3 - 3x - 1 \).

**Solution**

**Analysis**

Look at the graph of the function \( f \) in Figure 3.35. Notice, at \( x = -0.5 \), the graph bounces off the \( x \)-axis, indicating the even multiplicity \((2,4,6\ldots)\) for the zero \(-0.5\). At \( x = 1 \), the graph crosses the \( x \)-axis, indicating the odd multiplicity \((1,3,5\ldots)\) for the zero \( x = 1 \).

Using the Fundamental Theorem of Algebra

Now that we can find rational zeros for a polynomial function, we will look at a theorem that discusses the number of complex zeros of a polynomial function. The **Fundamental Theorem of Algebra** tells us that every polynomial function has at least one complex zero. This theorem forms the foundation for solving polynomial equations.

Suppose \( f \) is a polynomial function of degree four, and \( f(x) = 0 \). The Fundamental Theorem of Algebra states that there is at least one complex solution, call it \( c_1 \). By the Factor Theorem, we can write \( f(x) \) as a product of \( x - c_1 \) and a polynomial quotient. Since \( x - c_1 \) is linear, the polynomial quotient will be of degree three. Now we apply the Fundamental Theorem of Algebra to the third-degree polynomial quotient. It will have at least one complex zero, call it \( c_2 \).

So we can write the polynomial quotient as a product of \( x - c_2 \) and a new polynomial quotient of degree two. Continue to apply the Fundamental Theorem of Algebra until all of the zeros are found. There will be four of them and each one will yield a factor of \( f(x) \).

The Fundamental Theorem of Algebra states that, if \( f(x) \) is a polynomial of degree \( n > 0 \), then \( f(x) \) has at least one complex zero.

We can use this theorem to argue that, if \( f(x) \) is a polynomial of degree \( n > 0 \), and \( a \) is a non-zero real number, then \( f(x) \) has exactly \( n \) linear factors

\[
f(x) = a(x - c_1)(x - c_2)\ldots(x - c_n)
\]

where \( c_1, c_2, \ldots, c_n \) are complex numbers. Therefore, \( f(x) \) has \( n \) roots if we allow for multiplicities.

**Q&A**

**Does every polynomial have at least one imaginary zero?**

No. A complex number is not necessarily imaginary. Real numbers are also complex numbers.
Example 3.55

Finding the Zeros of a Polynomial Function with Complex Zeros

Find the zeros of \( f(x) = 3x^3 + 9x^2 + x + 3 \).

Solution

Analysis

Look at the graph of the function \( f \) in Figure 3.35. Notice that, at \( x = -3 \), the graph crosses the \( x \)-axis, indicating an odd multiplicity (1) for the zero \( x = -3 \). Also note the presence of the two turning points. This means that, since there is a 3rd degree polynomial, we are looking at the maximum number of turning points. Thus, all the \( x \)-intercepts for the function are shown. So either the multiplicity of \( x = -3 \) is 1 and there are two complex solutions, which is what we found, or the multiplicity at \( x = -3 \) is three. Either way, our result is correct.

Find the zeros of \( f(x) = 2x^3 + 5x^2 - 11x + 4 \).

Using the Linear Factorization Theorem to Find Polynomials with Given Zeros

A vital implication of the Fundamental Theorem of Algebra, as we stated above, is that a polynomial function of degree \( n \) will have \( n \) zeros in the set of complex numbers, if we allow for multiplicities. This means that we can factor the polynomial function into \( n \) factors. The Linear Factorization Theorem tells us that a polynomial function will have the same number of factors as its degree, and that each factor will be in the form \( (x - c) \), where \( c \) is a complex number.

Let \( f \) be a polynomial function with real coefficients, and suppose \( a + bi, b \neq 0 \), is a zero of \( f(x) \). Then, by the Factor Theorem, \( x - (a + bi) \) is a factor of \( f(x) \). For \( f \) to have real coefficients, \( x - (a - bi) \) must also be a factor of \( f(x) \). This is true because any factor other than \( x - (a - bi) \), when multiplied by \( x - (a + bi) \), will leave imaginary components in the product. Only multiplication with conjugate pairs will eliminate the imaginary parts and result in real coefficients. In other words, if a polynomial function \( f \) with real coefficients has a complex zero \( a + bi \), then the complex conjugate \( a - bi \) must also be a zero of \( f(x) \). This is called the Complex Conjugate Theorem.
Complex Conjugate Theorem

According to the **Linear Factorization Theorem**, a polynomial function will have the same number of factors as its degree, and each factor will be in the form \((x - c)\), where \(c\) is a complex number.

If the polynomial function \(f\) has real coefficients and a complex zero in the form \(a + bi\), then the complex conjugate of the zero, \(a - bi\), is also a zero.

**How To:**

Given the zeros of a polynomial function \(f\) and a point \((c, f(c))\) on the graph of \(f\), use the Linear Factorization Theorem to find the polynomial function.

1. Use the zeros to construct the linear factors of the polynomial.
2. Multiply the linear factors to expand the polynomial.
3. Substitute \((c, f(c))\) into the function to determine the leading coefficient.
4. Simplify.

**Example 3.56**

**Using the Linear Factorization Theorem to Find a Polynomial with Given Zeros**

Find a fourth degree polynomial with real coefficients that has zeros of \(-3, 2, i\), such that \(f(-2) = 100\).

**Solution**

**Analysis**

We found that both \(i\) and \(-i\) were zeros, but only one of these zeros needed to be given. If \(i\) is a zero of a polynomial with real coefficients, then \(-i\) must also be a zero of the polynomial because \(-i\) is the complex conjugate of \(i\).

**Q&A**

If \(2 + 3i\) were given as a zero of a polynomial with real coefficients, would \(2 - 3i\) also need to be a zero?

Yes. When any complex number with an imaginary component is given as a zero of a polynomial with real coefficients, the conjugate must also be a zero of the polynomial.

**Try It**

Find a third degree polynomial with real coefficients that has zeros of \(5\) and \(-2i\) such that \(f(1) = 10\).

**Using Descartes' Rule of Signs**

There is a straightforward way to determine the possible numbers of positive and negative real zeros for any polynomial function. If the polynomial is written in descending order, **Descartes' Rule of Signs** tells us of a relationship between the number of sign changes in \(f(x)\) and the number of positive real zeros. For example, the polynomial function below has one sign change.

\[
f(x) = x^4 + x^3 + x^2 + x - 1\]

This tells us that the function must have 1 positive real zero.

There is a similar relationship between the number of sign changes in \(f(-x)\) and the number of negative real zeros.
In this case, \( f(-x) \) has 3 sign changes. This tells us that \( f(x) \) could have 3 or 1 negative real zeros.

**Descartes’ Rule of Signs**

According to **Descartes’ Rule of Signs**, if we let \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \) be a polynomial function with real coefficients:

- The number of positive real zeros is either equal to the number of sign changes of \( f(x) \) or is less than the number of sign changes by an even integer.
- The number of negative real zeros is either equal to the number of sign changes of \( f(-x) \) or is less than the number of sign changes by an even integer.

**Example 3.57**

**Using Descartes’ Rule of Signs**

Use Descartes’ Rule of Signs to determine the possible numbers of positive and negative real zeros for \( f(x) = -x^4 - 3x^3 + 6x^2 - 4x - 12 \).

**Solution**

**Analysis**

We can confirm the numbers of positive and negative real roots by examining a graph of the function. See **Figure 3.35**. We can see from the graph that the function has 0 positive real roots and 2 negative real roots.

**Try It**

3.36 Use Descartes’ Rule of Signs to determine the maximum possible numbers of positive and negative real zeros for \( f(x) = 2x^4 - 10x^3 + 11x^2 - 15x + 12 \). Use a graph to verify the numbers of positive and negative real zeros for the function.
Solving Real-World Applications

We have now introduced a variety of tools for solving polynomial equations. Let’s use these tools to solve the bakery problem from the beginning of the section.

Example 3.58

Solving Polynomial Equations

A new bakery offers decorated sheet cakes for children’s birthday parties and other special occasions. The bakery wants the volume of a small cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

Solution

A shipping container in the shape of a rectangular solid must have a volume of 84 cubic meters. The client tells the manufacturer that, because of the contents, the length of the container must be one meter longer than the width, and the height must be one meter greater than twice the width. What should the dimensions of the container be?

3.37  A shipping container in the shape of a rectangular solid must have a volume of 84 cubic meters. The client tells the manufacturer that, because of the contents, the length of the container must be one meter longer than the width, and the height must be one meter greater than twice the width. What should the dimensions of the container be?

Access these online resources for additional instruction and practice with zeros of polynomial functions.

• Real Zeros, Factors, and Graphs of Polynomial Functions (http://openstaxcollege.org/l/realzeros)
• Complex Factorization Theorem (http://openstaxcollege.org/l/factortheorem)
• Find the Zeros of a Polynomial Function (http://openstaxcollege.org/l/findthezeros)
• Find the Zeros of a Polynomial Function 2 (http://openstaxcollege.org/l/findthezeros2)
• Find the Zeros of a Polynomial Function 3 (http://openstaxcollege.org/l/findthezeros3)
3.6 EXERCISES

Verbal

375. Describe a use for the Remainder Theorem.

376. Explain why the Rational Zero Theorem does not guarantee finding zeros of a polynomial function.

377. What is the difference between rational and real zeros?

378. If Descartes’ Rule of Signs reveals a no change of signs or one sign of changes, what specific conclusion can be drawn?

379. If synthetic division reveals a zero, why should we try that value again as a possible solution?

Algebraic

For the following exercises, use the Remainder Theorem to find the remainder.

380. \((x^4 - 9x^2 + 14) \div (x - 2)\)

381. \((3x^3 - 2x^2 + x - 4) \div (x + 3)\)

382. \((x^4 + 5x^3 - 4x - 17) \div (x + 1)\)

383. \((-3x^2 + 6x + 24) \div (x - 4)\)

384. \((5x^5 - 4x^4 + 3x^3 - 2x^2 + x - 1) \div (x + 6)\)

385. \((x^4 - 1) \div (x - 4)\)

386. \((3x^3 + 4x^2 - 8x + 2) \div (x - 3)\)

387. \((4x^3 + 5x^2 - 2x + 7) \div (x + 2)\)

For the following exercises, use the Factor Theorem to find all real zeros for the given polynomial function and one factor.

388. \(f(x) = 2x^3 - 9x^2 + 13x - 6; \ x - 1\)

389. \(f(x) = 2x^3 + x^2 - 5x + 2; \ x + 2\)

390. \(f(x) = 3x^3 + x^2 - 20x + 12; \ x + 3\)

391. \(f(x) = 2x^3 + 3x^2 + x + 6; \ x + 2\)

392. \(f(x) = -5x^3 + 16x^2 - 9; \ x - 3\)

393. \(x^3 + 3x^2 + 4x + 12; \ x + 3\)

394. \(4x^3 - 7x + 3; \ x - 1\)

395. \(2x^3 + 5x^2 - 12x - 30, \ 2x + 5\)

For the following exercises, use the Rational Zero Theorem to find all real zeros.

396. \(x^3 - 3x^2 - 10x + 24 = 0\)
397. \(2x^3 + 7x^2 - 10x - 24 = 0\)

398. \(x^3 + 2x^2 - 9x - 18 = 0\)

399. \(x^3 + 5x^2 - 16x - 80 = 0\)

400. \(x^3 - 3x^2 - 25x + 75 = 0\)

401. \(2x^3 - 3x^2 - 32x - 15 = 0\)

402. \(2x^3 + x^2 - 7x - 6 = 0\)

403. \(2x^3 - 3x^2 - x + 1 = 0\)

404. \(3x^3 - x^2 - 11x - 6 = 0\)

405. \(2x^3 - 5x^2 + 9x - 9 = 0\)

406. \(2x^3 - 3x^2 + 4x + 3 = 0\)

407. \(x^4 - 2x^3 - 7x^2 + 8x + 12 = 0\)

408. \(x^4 + 2x^3 - 9x^2 - 2x + 8 = 0\)

409. \(4x^4 + 4x^3 - 25x^2 - x + 6 = 0\)

410. \(2x^4 - 3x^3 - 15x^2 + 32x - 12 = 0\)

411. \(x^4 + 2x^3 - 4x^2 - 10x - 5 = 0\)

412. \(4x^3 - 3x + 1 = 0\)

413. \(8x^4 + 26x^3 + 39x^2 + 26x + 6\)

For the following exercises, find all complex solutions (real and non-real).

414. \(x^3 + x^2 + x + 1 = 0\)

415. \(x^3 - 8x^2 + 25x - 26 = 0\)

416. \(x^3 + 13x^2 + 57x + 85 = 0\)

417. \(3x^3 - 4x^2 + 11x + 10 = 0\)

418. \(x^4 + 2x^3 + 22x^2 + 50x - 75 = 0\)

419. \(2x^3 - 3x^2 + 32x + 17 = 0\)

**Graphical**

For the following exercises, use Descartes’ Rule to determine the possible number of positive and negative solutions. Confirm with the given graph.

420. \(f(x) = x^3 - 1\)

421. \(f(x) = x^4 - x^2 - 1\)
422. \( f(x) = x^3 - 2x^2 - 5x + 6 \)

423. \( f(x) = x^3 - 2x^2 + x - 1 \)

424. \( f(x) = x^4 + 2x^3 - 12x^2 + 14x - 5 \)

425. \( f(x) = 2x^3 + 37x^2 + 200x + 300 \)

426. \( f(x) = x^3 - 2x^2 - 16x + 32 \)

427. \( f(x) = 2x^4 - 5x^3 - 5x^2 + 5x + 3 \)

428. \( f(x) = 2x^4 - 5x^3 - 14x^2 + 20x + 8 \)

429. \( f(x) = 10x^4 - 21x^2 + 11 \)

**Numeric**

For the following exercises, list all possible rational zeros for the functions.

430. \( f(x) = x^4 + 3x^3 - 4x + 4 \)

431. \( f(x) = 2x^3 + 3x^2 - 8x + 5 \)

432. \( f(x) = 3x^3 + 5x^2 - 5x + 4 \)

433. \( f(x) = 6x^4 - 10x^2 + 13x + 1 \)

434. \( f(x) = 4x^5 - 10x^4 + 8x^3 + x^2 - 8 \)

**Technology**

For the following exercises, use your calculator to graph the polynomial function. Based on the graph, find the rational zeros. All real solutions are rational.

435. \( f(x) = 6x^3 - 7x^2 + 1 \)

436. \( f(x) = 4x^3 - 4x^2 - 13x - 5 \)

437. \( f(x) = 8x^3 - 6x^2 - 23x + 6 \)

438. \( f(x) = 12x^4 + 55x^3 + 12x^2 - 117x + 54 \)

439. \( f(x) = 16x^4 - 24x^3 + x^2 - 15x + 25 \)

**Extensions**

For the following exercises, construct a polynomial function of least degree possible using the given information.

440. Real roots: –1, 1, 3 and \((2, f(2)) = (2, 4)\)

441. Real roots: –1 (with multiplicity 2 and 1) and \((2, f(2)) = (2, 4)\)

442. Real roots: –2, \( \frac{1}{2} \) (with multiplicity 2) and \((-3, f(-3)) = (-3, 5)\)

443.
Real roots: \(-\frac{1}{2}, 0, \frac{1}{2}\) and \((-2, f(-2)) = (-2, 6)\)

444. Real roots: \(-4, -1, 1, 4\) and \((-2, f(-2)) = (-2, 10)\)

**Real-World Applications**

For the following exercises, find the dimensions of the box described.

445. The length is twice as long as the width. The height is 2 inches greater than the width. The volume is 192 cubic inches.

446. The length, width, and height are consecutive whole numbers. The volume is 120 cubic inches.

447. The length is one inch more than the width, which is one inch more than the height. The volume is 86.625 cubic inches.

448. The length is three times the height and the height is one inch less than the width. The volume is 108 cubic inches.

449. The length is 3 inches more than the width. The width is 2 inches more than the height. The volume is 120 cubic inches.

For the following exercises, find the dimensions of the right circular cylinder described.

450. The radius is 3 inches more than the height. The volume is \(16\pi\) cubic meters.

451. The height is one less than one half the radius. The volume is \(72\pi\) cubic meters.

452. The radius and height differ by one meter. The radius is larger and the volume is \(48\pi\) cubic meters.

453. The radius and height differ by two meters. The height is greater and the volume is \(28.125\pi\) cubic meters.

454. The radius is \(\frac{1}{3}\) meter greater than the height. The volume is \(\frac{98}{9}\pi\) cubic meters.
3.7 | Rational Functions

Learning Objectives

In this section, you will:

3.7.1 Use arrow notation.
3.7.2 Solve applied problems involving rational functions.
3.7.3 Find the domains of rational functions.
3.7.4 Identify vertical asymptotes.
3.7.5 Identify horizontal asymptotes.
3.7.6 Graph rational functions.

Suppose we know that the cost of making a product is dependent on the number of items, \( x \), produced. This is given by the equation \( C(x) = 15,000x - 0.1x^2 + 1000 \). If we want to know the average cost for producing \( x \) items, we would divide the cost function by the number of items, \( x \).

The average cost function, which yields the average cost per item for \( x \) items produced, is

\[
 f(x) = \frac{15,000x - 0.1x^2 + 1000}{x}.
\]

Many other application problems require finding an average value in a similar way, giving us variables in the denominator. Written without a variable in the denominator, this function will contain a negative integer power.

In the last few sections, we have worked with polynomial functions, which are functions with non-negative integers for exponents. In this section, we explore rational functions, which have variables in the denominator.

Using Arrow Notation

We have seen the graphs of the basic reciprocal function and the squared reciprocal function from our study of toolkit functions. Examine these graphs, as shown in Figure 3.36, and notice some of their features.

\[
 f(x) = \frac{1}{x}
\]

\[
 f(x) = \frac{1}{x^2}
\]

Figure 3.36

Several things are apparent if we examine the graph of \( f(x) = \frac{1}{x} \).

1. On the left branch of the graph, the curve approaches the x-axis (\( y = 0 \)) as \( x \to -\infty \).
2. As the graph approaches \( x = 0 \) from the left, the curve drops, but as we approach zero from the right, the curve rises.
3. Finally, on the right branch of the graph, the curves approaches the x-axis (\( y = 0 \)) as \( x \to \infty \).
To summarize, we use **arrow notation** to show that \( x \) or \( f(x) \) is approaching a particular value. See **Table 3.10**.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x \to a^- )</td>
<td>( x ) approaches ( a ) from the left (( x &lt; a ) but close to ( a ))</td>
</tr>
<tr>
<td>( x \to a^+ )</td>
<td>( x ) approaches ( a ) from the right (( x &gt; a ) but close to ( a ))</td>
</tr>
<tr>
<td>( x \to \infty )</td>
<td>( x ) approaches infinity (( x ) increases without bound)</td>
</tr>
<tr>
<td>( x \to -\infty )</td>
<td>( x ) approaches negative infinity (( x ) decreases without bound)</td>
</tr>
<tr>
<td>( f(x) \to \infty )</td>
<td>the output approaches infinity (the output increases without bound)</td>
</tr>
<tr>
<td>( f(x) \to -\infty )</td>
<td>the output approaches negative infinity (the output decreases without bound)</td>
</tr>
<tr>
<td>( f(x) \to a )</td>
<td>the output approaches ( a )</td>
</tr>
</tbody>
</table>

**Table 3.10 Arrow Notation**

**Local Behavior of** \( f(x) = \frac{1}{x} \)

Let's begin by looking at the reciprocal function, \( f(x) = \frac{1}{x} \). We cannot divide by zero, which means the function is undefined at \( x = 0 \); so zero is not in the domain. As the input values approach zero from the left side (becoming very small, negative values), the function values decrease without bound (in other words, they approach negative infinity). We can see this behavior in **Table 3.11**.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-0.1</th>
<th>-0.01</th>
<th>-0.001</th>
<th>-0.0001</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = \frac{1}{x} )</td>
<td>-10</td>
<td>-100</td>
<td>-1000</td>
<td>-10,000</td>
</tr>
</tbody>
</table>

**Table 3.11**

We write in arrow notation

\[
\text{as } x \to 0^- , \ f(x) \to -\infty
\]  \( \text{(3.50)} \)

As the input values approach zero from the right side (becoming very small, positive values), the function values increase without bound (approaching infinity). We can see this behavior in **Table 3.12**.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0.1</th>
<th>0.01</th>
<th>0.001</th>
<th>0.0001</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = \frac{1}{x} )</td>
<td>10</td>
<td>100</td>
<td>1000</td>
<td>10,000</td>
</tr>
</tbody>
</table>

**Table 3.12**
We write in arrow notation

\[
\text{As } x \to 0^+, \quad f(x) \to \infty.
\]

(3.51)

See Figure 3.37.

This behavior creates a **vertical asymptote**, which is a vertical line that the graph approaches but never crosses. In this case, the graph is approaching the vertical line \( x = 0 \) as the input becomes close to zero. See Figure 3.38.

**Figure 3.37**

**Figure 3.38**

**Vertical Asymptote**

A **vertical asymptote** of a graph is a vertical line \( x = a \) where the graph tends toward positive or negative infinity as the inputs approach \( a \). We write

\[
\text{As } x \to a, \quad f(x) \to \infty, \quad \text{or as } x \to a, \quad f(x) \to - \infty.
\]

(3.52)

**End Behavior of** \( f(x) = \frac{1}{x} \)

As the values of \( x \) approach infinity, the function values approach 0. As the values of \( x \) approach negative infinity, the function values approach 0. See Figure 3.39. Symbolically, using arrow notation
As \( x \to \infty \), \( f(x) \to 0 \), and as \( x \to -\infty \), \( f(x) \to 0 \).

Based on this overall behavior and the graph, we can see that the function approaches 0 but never actually reaches 0; it seems to level off as the inputs become large. This behavior creates a horizontal asymptote, a horizontal line that the graph approaches as the input increases or decreases without bound. In this case, the graph is approaching the horizontal line \( y = 0 \). See Figure 3.40.

**Horizontal Asymptote**

A horizontal asymptote of a graph is a horizontal line \( y = b \) where the graph approaches the line as the inputs increase or decrease without bound. We write

\[
\text{As } x \to \infty \text{ or } x \to -\infty, \ f(x) \to b.
\]  

**Example 3.59**

**Using Arrow Notation**
Use arrow notation to describe the end behavior and local behavior of the function graphed in Figure 3.41.

Solution

Example 3.60

Using Transformations to Graph a Rational Function

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

Solution

Analysis

Notice that horizontal and vertical asymptotes are shifted left 2 and up 3 along with the function.

3.38 Use arrow notation to describe the end behavior and local behavior for the reciprocal squared function.

3.39 Sketch the graph, and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

Solving Applied Problems Involving Rational Functions

In Example 3.60, we shifted a toolkit function in a way that resulted in the function \( f(x) = \frac{3x + 7}{x + 2} \). This is an example of a rational function. A rational function is a function that can be written as the quotient of two polynomial functions. Many real-world problems require us to find the ratio of two polynomial functions. Problems involving rates and concentrations often involve rational functions.
A rational function is a function that can be written as the quotient of two polynomial functions \( P(x) \) and \( Q(x) \).

\[
f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1} x^{p-1} + \ldots + a_1 x + a_0}{b_q x^q + b_{q-1} x^{q-1} + \ldots + b_1 x + b_0}, \quad Q(x) \neq 0
\]

### Example 3.61

**Solving an Applied Problem Involving a Rational Function**

A large mixing tank currently contains 100 gallons of water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after 12 minutes. Is that a greater concentration than at the beginning?

**Solution**

**Analysis**
To find the horizontal asymptote, divide the leading coefficient in the numerator by the leading coefficient in the denominator:

\[
\frac{1}{10} = 0.1
\]

Notice the horizontal asymptote is \( y = 0.1 \). This means the concentration, \( C \), the ratio of pounds of sugar to gallons of water, will approach 0.1 in the long term.

**Try It**

There are 1,200 freshmen and 1,500 sophomores at a prep rally at noon. After 12 p.m., 20 freshmen arrive at the rally every five minutes while 15 sophomores leave the rally. Find the ratio of freshmen to sophomores at 1 p.m.

**Finding the Domains of Rational Functions**

A vertical asymptote represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

**Domain of a Rational Function**

The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.

**How To: Given a rational function, find the domain.**

1. Set the denominator equal to zero.
2. Solve to find the \( x \)-values that cause the denominator to equal zero.
3. The domain is all real numbers except those found in Step 2.

**Example 3.62**
Finding the Domain of a Rational Function

Find the domain of \( f(x) = \frac{x + 3}{x^2 - 9} \)

Solution

Analysis

A graph of this function, as shown in Figure 3.41, confirms that the function is not defined when \( x = \pm 3 \).

There is a vertical asymptote at \( x = 3 \) and a hole in the graph at \( x = -3 \). We will discuss these types of holes in greater detail later in this section.

Try it 3.41 Find the domain of \( f(x) = \frac{4x}{5(x - 1)(x - 5)} \).

Identifying Vertical Asymptotes of Rational Functions

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

Given a rational function, identify any vertical asymptotes of its graph.

1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.
4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are removable discontinuities.

Example 3.63
Identifying Vertical Asymptotes

Find the vertical asymptotes of the graph of \( k(x) = \frac{5 + 2x^2}{2 - x - x^2} \).

Solution

Removable Discontinuities

Occasionally, a graph will contain a hole: a single point where the graph is not defined, indicated by an open circle. We call such a hole a **removable discontinuity**.

For example, the function \( f(x) = \frac{x^2 - 1}{x^2 - 2x - 3} \) may be re-written by factoring the numerator and the denominator.

\[
f(x) = \frac{(x + 1)(x - 1)}{(x + 1)(x - 3)}
\]

Notice that \( x + 1 \) is a common factor to the numerator and the denominator. The zero of this factor, \( x = -1 \), is the location of the removable discontinuity. Notice also that \( x - 3 \) is not a factor in both the numerator and denominator. The zero of this factor, \( x = 3 \), is the vertical asymptote. See Figure 3.42.

![Figure 3.42](image)

Removable Discontinuities of Rational Functions

A **removable discontinuity** occurs in the graph of a rational function at \( x = a \) if \( a \) is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This is the location of the removable discontinuity. This is true if the multiplicity of this factor is greater than or equal to that in the denominator. If the multiplicity of this factor is greater in the denominator, then there is still an asymptote at that value.

Example 3.64

**Identifying Vertical Asymptotes and Removable Discontinuities for a Graph**

Find the vertical asymptotes and removable discontinuities of the graph of \( k(x) = \frac{x - 2}{x^3 - 4} \).
Find the vertical asymptotes and removable discontinuities of the graph of \( f(x) = \frac{x^2 - 25}{x^3 - 6x^2 + 5x} \).

### Identifying Horizontal Asymptotes of Rational Functions

While vertical asymptotes describe the behavior of a graph as the output gets very large or very small, horizontal asymptotes help describe the behavior of a graph as the input gets very large or very small. Recall that a polynomial’s end behavior will mirror that of the leading term. Likewise, a rational function’s end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions.

There are three distinct outcomes when checking for horizontal asymptotes:

**Case 1:** If the degree of the denominator > degree of the numerator, there is a horizontal asymptote at \( y = 0 \).

Example: \( f(x) = \frac{4x + 2}{x^2 + 4x - 5} \)

In this case, the end behavior is \( f(x) \approx \frac{4x}{x^2} = \frac{4}{x} \). This tells us that, as the inputs increase or decrease without bound, this function will behave similarly to the function \( g(x) = \frac{4}{x} \), and the outputs will approach zero, resulting in a horizontal asymptote at \( y = 0 \). See **Figure 3.43**. Note that this graph crosses the horizontal asymptote.

![Figure 3.43](horizontal-asymptote-y-0.png)

**Figure 3.43** Horizontal Asymptote \( y = 0 \) when \( f(x) = \frac{p(x)}{q(x)} \), \( q(x) \neq 0 \) where degree of \( p < \text{degree of } q \).

**Case 2:** If the degree of the denominator < degree of the numerator by one, we get a slant asymptote.

Example: \( f(x) = \frac{3x^2 - 2x + 1}{x - 1} \)

In this case, the end behavior is \( f(x) \approx \frac{3x^2}{x} = 3x \). This tells us that as the inputs increase or decrease without bound, this function will behave similarly to the function \( g(x) = 3x \). As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote. However, the graph of \( g(x) = 3x \) looks like a diagonal line, and since \( f \) will behave similarly to \( g \), it will approach a line close to \( y = 3x \). This line is a slant asymptote.

To find the equation of the slant asymptote, divide \( \frac{3x^2 - 2x + 1}{x - 1} \). The quotient is \( 3x + 1 \), and the remainder is 2. The slant asymptote is the graph of the line \( g(x) = 3x + 1 \). See **Figure 3.44**.

![Figure 3.44](slant-asymptote.png)

**Figure 3.44** Slant Asymptote \( y = 3x + 1 \) when \( f(x) = \frac{p(x)}{q(x)} \), \( q(x) \neq 0 \) where degree of \( p = \text{degree of } q + 1 \).
Figure 3.44 Slant Asymptote when \( f(x) = \frac{p(x)}{q(x)} \), \( q(x) \neq 0 \)

where degree of \( p > \) degree of \( q \) by 1.

**Case 3:** If the degree of the denominator = degree of the numerator, there is a horizontal asymptote at \( y = \frac{a_n}{b_n} \) where \( a_n \) and \( b_n \) are the leading coefficients of \( p(x) \) and \( q(x) \) for \( f(x) = \frac{p(x)}{q(x)} \), \( q(x) \neq 0 \).

Example: \( f(x) = \frac{3x^2 + 2}{x^2 + 4x - 5} \)

In this case, the end behavior is \( f(x) \approx \frac{3x^2}{x^2} = 3 \). This tells us that as the inputs grow large, this function will behave like the function \( g(x) = 3 \), which is a horizontal line. As \( x \to \pm \infty \), \( f(x) \to 3 \), resulting in a horizontal asymptote at \( y = 3 \). See Figure 3.45. Note that this graph crosses the horizontal asymptote.

Figure 3.45 Horizontal Asymptote when \( f(x) = \frac{p(x)}{q(x)} \), \( q(x) \neq 0 \) where degree of \( p = \) degree of \( q \).

Notice that, while the graph of a rational function will never cross a vertical asymptote, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph will have at most one horizontal (or slant) asymptote.

It should be noted that, if the degree of the numerator is larger than the degree of the denominator by more than one, the end behavior of the graph will mimic the behavior of the reduced end behavior fraction. For instance, if we had the function
\[ f(x) = \frac{3x^5 - x^2}{x + 3} \] (3.59)

with end behavior

\[ f(x) \approx \frac{3x^5}{x} = 3x^4 \] (3.60)

the end behavior of the graph would look similar to that of an even polynomial with a positive leading coefficient.

\[ x \to \pm \infty, \quad f(x) \to \infty \] (3.61)

### Horizontal Asymptotes of Rational Functions

The horizontal asymptote of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of numerator is less than degree of denominator: horizontal asymptote at \( y = 0 \).
- Degree of numerator is greater than degree of denominator by one: no horizontal asymptote; slant asymptote.
- Degree of numerator is equal to degree of denominator: horizontal asymptote at ratio of leading coefficients.

### Example 3.65

#### Identifying Horizontal and Slant Asymptotes

For the functions below, identify the horizontal or slant asymptote.

- a. \( g(x) = \frac{6x^3 - 10x}{2x^3 + 5x^2} \)
- b. \( h(x) = \frac{x^2 - 4x + 1}{x + 2} \)
- c. \( k(x) = \frac{x^2 + 4x}{x^3 - 8} \)

### Solution

#### Example 3.66

#### Identifying Horizontal Asymptotes

In the sugar concentration problem earlier, we created the equation \( C(t) = \frac{5 + t}{100 + 10t} \).

Find the horizontal asymptote and interpret it in context of the problem.

### Solution

#### Example 3.67
Identifying Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes of the function

\[ f(x) = \frac{(x - 2)(x + 3)}{(x - 1)(x + 2)(x - 5)} \]

**Solution**

**Example 3.68**

**Finding the Intercepts of a Rational Function**

Find the intercepts of \( f(x) = \frac{(x - 2)(x + 3)}{(x - 1)(x + 2)(x - 5)} \).

**Solution**

**3.43** Find the vertical and horizontal asymptotes of the function:

\[ f(x) = \frac{(2x - 1)(2x + 1)}{(x - 2)(x + 3)} \]

**Intercepts of Rational Functions**

A rational function will have a y-intercept when the input is zero, if the function is defined at zero. A rational function will not have a y-intercept if the function is not defined at zero.

Likewise, a rational function will have x-intercepts at the inputs that cause the output to be zero. Since a fraction is only equal to zero when the numerator is zero, x-intercepts can only occur when the numerator of the rational function is equal to zero.

**3.44** Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function. Then, find the x- and y-intercepts and the horizontal and vertical asymptotes.

**Graphing Rational Functions**

In **Example 3.67**, we see that the numerator of a rational function reveals the x-intercepts of the graph, whereas the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Fortunately, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

The vertical asymptotes associated with the factors of the denominator will mirror one of the two toolkit reciprocal functions. When the degree of the factor in the denominator is odd, the distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity. See **Figure 3.46**.
When the degree of the factor in the denominator is even, the distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote or heads toward negative infinity on both sides. See Figure 3.47.

For example, the graph of \( f(x) = \frac{(x + 1)^2(x - 3)}{(x + 3)^2(x - 2)} \) is shown in Figure 3.48.
At the x-intercept \( x = -1 \) corresponding to the \((x + 1)^2\) factor of the numerator, the graph bounces, consistent with the quadratic nature of the factor.

At the x-intercept \( x = 3 \) corresponding to the \((x - 3)\) factor of the numerator, the graph passes through the axis as we would expect from a linear factor.

At the vertical asymptote \( x = -3 \) corresponding to the \((x + 3)^2\) factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the function \( f(x) = \frac{1}{x^2} \).

At the vertical asymptote \( x = 2 \), corresponding to the \((x - 2)\) factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of the function \( f(x) = \frac{1}{x} \).

**Given a rational function, sketch a graph.**

1. Evaluate the function at 0 to find the y-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the x-intercepts.
4. Find the multiplicities of the x-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.
Graphing a Rational Function

Sketch a graph of \( f(x) = \frac{(x + 2)(x - 3)}{(x + 1)^2(x - 2)} \).

Solution

Solution

Given the function \( f(x) = \frac{(x + 2)^2(x - 2)}{2(x - 1)^2(x - 3)} \), use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

Writing Rational Functions

Now that we have analyzed the equations for rational functions and how they relate to a graph of the function, we can use information given by a graph to write the function. A rational function written in factored form will have an \( x \)-intercept where each factor of the numerator is equal to zero. (An exception occurs in the case of a removable discontinuity.) As a result, we can form a numerator of a function whose graph will pass through a set of \( x \)-intercepts by introducing a corresponding set of factors. Likewise, because the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

Writing Rational Functions from Intercepts and Asymptotes

If a rational function has \( x \)-intercepts at \( x = x_1, \ x_2, \ldots, \ x_n \), vertical asymptotes at \( x = v_1, \ v_2, \ldots, \ v_m \), and no \( x_i = \) any \( v_j \), then the function can be written in the form:

\[
f(x) = \frac{a(x - x_1)^{p_1}(x - x_2)^{p_2} \ldots (x - x_n)^{p_n}}{(x - v_1)^{q_1}(x - v_2)^{q_2} \ldots (x - v_m)^{q_m}}
\]  

(3.62)

where the powers \( p_i \) or \( q_i \) on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor \( a \) can be determined given a value of the function other than the \( x \)-intercept or by the horizontal asymptote if it is nonzero.

Given a graph of a rational function, write the function.

1. Determine the factors of the numerator. Examine the behavior of the graph at the \( x \)-intercepts to determine the zeroes and their multiplicities. (This is easy to do when finding the “simplest” function with small multiplicities—such as 1 or 3—but may be difficult for larger multiplicities—such as 5 or 7, for example.)
2. Determine the factors of the denominator. Examine the behavior on both sides of each vertical asymptote to determine the factors and their powers.
3. Use any clear point on the graph to find the stretch factor.

Example 3.70

Writing a Rational Function from Intercepts and Asymptotes

Write an equation for the rational function shown in Figure 3.49.
Access these online resources for additional instruction and practice with rational functions.

- Graphing Rational Functions (http://openstaxcollege.org/l/graphrational)
- Find the Equation of a Rational Function (http://openstaxcollege.org/l/equatrational)
- Determining Vertical and Horizontal Asymptotes (http://openstaxcollege.org/l/asymptote)
- Find the Intercepts, Asymptotes, and Hole of a Rational Function (http://openstaxcollege.org/l/interasymptote)
3.7 EXERCISES

Verbal

455. What is the fundamental difference in the algebraic representation of a polynomial function and a rational function?

456. What is the fundamental difference in the graphs of polynomial functions and rational functions?

457. If the graph of a rational function has a removable discontinuity, what must be true of the functional rule?

458. Can a graph of a rational function have no vertical asymptote? If so, how?

459. Can a graph of a rational function have no x-intercepts? If so, how?

Algebraic

For the following exercises, find the domain of the rational functions.

460. \( f(x) = \frac{x - 1}{x + 2} \)

461. \( f(x) = \frac{x + 1}{x^2 - 1} \)

462. \( f(x) = \frac{x^2 + 4}{x^2 - 2x - 8} \)

463. \( f(x) = \frac{x^2 + 4x - 3}{x^4 - 5x^2 + 4} \)

For the following exercises, find the domain, vertical asymptotes, and horizontal asymptotes of the functions.

464. \( f(x) = \frac{4}{x - 1} \)

465. \( f(x) = \frac{2}{5x + 2} \)

466. \( f(x) = \frac{x}{x^2 - 9} \)

467. \( f(x) = \frac{x}{x^2 + 5x - 36} \)

468. \( f(x) = \frac{3 + x}{x^3 - 27} \)

469. \( f(x) = \frac{3x - 4}{x^3 - 16x} \)

470. \( f(x) = \frac{x^2 - 1}{x^3 + 9x^2 + 14x} \)

471. \( f(x) = \frac{x + 5}{x^2 - 25} \)

472. \( f(x) = \frac{x - 4}{x - 6} \)

473. \( f(x) = \frac{4 - 2x}{3x - 1} \)

For the following exercises, find the x- and y-intercepts for the functions.
474. \( f(x) = \frac{x + 5}{x^2 + 4} \)

475. \( f(x) = \frac{x}{x^2 - x} \)

476. \( f(x) = \frac{x^2 + 8x + 7}{x^2 + 11x + 30} \)

477. \( f(x) = \frac{x^2 + x + 6}{x^2 - 10x + 24} \)

478. \( f(x) = \frac{94 - 2x^2}{3x^2 - 12} \)

For the following exercises, describe the local and end behavior of the functions.

479. \( f(x) = \frac{x}{2x + 1} \)

480. \( f(x) = \frac{2x}{x - 6} \)

481. \( f(x) = \frac{-2x}{x - 6} \)

482. \( f(x) = \frac{x^2 - 4x + 3}{x^2 - 4x - 5} \)

483. \( f(x) = \frac{2x^2 - 32}{6x^2 + 13x - 5} \)

For the following exercises, find the slant asymptote of the functions.

484. \( f(x) = \frac{24x^2 + 6x}{2x + 1} \)

485. \( f(x) = \frac{4x^2 - 10}{2x - 4} \)

486. \( f(x) = \frac{81x^2 - 18}{3x - 2} \)

487. \( f(x) = \frac{6x^3 - 5x}{3x^2 + 4} \)

488. \( f(x) = \frac{x^2 + 5x + 4}{x - 1} \)

**Graphical**

For the following exercises, use the given transformation to graph the function. Note the vertical and horizontal asymptotes.

489. The reciprocal function shifted up two units.

490. The reciprocal function shifted down one unit and left three units.

491. The reciprocal squared function shifted to the right 2 units.

492. The reciprocal squared function shifted down 2 units and right 1 unit.
For the following exercises, find the horizontal intercepts, the vertical intercept, the vertical asymptotes, and the horizontal or slant asymptote of the functions. Use that information to sketch a graph.

493. \( p(x) = \frac{2x - 3}{x + 4} \)

494. \( q(x) = \frac{x - 5}{3x - 1} \)

495. \( s(x) = \frac{4}{(x - 2)^2} \)

496. \( r(x) = \frac{5}{(x + 1)^2} \)

497. \( f(x) = \frac{3x^2 - 14x - 5}{3x^2 + 8x - 16} \)

498. \( g(x) = \frac{2x^2 + 7x - 15}{3x^2 - 14 + 15} \)

499. \( a(x) = \frac{x^2 + 2x - 3}{x^2 - 1} \)

500. \( b(x) = \frac{x^2 - x - 6}{x^2 - 4} \)

501. \( h(x) = \frac{2x^2 + x - 1}{x - 4} \)

502. \( k(x) = \frac{2x^2 - 3x - 20}{x - 5} \)

503. \( w(x) = \frac{(x - 1)(x + 3)(x - 5)}{(x + 2)^2(x - 4)} \)

504. \( z(x) = \frac{(x + 2)^2(x - 5)}{(x - 3)(x + 1)(x + 4)} \)

For the following exercises, write an equation for a rational function with the given characteristics.

505. Vertical asymptotes at \( x = 5 \) and \( x = -5 \), x-intercepts at \( (2, 0) \) and \( (-1, 0) \), y-intercept at \( (0, 4) \)

506. Vertical asymptotes at \( x = -4 \) and \( x = -1 \), x-intercepts at \( (1, 0) \) and \( (5, 0) \), y-intercept at \( (0, 7) \)

507. Vertical asymptotes at \( x = -4 \) and \( x = -5 \), x-intercepts at \( (4, 0) \) and \( (-6, 0) \), Horizontal asymptote at \( y = 7 \)

508. Vertical asymptotes at \( x = -3 \) and \( x = 6 \), x-intercepts at \( (-2, 0) \) and \( (1, 0) \), Horizontal asymptote at \( y = -2 \)

509. Vertical asymptote at \( x = -1 \), Double zero at \( x = 2 \), y-intercept at \( (0, 2) \)

510. Vertical asymptote at \( x = 3 \), Double zero at \( x = 1 \), y-intercept at \( (0, 4) \)

For the following exercises, use the graphs to write an equation for the function.

511.
For the following exercises, make tables to show the behavior of the function near the vertical asymptote and reflecting the horizontal asymptote

518. $f(x) = \frac{1}{x-2}$

519. $f(x) = \frac{x}{x-3}$

520. $f(x) = \frac{2x}{x+4}$

521. $f(x) = \frac{2x}{(x-3)^2}$

522. $f(x) = \frac{x^2}{x^2 + 2x + 1}$

**Technology**

For the following exercises, use a calculator to graph $f(x)$. Use the graph to solve $f(x) > 0$.

524. $f(x) = \frac{2}{x+1}$
Extensions

For the following exercises, identify the removable discontinuity.

529. \( f(x) = \frac{x^2 - 4}{x - 2} \)

530. \( f(x) = \frac{x^3 + 1}{x + 1} \)

531. \( f(x) = \frac{x^2 + x - 6}{x - 2} \)

532. \( f(x) = \frac{2x^2 + 5x - 3}{x + 3} \)

533. \( f(x) = \frac{x^3 + x^2}{x + 1} \)

Real-World Applications

For the following exercises, express a rational function that describes the situation.

534. A large mixing tank currently contains 200 gallons of water, into which 10 pounds of sugar have been mixed. A tap will open, pouring 10 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 3 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after \( t \) minutes.

535. A large mixing tank currently contains 300 gallons of water, into which 8 pounds of sugar have been mixed. A tap will open, pouring 20 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 2 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after \( t \) minutes.

For the following exercises, use the given rational function to answer the question.

536. The concentration \( C \) of a drug in a patient’s bloodstream \( t \) hours after injection in given by \( C(t) = \frac{2t}{3 + t^2} \). What happens to the concentration of the drug as \( t \) increases?

537. The concentration \( C \) of a drug in a patient’s bloodstream \( t \) hours after injection is given by \( C(t) = \frac{100t}{2t^2 + 75} \). Use a calculator to approximate the time when the concentration is highest.

For the following exercises, construct a rational function that will help solve the problem. Then, use a calculator to answer the question.

538. An open box with a square base is to have a volume of 108 cubic inches. Find the dimensions of the box that will have minimum surface area. Let \( x \) = length of the side of the base.

539. A rectangular box with a square base is to have a volume of 20 cubic feet. The material for the base costs 30 cents/square foot. The material for the sides costs 10 cents/square foot. The material for the top costs 20 cents/square foot. Determine the dimensions that will yield minimum cost. Let \( x \) = length of the side of the base.
540. A right circular cylinder has volume of 100 cubic inches. Find the radius and height that will yield minimum surface area. Let \( x \) = radius.

541. A right circular cylinder with no top has a volume of 50 cubic meters. Find the radius that will yield minimum surface area. Let \( x \) = radius.

542. A right circular cylinder is to have a volume of 40 cubic inches. It costs 4 cents/square inch to construct the top and bottom and 1 cent/square inch to construct the rest of the cylinder. Find the radius to yield minimum cost. Let \( x \) = radius.
In this section, you will:

3.8.1 Find the inverse of a polynomial function.
3.8.2 Restrict the domain to find the inverse of a polynomial function.

A mound of gravel is in the shape of a cone with the height equal to twice the radius.

Figure 3.50

The volume is found using a formula from elementary geometry.

\[ V = \frac{1}{3} \pi r^2 h \]

\[ = \frac{1}{3} \pi r^2 (2r) \]

\[ = \frac{2}{3} \pi r^3 \]

We have written the volume \( V \) in terms of the radius \( r \). However, in some cases, we may start out with the volume and want to find the radius. For example: A customer purchases 100 cubic feet of gravel to construct a cone shape mound with a height twice the radius. What are the radius and height of the new cone? To answer this question, we use the formula

\[ r = \sqrt[3]{\frac{3V}{2\pi}} \]

This function is the inverse of the formula for \( V \) in terms of \( r \).

In this section, we will explore the inverses of polynomial and rational functions and in particular the radical functions we encounter in the process.

**Finding the Inverse of a Polynomial Function**

Two functions \( f \) and \( g \) are inverse functions if for every coordinate pair in \( f \), \((a, b)\), there exists a corresponding coordinate pair in the inverse function, \( g \), \((b, a)\). In other words, the coordinate pairs of the inverse functions have the input and output interchanged.

For a function to have an inverse function the function to create a new function that is one-to-one and would have an inverse function.

For example, suppose a water runoff collector is built in the shape of a parabolic trough as shown in Figure 3.51. We can use the information in the figure to find the surface area of the water in the trough as a function of the depth of the water.
Because it will be helpful to have an equation for the parabolic cross-sectional shape, we will impose a coordinate system at the cross section, with $x$ measured horizontally and $y$ measured vertically, with the origin at the vertex of the parabola. See Figure 3.52.

From this we find an equation for the parabolic shape. We placed the origin at the vertex of the parabola, so we know the equation will have form $y(x) = ax^2$. Our equation will need to pass through the point $(6, 18)$, from which we can solve for the stretch factor $a$.

\[ 18 = a6^2 \]
\[ a = \frac{18}{36} \]
\[ a = \frac{1}{2} \]

Our parabolic cross section has the equation

\[ y(x) = \frac{1}{2}x^2. \]  

We are interested in the surface area of the water, so we must determine the width at the top of the water as a function of the water depth. For any depth $y$ the width will be given by $2x$, so we need to solve the equation above for $x$ and find the inverse function. However, notice that the original function is not one-to-one, and indeed, given any output there are two inputs that produce the same output, one positive and one negative.

To find an inverse, we can restrict our original function to a limited domain on which it is one-to-one. In this case, it makes sense to restrict ourselves to positive $x$ values. On this domain, we can find an inverse by solving for the input variable:
\[ y = \frac{1}{2}x^2 \]
\[ 2y = x^2 \]
\[ x = \pm \sqrt{2y} \]

This is not a function as written. We are limiting ourselves to positive \( x \) values, so we eliminate the negative solution, giving us the inverse function we’re looking for.

\[ y = \frac{2}{x^2}, \quad x > 0 \]  

(3.68)

Because \( x \) is the distance from the center of the parabola to either side, the entire width of the water at the top will be \( 2x \). The trough is 3 feet (36 inches) long, so the surface area will then be:

\[ \text{Area} = l \cdot w = 36 \cdot 2x = 72x \]

\[ = 72\sqrt{2y} \]

(3.69)

This example illustrates two important points:

1. When finding the inverse of a quadratic, we have to limit ourselves to a domain on which the function is one-to-one.
2. The inverse of a quadratic function is a square root function. Both are toolkit functions and different types of power functions.

Functions involving roots are often called radical functions. While it is not possible to find an inverse of most polynomial functions, some basic polynomials do have inverses. Such functions are called invertible functions, and we use the notation \( f^{-1}(x) \).

Warning: \( f^{-1}(x) \) is not the same as the reciprocal of the function \( f(x) \). This use of “–1” is reserved to denote inverse functions. To denote the reciprocal of a function \( f(x) \), we would need to write \( (f(x))^{-1} = \frac{1}{f(x)} \).

An important relationship between inverse functions is that they “undo” each other. If \( f^{-1} \) is the inverse of a function \( f \), then \( f \) is the inverse of the function \( f^{-1} \). In other words, whatever the function \( f \) does to \( x \), \( f^{-1} \) undoes it—and vice-versa. More formally, we write

\[ f^{-1}(f(x)) = x, \quad \text{for all } x \text{ in the domain of } f \]  

(3.70)

and

\[ f(f^{-1}(x)) = x, \quad \text{for all } x \text{ in the domain of } f^{-1}. \]  

(3.71)

### Verifying Two Functions Are Inverses of One Another

Two functions, \( f \) and \( g \), are inverses of one another if for all \( x \) in the domain of \( f \) and \( g \):

\[ g(f(x)) = f(g(x)) = x \]  

(3.72)

**Example 3.71**

**How To:** Given a polynomial function, find the inverse of the function by restricting the domain in such a way that the new function is one-to-one.

1. Replace \( f(x) \) with \( y \).
2. Interchange \( x \) and \( y \).
3. Solve for \( y \), and rename the function \( f^{-1}(x) \).
Verifying Inverse Functions

Show that \( f(x) = \frac{1}{x + 1} \) and \( f^{-1}(x) = \frac{1}{x} - 1 \) are inverses, for \( x \neq 0, -1 \).

Solution

Example 3.72

Finding the Inverse of a Cubic Function

Find the inverse of the function \( f(x) = 5x^3 + 1 \).

Solution

Analysis

Look at the graph of \( f \) and \( f^{-1} \). Notice that the two graphs are symmetrical about the line \( y = x \). This is always the case when graphing a function and its inverse function.

Also, since the method involved interchanging \( x \) and \( y \), notice corresponding points. If \((a, b)\) is on the graph of \( f \), then \((b, a)\) is on the graph of \( f^{-1} \). Since \((0, 1)\) is on the graph of \( f \), then \((1, 0)\) is on the graph of \( f^{-1} \). Similarly, since \((1, 6)\) is on the graph of \( f \), then \((6, 1)\) is on the graph of \( f^{-1} \). See Figure 3.52.
Find the inverse function of $f(x) = \frac{3}{\sqrt{x}} + 4$.

### Restricting the Domain to Find the Inverse of a Polynomial Function

So far, we have been able to find the inverse functions of cubic functions without having to restrict their domains. However, as we know, not all cubic polynomials are one-to-one. Some functions that are not one-to-one may have their domain restricted so that they are one-to-one, but only over that domain. The function over the restricted domain would then have an inverse function. Since quadratic functions are not one-to-one, we must restrict their domain in order to find their inverses.

---

**Restricting the Domain**

If a function is not one-to-one, it cannot have an inverse. If we restrict the domain of the function so that it becomes one-to-one, thus creating a new function, this new function will have an inverse.

**How To**

Given a polynomial function, restrict the domain of a function that is not one-to-one and then find the inverse.

1. Restrict the domain by determining a domain on which the original function is one-to-one.
2. Replace $f(x)$ with $y$.
3. Interchange $x$ and $y$.
4. Solve for $y$, and rename the function or pair of function $f^{-1}(x)$.
5. Revise the formula for $f^{-1}(x)$ by ensuring that the outputs of the inverse function correspond to the restricted domain of the original function.

---

**Example 3.73**
Restricting the Domain to Find the Inverse of a Polynomial Function

Find the inverse function of $f$:

a. $f(x) = (x - 4)^2, \quad x \geq 4$

b. $f(x) = (x - 4)^2, \quad x \leq 4$

Solution

Analysis

On the graphs in Figure 3.52, we see the original function graphed on the same set of axes as its inverse function. Notice that together the graphs show symmetry about the line $y = x$. The coordinate pair $(4, 0)$ is on the graph of $f$ and the coordinate pair $(0, 4)$ is on the graph of $f^{-1}$. For any coordinate pair, if $(a, b)$ is on the graph of $f$, then $(b, a)$ is on the graph of $f^{-1}$. Finally, observe that the graph of $f$ intersects the graph of $f^{-1}$ on the line $y = x$. Points of intersection for the graphs of $f$ and $f^{-1}$ will always lie on the line $y = x$.

Example 3.74

Finding the Inverse of a Quadratic Function When the Restriction Is Not Specified

Restrict the domain and then find the inverse of

$$f(x) = (x - 2)^2 - 3.$$

Solution

Analysis

Notice that we arbitrarily decided to restrict the domain on $x \geq 2$. We could just have easily opted to restrict the domain on $x \leq 2$, in which case $f^{-1}(x) = 2 - \sqrt{x + 3}$. Observe the original function graphed on the same set
of axes as its inverse function in Figure 3.52. Notice that both graphs show symmetry about the line \( y = x \). The coordinate pair \((-3, 2)\) is on the graph of \( f \) and the coordinate pair \((2, -3)\) is on the graph of \( f^{-1} \). Observe from the graph of both functions on the same set of axes that

\[
\text{domain of } f = \text{range of } f^{-1} = [2, \infty)
\]

and

\[
\text{domain of } f^{-1} = \text{range of } f = [-3, \infty).
\]

Finally, observe that the graph of \( f \) intersects the graph of \( f^{-1} \) along the line \( y = x \).

Find the inverse of the function \( f(x) = x^2 + 1 \), on the domain \( x \geq 0 \).

**Solving Applications of Radical Functions**

Notice that the functions from previous examples were all polynomials, and their inverses were radical functions. If we want to find the inverse of a radical function, we will need to restrict the domain of the answer because the range of the original function is limited.

**Given a radical function, find the inverse.**

1. Determine the range of the original function.
2. Replace \( f(x) \) with \( y \), then solve for \( x \).
3. If necessary, restrict the domain of the inverse function to the range of the original function.

**Example 3.75**

**Finding the Inverse of a Radical Function**
Restrict the domain and then find the inverse of the function \( f(x) = \sqrt{x - 4} \).

**Solution**

**Analysis**

Notice in Figure 3.52 that the inverse is a reflection of the original function over the line \( y = x \). Because the original function has only positive outputs, the inverse function has only positive inputs.

---

3.49 Restrict the domain and then find the inverse of the function \( f(x) = \sqrt{2x + 3} \).

**Solving Applications of Radical Functions**

Radical functions are common in physical models, as we saw in the section opener. We now have enough tools to be able to solve the problem posed at the start of the section.

**Example 3.76**

**Solving an Application with a Cubic Function**

A mound of gravel is in the shape of a cone with the height equal to twice the radius. The volume of the cone in terms of the radius is given by

\[ V = \frac{2}{3} \pi r^3. \]

Find the inverse of the function \( V = \frac{2}{3} \pi r^3 \) that determines the volume \( V \) of a cone and is a function of the radius \( r \). Then use the inverse function to calculate the radius of such a mound of gravel measuring 100 cubic feet. Use \( \pi = 3.14 \).

**Solution**

---
Determining the Domain of a Radical Function Composed with Other Functions

When radical functions are composed with other functions, determining domain can become more complicated.

Example 3.77

Finding the Domain of a Radical Function Composed with a Rational Function

Find the domain of the function \( f(x) = \sqrt{\frac{(x + 2)(x - 3)}{x - 1}} \).

Solution

Finding Inverses of Rational Functions

As with finding inverses of quadratic functions, it is sometimes desirable to find the inverse of a rational function, particularly of rational functions that are the ratio of linear functions, such as in concentration applications.

Example 3.78

Finding the Inverse of a Rational Function

The function \( C = \frac{20 + 0.4n}{100 + n} \) represents the concentration \( C \) of an acid solution after \( n \) mL of 40% solution has been added to 100 mL of a 20% solution. First, find the inverse of the function; that is, find an expression for \( n \) in terms of \( C \). Then use your result to determine how much of the 40% solution should be added so that the final mixture is a 35% solution.

Solution

Access these online resources for additional instruction and practice with inverses and radical functions.

- Graphing the Basic Square Root Function (http://openstaxcollege.org/l/graphsquareroot)
- Find the Inverse of a Square Root Function (http://openstaxcollege.org/l/inversesquare)
- Find the Inverse of a Rational Function (http://openstaxcollege.org/l/inverserational)
- Find the Inverse of a Rational Function and an Inverse Function Value (http://openstaxcollege.org/l/rationalinverse)
- Inverse Functions (http://openstaxcollege.org/l/inversefunction)
3.8 EXERCISES

Verbal

543. Explain why we cannot find inverse functions for all polynomial functions.

544. Why must we restrict the domain of a quadratic function when finding its inverse?

545. When finding the inverse of a radical function, what restriction will we need to make?

546. The inverse of a quadratic function will always take what form?

Algebraic

For the following exercises, find the inverse of the function on the given domain.

547. \( f(x) = (x - 4)^2, \ [4, \infty) \)

548. \( f(x) = (x + 2)^2, \ [-2, \infty) \)

549. \( f(x) = (x + 1)^2 - 3, \ [-1, \infty) \)

550. \( f(x) = 2 - \sqrt{3 + x} \)

551. \( f(x) = 3x^2 + 5, \ [\infty, 0] \)

552. \( f(x) = 12 - x^2, \ [0, \infty) \)

553. \( f(x) = 9 - x^2, \ [0, \infty) \)

554. \( f(x) = 2x^2 + 4, \ [0, \infty) \)

For the following exercises, find the inverse of the functions.

555. \( f(x) = x^3 + 5 \)

556. \( f(x) = 3x^3 + 1 \)

557. \( f(x) = 4 - x^3 \)

558. \( f(x) = 4 - 2x^3 \)

For the following exercises, find the inverse of the functions.

559. \( f(x) = \sqrt{2x + 1} \)

560. \( f(x) = \sqrt{3 - 4x} \)

561. \( f(x) = 9 + \sqrt{4x - 4} \)

562. \( f(x) = \sqrt{6x - 8} + 5 \)

563. \( f(x) = 9 + 2\sqrt[3]{x} \)

564. \( f(x) = 3 - \frac{1}{3}x \)

565.
\[ f(x) = \frac{2}{x+8} \]

566. \[ f(x) = \frac{3}{x-4} \]

567. \[ f(x) = \frac{x+3}{x+7} \]

568. \[ f(x) = \frac{x-2}{x+7} \]

569. \[ f(x) = \frac{3x+4}{5-4x} \]

570. \[ f(x) = \frac{5x+1}{2-5x} \]

571. \[ f(x) = x^2 + 2x, \quad [-1, \infty) \]

572. \[ f(x) = x^2 + 4x + 1, \quad [-2, \infty) \]

573. \[ f(x) = x^2 - 6x + 3, \quad [3, \infty) \]

**Graphical**

For the following exercises, find the inverse of the function and graph both the function and its inverse.

574. \[ f(x) = x^2 + 2, \quad x \geq 0 \]

575. \[ f(x) = 4 - x^2, \quad x \geq 0 \]

576. \[ f(x) = (x + 3)^2, \quad x \geq -3 \]

577. \[ f(x) = (x - 4)^2, \quad x \geq 4 \]

578. \[ f(x) = x^3 + 3 \]

579. \[ f(x) = 1 - x^3 \]

580. \[ f(x) = x^2 + 4x, \quad x \geq -2 \]

581. \[ f(x) = x^2 - 6x + 1, \quad x \geq 3 \]

582. \[ f(x) = \frac{2}{x} \]

583. \[ f(x) = \frac{1}{x^2}, \quad x \geq 0 \]

For the following exercises, use a graph to help determine the domain of the functions.

584. \[ f(x) = \sqrt{\frac{(x+1)(x-1)}{x}} \]

585. \[ f(x) = \sqrt{\frac{(x+2)(x-3)}{x-1}} \]

586. \[ f(x) = \sqrt{\frac{x(x+3)}{x-4}} \]
587. \(f(x) = \sqrt{\frac{x^2 - x - 20}{x - 2}}\)

588. \(f(x) = \sqrt{\frac{9 - x^2}{x + 4}}\)

**Technology**

For the following exercises, use a calculator to graph the function. Then, using the graph, give three points on the graph of the inverse with \(y\)-coordinates given.

589. \(f(x) = x^3 - x - 2, \quad y = 1, \quad 2, \quad 3\)

590. \(f(x) = x^3 + x - 2, \quad y = 0, \quad 1, \quad 2\)

591. \(f(x) = x^3 + 3x - 4, \quad y = 0, \quad 1, \quad 2\)

592. \(f(x) = x^3 + 8x - 4, \quad y = -1, \quad 0, \quad 1\)

593. \(f(x) = x^4 + 5x + 1, \quad y = -1, \quad 0, \quad 1\)

**Extensions**

For the following exercises, find the inverse of the functions with \(a, \ b, \ c\) positive real numbers.

594. \(f(x) = ax^3 + b\)

595. \(f(x) = x^2 + bx\)

596. \(f(x) = \sqrt[4]{ax^2 + b}\)

597. \(f(x) = \sqrt[3]{ax + b}\)

598. \(f(x) = \frac{ax + b}{x + c}\)

**Real-World Applications**

For the following exercises, determine the function described and then use it to answer the question.

599. An object dropped from a height of 200 meters has a height, \(h(t)\), in meters after \(t\) seconds have lapsed, such that \(h(t) = 200 - 4.9t^2\). Express \(t\) as a function of height, \(h\), and find the time to reach a height of 50 meters.

600. An object dropped from a height of 600 feet has a height, \(h(t)\), in feet after \(t\) seconds have elapsed, such that \(h(t) = 600 - 16t^2\). Express \(t\) as a function of height \(h\), and find the time to reach a height of 400 feet.

601. The volume, \(V\), of a sphere in terms of its radius, \(r\), is given by \(V(r) = \frac{4}{3}\pi r^3\). Express \(r\) as a function of \(V\), and find the radius of a sphere with volume of 200 cubic feet.

602. The surface area, \(A\), of a sphere in terms of its radius, \(r\), is given by \(A(r) = 4\pi r^2\). Express \(r\) as a function of \(V\), and find the radius of a sphere with a surface area of 1000 square inches.

603. A container holds 100 ml of a solution that is 25 ml acid. If \(n\) ml of a solution that is 60% acid is added, the function \(C(n) = \frac{25 + 0.6n}{100 + n}\) gives the concentration, \(C\), as a function of the number of ml added, \(n\). Express \(n\) as a function of \(C\) and determine the number of mL that need to be added to have a solution that is 50% acid.
604. The period $T$, in seconds, of a simple pendulum as a function of its length $l$, in feet, is given by $T(l) = 2\pi \sqrt{\frac{l}{32.2}}$. Express $l$ as a function of $T$ and determine the length of a pendulum with period of 2 seconds.

605. The volume of a cylinder, $V$, in terms of radius, $r$, and height, $h$, is given by $V = \pi r^2 h$. If a cylinder has a height of 6 meters, express the radius as a function of $V$ and find the radius of a cylinder with volume of 300 cubic meters.

606. The surface area, $A$, of a cylinder in terms of its radius, $r$, and height, $h$, is given by $A = 2\pi r^2 + 2\pi rh$. If the height of the cylinder is 4 feet, express the radius as a function of $V$ and find the radius if the surface area is 200 square feet.

607. The volume of a right circular cone, $V$, in terms of its radius, $r$, and its height, $h$, is given by $V = \frac{1}{3}\pi r^2 h$. Express $r$ in terms of $h$ if the height of the cone is 12 feet and find the radius of a cone with volume of 50 cubic inches.

608. Consider a cone with height of 30 feet. Express the radius, $r$, in terms of the volume, $V$, and find the radius of a cone with volume of 1000 cubic feet.
### 3.9 | Modeling Using Variation

#### Learning Objectives

In this section, you will:

- **3.9.1** Solve direct variation problems.
- **3.9.2** Solve inverse variation problems.
- **3.9.3** Solve problems involving joint variation.

A used-car company has just offered their best candidate, Nicole, a position in sales. The position offers 16% commission on her sales. Her earnings depend on the amount of her sales. For instance, if she sells a vehicle for $4,600, she will earn $736. She wants to evaluate the offer, but she is not sure how. In this section, we will look at relationships, such as this one, between earnings, sales, and commission rate.

#### Solving Direct Variation Problems

In the example above, Nicole’s earnings can be found by multiplying her sales by her commission. The formula $e = 0.16s$ tells us her earnings, $e$, come from the product of 0.16, her commission, and the sale price of the vehicle. If we create a table, we observe that as the sales price increases, the earnings increase as well, which should be intuitive. See Table 3.13.

<table>
<thead>
<tr>
<th>$s$, sales prices</th>
<th>$e = 0.16s$</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4,600$</td>
<td>$e = 0.16(4,600) = 736$</td>
<td>A sale of a $4,600 vehicle results in $736 earnings.</td>
</tr>
<tr>
<td>$9,200$</td>
<td>$e = 0.16(9,200) = 1,472$</td>
<td>A sale of a $9,200 vehicle results in $1,472 earnings.</td>
</tr>
<tr>
<td>$18,400$</td>
<td>$e = 0.16(18,400) = 2,944$</td>
<td>A sale of a $18,400 vehicle results in $2,944 earnings.</td>
</tr>
</tbody>
</table>

Table 3.13

Notice that earnings are a multiple of sales. As sales increase, earnings increase in a predictable way. Double the sales of the vehicle from $4,600 to $9,200, and we double the earnings from $736 to $1,472. As the input increases, the output increases as a multiple of the input. A relationship in which one quantity is a constant multiplied by another quantity is called **direct variation**. Each variable in this type of relationship varies directly with the other.

**Figure 3.53** represents the data for Nicole’s potential earnings. We say that earnings vary directly with the sales price of the car. The formula $y = kx^n$ is used for direct variation. The value $k$ is a nonzero constant greater than zero and is called the **constant of variation**. In this case, $k = 0.16$. and $n = 1$. 
Figure 3.53

Direct Variation

If \( x \) and \( y \) are related by an equation of the form

\[
y = kx^n
\]

then we say that the relationship is **direct variation** and \( y \) **varies directly** with the \( n \)th power of \( x \). In direct variation relationships, there is a nonzero constant ratio \( k = \frac{y}{x^n} \), where \( k \) is called the **constant of variation**, which help defines the relationship between the variables.

**How To**: Given a description of a direct variation problem, solve for an unknown.

1. Identify the input, \( x \), and the output, \( y \).
2. Determine the constant of variation. You may need to divide \( y \) by the specified power of \( x \) to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

**Example 3.79**

**Solving a Direct Variation Problem**

The quantity \( y \) varies directly with the cube of \( x \). If \( y = 25 \) when \( x = 2 \), find \( y \) when \( x \) is 6.

**Solution**

**Analysis**

The graph of this equation is a simple cubic, as shown in Figure 3.53.
3.51 The quantity $y$ varies directly with the square of $x$. If $y = 24$ when $x = 3$, find $y$ when $x$ is 4.

Solving Inverse Variation Problems

Water temperature in an ocean varies inversely to the water’s depth. The formula $T = \frac{14,000}{d}$ gives us the temperature in degrees Fahrenheit at a depth in feet below Earth’s surface. Consider the Atlantic Ocean, which covers 22% of Earth’s surface. At a certain location, at the depth of 500 feet, the temperature may be 28°F.

If we create Table 3.14, we observe that, as the depth increases, the water temperature decreases.

<table>
<thead>
<tr>
<th>$d$, depth</th>
<th>$T = \frac{14,000}{d}$</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>500 ft</td>
<td>$\frac{14,000}{500} = 28$</td>
<td>At a depth of 500 ft, the water temperature is 28° F.</td>
</tr>
<tr>
<td>1000 ft</td>
<td>$\frac{14,000}{1000} = 14$</td>
<td>At a depth of 1,000 ft, the water temperature is 14° F.</td>
</tr>
<tr>
<td>2000 ft</td>
<td>$\frac{14,000}{2000} = 7$</td>
<td>At a depth of 2,000 ft, the water temperature is 7° F.</td>
</tr>
</tbody>
</table>

Table 3.14

We notice in the relationship between these variables that, as one quantity increases, the other decreases. The two quantities are said to be inversely proportional and each term varies inversely with the other. Inversely proportional relationships are also called inverse variations.
For our example, Figure 3.54 depicts the inverse variation. We say the water temperature varies inversely with the depth of the water because, as the depth increases, the temperature decreases. The formula \( y = \frac{k}{x} \) for inverse variation in this case uses \( k = 14,000 \).

![Figure 3.54](image)

### Inverse Variation

If \( x \) and \( y \) are related by an equation of the form

\[
y = \frac{k}{x^n}.
\]

(3.74)

where \( k \) is a nonzero constant, then we say that \( y \) varies inversely with the \( n \)th power of \( x \). In inversely proportional relationships, or inverse variations, there is a constant multiple \( k = x^n y \).

### Example 3.80

#### Writing a Formula for an Inversely Proportional Relationship

A tourist plans to drive 100 miles. Find a formula for the time the trip will take as a function of the speed the tourist drives.

**Solution**

- **Given**: A description of an indirect variation problem, solve for an unknown.
- **How To**:
  1. Identify the input, \( x \), and the output, \( y \).
  2. Determine the constant of variation. You may need to multiply \( y \) by the specified power of \( x \) to determine the constant of variation.
  3. Use the constant of variation to write an equation for the relationship.
  4. Substitute known values into the equation to find the unknown.

### Example 3.81

#### Solving an Inverse Variation Problem
A quantity \( y \) varies inversely with the cube of \( x \). If \( y = 25 \) when \( x = 2 \), find \( y \) when \( x = 6 \).

**Solution**

**Analysis**

The graph of this equation is a rational function, as shown in Figure 3.54.

3.52 A quantity \( y \) varies inversely with the square of \( x \). If \( y = 8 \) when \( x = 3 \), find \( y \) when \( x = 4 \).

**Solving Problems Involving Joint Variation**

Many situations are more complicated than a basic direct variation or inverse variation model. One variable often depends on multiple other variables. When a variable is dependent on the product or quotient of two or more variables, this is called joint variation. For example, the cost of busing students for each school trip varies with the number of students attending and the distance from the school. The variable \( c \), cost, varies jointly with the number of students, \( n \), and the distance, \( d \).

**Joint Variation**

Joint variation occurs when a variable varies directly or inversely with multiple variables.

For instance, if \( x \) varies directly with both \( y \) and \( z \), we have \( x = kyz \). If \( x \) varies directly with \( y \) and inversely with \( z \), we have \( x = \frac{ky}{z} \). Notice that we only use one constant in a joint variation equation.

**Example 3.82**

**Solving Problems Involving Joint Variation**

A quantity \( x \) varies directly with the square of \( y \) and inversely with the cube root of \( z \). If \( x = 6 \) when \( y = 2 \) and \( z = 8 \), find \( x \) when \( y = 1 \) and \( z = 27 \).
3.53 \( x \) varies directly with the square of \( y \) and inversely with \( z \). If \( x = 40 \) when \( y = 4 \) and \( z = 2 \), find \( x \) when \( y = 10 \) and \( z = 25 \).

Access these online resources for additional instruction and practice with direct and inverse variation.

- Direct Variation (http://openstaxcollege.org/l/directvariation)
- Inverse Variation (http://openstaxcollege.org/l/inversevariatio)
- Direct and Inverse Variation (http://openstaxcollege.org/l/directinverse)
3.9 EXERCISES

Verbal

609. What is true of the appearance of graphs that reflect a direct variation between two variables?
610. If two variables vary inversely, what will an equation representing their relationship look like?
611. Is there a limit to the number of variables that can jointly vary? Explain.

Algebraic
For the following exercises, write an equation describing the relationship of the given variables.

612. \( y \) varies directly as \( x \) and when \( x = 6 \), \( y = 12 \).
613. \( y \) varies directly as the square of \( x \) and when \( x = 4 \), \( y = 80 \).
614. \( y \) varies directly as the square root of \( x \) and when \( x = 36 \), \( y = 24 \).
615. \( y \) varies directly as the cube of \( x \) and when \( x = 36 \), \( y = 24 \).
616. \( y \) varies directly as the cube root of \( x \) and when \( x = 27 \), \( y = 15 \).
617. \( y \) varies directly as the fourth power of \( x \) and when \( x = 1 \), \( y = 6 \).
618. \( y \) varies inversely as \( x \) and when \( x = 4 \), \( y = 2 \).
619. \( y \) varies inversely as the square of \( x \) and when \( x = 3 \), \( y = 2 \).
620. \( y \) varies inversely as the cube of \( x \) and when \( x = 2 \), \( y = 5 \).
621. \( y \) varies inversely as the fourth power of \( x \) and when \( x = 3 \), \( y = 1 \).
622. \( y \) varies inversely as the square root of \( x \) and when \( x = 25 \), \( y = 3 \).
623. \( y \) varies inversely as the cube root of \( x \) and when \( x = 64 \), \( y = 5 \).
624. \( y \) varies jointly with \( x \) and \( z \) and when \( x = 2 \) and \( z = 3 \), \( y = 36 \).
625. \( y \) varies jointly as \( x \), \( z \), and \( w \) and when \( x = 1 \), \( z = 2 \), \( w = 5 \), then \( y = 100 \).
626. \( y \) varies jointly as the square of \( x \) and the square of \( z \) and when \( x = 3 \) and \( z = 4 \), then \( y = 72 \).
627. \( y \) varies jointly as \( x \) and the square root of \( z \) and when \( x = 2 \) and \( z = 25 \), then \( y = 100 \).
628. \( y \) varies jointly as the square of \( x \) the cube of \( z \) and the square root of \( W \). When \( x = 1 \), \( z = 2 \), and \( W = 36 \), then \( y = 48 \).
629. \( y \) varies jointly as \( x \) and \( z \) and inversely as \( w \). When \( x = 3 \), \( z = 5 \), and \( w = 6 \), then \( y = 10 \).
630. \( y \) varies jointly as the square of \( x \) and the square root of \( z \) and inversely as the cube of \( w \). When \( x = 3 \), \( z = 4 \), and \( w = 3 \), then \( y = 6 \).
631. \( y \) varies jointly as \( x \) and \( z \) and inversely as the square root of \( w \) and the square of \( t \). When \( x = 3 \), \( z = 1 \), \( w = 25 \), and \( t = 2 \), then \( y = 6 \).
Numeric
For the following exercises, use the given information to find the unknown value.

632. \( y \) varies directly as \( x \). When \( x = 3 \), then \( y = 12 \). Find \( y \) when \( x = 20 \).

633. \( y \) varies directly as the square of \( x \). When \( x = 2 \), then \( y = 16 \). Find \( y \) when \( x = 8 \).

634. \( y \) varies directly as the cube of \( x \). When \( x = 3 \), then \( y = 5 \). Find \( y \) when \( x = 4 \).

635. \( y \) varies directly as the square root of \( x \). When \( x = 16 \), then \( y = 4 \). Find \( y \) when \( x = 36 \).

636. \( y \) varies directly as the cube root of \( x \). When \( x = 125 \), then \( y = 15 \). Find \( y \) when \( x = 1,000 \).

637. \( y \) varies inversely with \( x \). When \( x = 3 \), then \( y = 2 \). Find \( y \) when \( x = 1 \).

638. \( y \) varies inversely with the square of \( x \). When \( x = 4 \), then \( y = 3 \). Find \( y \) when \( x = 2 \).

639. \( y \) varies inversely with the cube of \( x \). When \( x = 3 \), then \( y = 1 \). Find \( y \) when \( x = 1 \).

640. \( y \) varies inversely with the square root of \( x \). When \( x = 64 \), then \( y = 12 \). Find \( y \) when \( x = 36 \).

641. \( y \) varies inversely with the cube root of \( x \). When \( x = 27 \), then \( y = 5 \). Find \( y \) when \( x = 125 \).

642. \( y \) varies jointly as \( x \) and \( z \). When \( x = 4 \) and \( z = 2 \), then \( y = 16 \). Find \( y \) when \( x = 3 \) and \( z = 3 \).

643. \( y \) varies jointly as \( x \), \( z \), and \( w \). When \( x = 2 \), \( z = 1 \), and \( w = 12 \), then \( y = 72 \). Find \( y \) when \( x = 1 \), \( z = 2 \), and \( w = 3 \).

644. \( y \) varies jointly as \( x \) and the square of \( z \). When \( x = 2 \) and \( z = 4 \), then \( y = 144 \). Find \( y \) when \( x = 4 \) and \( z = 5 \).

645. \( y \) varies jointly as the square of \( x \) and the square root of \( z \). When \( x = 2 \) and \( z = 9 \), then \( y = 24 \). Find \( y \) when \( x = 3 \) and \( z = 25 \).

646. \( y \) varies jointly as \( x \) and \( z \) and inversely as \( w \). When \( x = 5 \), \( z = 2 \), and \( w = 20 \), then \( y = 4 \). Find \( y \) when \( x = 3 \) and \( z = 8 \), and \( w = 48 \).

647. \( y \) varies jointly as the square of \( x \) and the cube of \( z \) and inversely as the square root of \( w \). When \( x = 2 \), \( z = 2 \), and \( w = 64 \), then \( y = 12 \). Find \( y \) when \( x = 1 \), \( z = 3 \), and \( w = 4 \).

648. \( y \) varies jointly as the square of \( x \) and of \( z \) and inversely as the square root of \( w \) and of \( t \). When \( x = 2 \), \( z = 3 \), \( w = 16 \), and \( t = 3 \), then \( y = 1 \). Find \( y \) when \( x = 3 \), \( z = 2 \), \( w = 36 \), and \( t = 5 \).

Technology
For the following exercises, use a calculator to graph the equation implied by the given variation.

649. \( y \) varies directly with the square of \( x \) and when \( x = 2 \), \( y = 3 \).

650. \( y \) varies directly as the cube of \( x \) and when \( x = 2 \), \( y = 4 \).

651. \( y \) varies directly as the square root of \( x \) and when \( x = 36 \), \( y = 2 \).

652. \( y \) varies inversely with \( x \) and when \( x = 6 \), \( y = 2 \).

653. \( y \) varies inversely as the square of \( x \) and when \( x = 1 \), \( y = 4 \).
Extensions

For the following exercises, use Kepler’s Law, which states that the square of the time, \( T \), required for a planet to orbit the Sun varies directly with the cube of the mean distance, \( a \), that the planet is from the Sun.

654. Using the Earth’s time of 1 year and mean distance of 93 million miles, find the equation relating \( T \) and \( a \).

655. Use the result from the previous exercise to determine the time required for Mars to orbit the Sun if its mean distance is 142 million miles.

656. Using Earth’s distance of 150 million kilometers, find the equation relating \( T \) and \( a \).

657. Use the result from the previous exercise to determine the time required for Venus to orbit the Sun if its mean distance is 108 million kilometers.

658. Using Earth’s distance of 1 astronomical unit (A.U.), determine the time for Saturn to orbit the Sun if its mean distance is 9.54 A.U.

Real-World Applications

For the following exercises, use the given information to answer the questions.

659. The distance \( s \) that an object falls varies directly with the square of the time, \( t \), of the fall. If an object falls 16 feet in one second, how long for it to fall 144 feet?

660. The velocity \( v \) of a falling object varies directly to the time, \( t \), of the fall. If after 2 seconds, the velocity of the object is 64 feet per second, what is the velocity after 5 seconds?

661. The rate of vibration of a string under constant tension varies inversely with the length of the string. If a string is 24 inches long and vibrates 128 times per second, what is the length of a string that vibrates 64 times per second?

662. The volume of a gas held at constant temperature varies indirectly as the pressure of the gas. If the volume of a gas is 1200 cubic centimeters when the pressure is 200 millimeters of mercury, what is the volume when the pressure is 300 millimeters of mercury?

663. The weight of an object above the surface of the Earth varies inversely with the square of the distance from the center of the Earth. If a body weighs 50 pounds when it is 3960 miles from Earth’s center, what would it weigh it were 3970 miles from Earth’s center?

664. The intensity of light measured in foot-candles varies inversely with the square of the distance from the light source. Suppose the intensity of a light bulb is 0.08 foot-candles at a distance of 3 meters. Find the intensity level at 8 meters.

665. The current in a circuit varies inversely with its resistance measured in ohms. When the current in a circuit is 40 amperes, the resistance is 10 ohms. Find the current if the resistance is 12 ohms.

666. The force exerted by the wind on a plane surface varies jointly with the square of the velocity of the wind and with the area of the plane surface. If the area of the surface is 40 square feet surface and the wind velocity is 20 miles per hour, the resulting force is 15 pounds. Find the force on a surface of 65 square feet with a velocity of 30 miles per hour.

667. The horsepower (hp) that a shaft can safely transmit varies jointly with its speed (in revolutions per minute (rpm)) and the cube of the diameter. If the shaft of a certain material 3 inches in diameter can transmit 45 hp at 100 rpm, what must the diameter be in order to transmit 60 hp at 150 rpm?

668. The kinetic energy \( K \) of a moving object varies jointly with its mass \( m \) and the square of its velocity \( v \). If an object weighing 40 kilograms with a velocity of 15 meters per second has a kinetic energy of 1000 joules, find the kinetic energy if the velocity is increased to 20 meters per second.
CHAPTER 3 REVIEW

KEY TERMS

**arrow notation:** a way to symbolically represent the local and end behavior of a function by using arrows to indicate that an input or output approaches a value

**axis of symmetry:** a vertical line drawn through the vertex of a parabola around which the parabola is symmetric; it is defined by \( x = -\frac{b}{2a} \).

**coefficient:** a nonzero real number multiplied by a variable raised to an exponent

**complex conjugate:** the complex number in which the sign of the imaginary part is changed and the real part of the number is left unchanged; when added to or multiplied by the original complex number, the result is a real number

**complex number:** the sum of a real number and an imaginary number, written in the standard form \( a + bi \), where \( a \) is the real part, and \( bi \) is the imaginary part

**complex plane:** a coordinate system in which the horizontal axis is used to represent the real part of a complex number and the vertical axis is used to represent the imaginary part of a complex number

**constant of variation:** the non-zero value \( k \) that helps define the relationship between variables in direct or inverse variation

**continuous function:** a function whose graph can be drawn without lifting the pen from the paper because there are no breaks in the graph

**Descartes’ Rule of Signs:** a rule that determines the maximum possible numbers of positive and negative real zeros based on the number of sign changes of \( f(x) \) and \( f(-x) \)

**Division Algorithm:** given a polynomial dividend \( f(x) \) and a non-zero polynomial divisor \( d(x) \) where the degree of \( d(x) \) is less than or equal to the degree of \( f(x) \), there exist unique polynomials \( q(x) \) and \( r(x) \) such that \( f(x) = d(x)q(x) + r(x) \) where \( q(x) \) is the quotient and \( r(x) \) is the remainder. The remainder is either equal to zero or has degree strictly less than \( d(x) \).

**degree:** the highest power of the variable that occurs in a polynomial

**direct variation:** the relationship between two variables that are a constant multiple of each other; as one quantity increases, so does the other

**end behavior:** the behavior of the graph of a function as the input decreases without bound and increases without bound

**Factor Theorem:** \( k \) is a zero of polynomial function \( f(x) \) if and only if \( (x - k) \) is a factor of \( f(x) \)

**Fundamental Theorem of Algebra:** a polynomial function with degree greater than 0 has at least one complex zero

**general form of a quadratic function:** the function that describes a parabola, written in the form \( f(x) = ax^2 + bx + c \), where \( a \), \( b \), and \( c \) are real numbers and \( a \neq 0 \).

**global maximum:** highest turning point on a graph; \( f(a) \) where \( f(a) \geq f(x) \) for all \( x \).

**global minimum:** lowest turning point on a graph; \( f(a) \) where \( f(a) \leq f(x) \) for all \( x \).

**horizontal asymptote:** a horizontal line \( y = b \) where the graph approaches the line as the inputs increase or decrease without bound.
**Intermediate Value Theorem:** for two numbers \( a \) and \( b \) in the domain of \( f \), if \( a < b \) and \( f(a) \neq f(b) \), then the function \( f \) takes on every value between \( f(a) \) and \( f(b) \); specifically, when a polynomial function changes from a negative value to a positive value, the function must cross the \( x \)-axis.

**imaginary number:** a number in the form \( bi \) where \( i = \sqrt{-1} \)

**inverse variation:** the relationship between two variables in which the product of the variables is a constant

**inversely proportional:** a relationship where one quantity is a constant divided by the other quantity; as one quantity increases, the other decreases

**invertible function:** any function that has an inverse function

**joint variation:** a relationship where a variable varies directly or inversely with multiple variables

**Linear Factorization Theorem:** allowing for multiplicities, a polynomial function will have the same number of factors as its degree, and each factor will be in the form \((x - c)\), where \( c \) is a complex number

**leading coefficient:** the coefficient of the leading term

**leading term:** the term containing the highest power of the variable

**multiplicity:** the number of times a given factor appears in the factored form of the equation of a polynomial; if a polynomial contains a factor of the form \((x - h)^p\), \( x = h \) is a zero of multiplicity \( p \).

**polynomial function:** a function that consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

**power function:** a function that can be represented in the form \( f(x) = kx^p \) where \( k \) is a constant, the base is a variable, and the exponent, \( p \), is a constant

**Rational Zero Theorem:** the possible rational zeros of a polynomial function have the form \( \frac{p}{q} \) where \( p \) is a factor of the constant term and \( q \) is a factor of the leading coefficient.

**Remainder Theorem:** if a polynomial \( f(x) \) is divided by \( x - k \), then the remainder is equal to the value \( f(k) \)

**rational function:** a function that can be written as the ratio of two polynomials

**removable discontinuity:** a single point at which a function is undefined that, if filled in, would make the function continuous; it appears as a hole on the graph of a function

**roots:** in a given function, the values of \( x \) at which \( y = 0 \), also called zeros

**smooth curve:** a graph with no sharp corners

**standard form of a quadratic function:** the function that describes a parabola, written in the form \( f(x) = a(x - h)^2 + k \), where \( (h, k) \) is the vertex.

**synthetic division:** a shortcut method that can be used to divide a polynomial by a binomial of the form \( x - k \)

**term of a polynomial function:** any \( a_i x^i \) of a polynomial function in the form \( f(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0 \)

**turning point:** the location at which the graph of a function changes direction

**varies directly:** a relationship where one quantity is a constant multiplied by the other quantity

**varies inversely:** a relationship where one quantity is a constant divided by the other quantity
vertex form of a quadratic function: another name for the standard form of a quadratic function

vertex: the point at which a parabola changes direction, corresponding to the minimum or maximum value of the quadratic function

vertical asymptote: a vertical line \( x = a \) where the graph tends toward positive or negative infinity as the inputs approach \( a \)

zeros: in a given function, the values of \( x \) at which \( y = 0 \), also called roots

**KEY EQUATIONS**

<table>
<thead>
<tr>
<th>general form of a quadratic function</th>
<th>( f(x) = ax^2 + bx + c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>the quadratic formula</td>
<td>( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} )</td>
</tr>
<tr>
<td>standard form of a quadratic function</td>
<td>( f(x) = a(x - h)^2 + k )</td>
</tr>
</tbody>
</table>

Table 3.15

<table>
<thead>
<tr>
<th>general form of a polynomial function</th>
<th>( f(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0 )</th>
</tr>
</thead>
</table>

Table 3.16

<table>
<thead>
<tr>
<th>Division Algorithm</th>
<th>( f(x) = d(x)q(x) + r(x) ) where ( q(x) \neq 0 )</th>
</tr>
</thead>
</table>

Table 3.17

<table>
<thead>
<tr>
<th>Rational Function</th>
<th>( f(x) = \frac{p(x)}{Q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0}{b_q x^q + b_{q-1} x^{q-1} + \ldots + b_1 x + b_0} ) where ( Q(x) \neq 0 )</th>
</tr>
</thead>
</table>

Table 3.18

<table>
<thead>
<tr>
<th>Direct variation</th>
<th>( y = kx^n ), ( k ) is a nonzero constant.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inverse variation</td>
<td>( y = \frac{k}{x^n} ), ( k ) is a nonzero constant.</td>
</tr>
</tbody>
</table>

Table 3.19
KEY CONCEPTS

3.1 Complex Numbers

- The square root of any negative number can be written as a multiple of \( i \). See Example 3.1.
- To plot a complex number, we use two number lines, crossed to form the complex plane. The horizontal axis is the real axis, and the vertical axis is the imaginary axis. See Example 3.2.
- Complex numbers can be added and subtracted by combining the real parts and combining the imaginary parts. See Example 3.3.
- Complex numbers can be multiplied and divided.
- To multiply complex numbers, distribute just as with polynomials. See Example 3.4, Example 3.5, and Example 3.8.
- To divide complex numbers, multiply both the numerator and denominator by the complex conjugate of the denominator to eliminate the complex number from the denominator. See Example 3.6, Example 3.7, and Example 3.9.
- The powers of \( i \) are cyclic, repeating every fourth one. See Example 3.10.

3.2 Quadratic Functions

- A polynomial function of degree two is called a quadratic function.
- The graph of a quadratic function is a parabola. A parabola is a U-shaped curve that can open either up or down.
- The axis of symmetry is the vertical line passing through the vertex. The zeros, or \( x \)-intercepts, are the points at which the parabola crosses the \( x \)-axis. The \( y \)-intercept is the point at which the parabola crosses the \( y \)-axis. See Example 3.11, Example 3.17, and Example 3.18.
- Quadratic functions are often written in general form. Standard or vertex form is useful to easily identify the vertex of a parabola. Either form can be written from a graph. See Example 3.12.
- The vertex can be found from an equation representing a quadratic function. See Example 3.13.
- The domain of a quadratic function is all real numbers. The range varies with the function. See Example 3.14.
- A quadratic function’s minimum or maximum value is given by the \( y \)-value of the vertex.
- The minimum or maximum value of a quadratic function can be used to determine the range of the function and to solve many kinds of real-world problems, including problems involving area and revenue. See Example 3.15 and Example 3.16.
- Some quadratic equations must be solved by using the quadratic formula. See Example 3.19.
- The vertex and the intercepts can be identified and interpreted to solve real-world problems. See Example 3.20.

3.3 Power Functions and Polynomial Functions

- A power function is a variable base raised to a number power. See Example 3.21.
- The behavior of a graph as the input decreases beyond bound and increases beyond bound is called the end behavior.
- The end behavior depends on whether the power is even or odd. See Example 3.22 and Example 3.23.
- A polynomial function is the sum of terms, each of which consists of a transformed power function with positive whole number power. See Example 3.24.
- The degree of a polynomial function is the highest power of the variable that occurs in a polynomial. The term containing the highest power of the variable is called the leading term. The coefficient of the leading term is called the leading coefficient. See Example 3.25.
- The end behavior of a polynomial function is the same as the end behavior of the power function represented by the leading term of the function. See Example 3.26 and Example 3.27.
- A polynomial of degree \( n \) will have at most \( n \) \( x \)-intercepts and at most \( n - 1 \) turning points. See Example 3.28, Example 3.29, Example 3.30, Example 3.31, and Example 3.32.
3.4 Graphs of Polynomial Functions

- Polynomial functions of degree 2 or more are smooth, continuous functions. See Example 3.33.
- To find the zeros of a polynomial function, if it can be factored, factor the function and set each factor equal to zero. See Example 3.34, Example 3.35, and Example 3.36.
- Another way to find the x-intercepts of a polynomial function is to graph the function and identify the points at which the graph crosses the x-axis. See Example 3.37.
- The multiplicity of a zero determines how the graph behaves at the x-intercepts. See Example 3.38.
- The graph of a polynomial will cross the horizontal axis at a zero with odd multiplicity.
- The graph of a polynomial will touch the horizontal axis at a zero with even multiplicity.
- The end behavior of a polynomial function depends on the leading term.
- The graph of a polynomial function changes direction at its turning points.
- A polynomial function of degree \( n \) has at most \( n - 1 \) turning points. See Example 3.39.
- To graph polynomial functions, find the zeros and their multiplicities, determine the end behavior, and ensure that the final graph has at most \( n - 1 \) turning points. See Example 3.40 and Example 3.42.
- Graphing a polynomial function helps to estimate local and global extremas. See Example 3.43.
- The Intermediate Value Theorem tells us that if \( f(a) \) and \( f(b) \) have opposite signs, then there exists at least one value \( c \) between \( a \) and \( b \) for which \( f(c) = 0 \). See Example 3.41.

3.5 Dividing Polynomials

- Polynomial long division can be used to divide a polynomial by any polynomial with equal or lower degree. See Example 3.44 and Example 3.45.
- The Division Algorithm tells us that a polynomial dividend can be written as the product of the divisor and the quotient added to the remainder.
- Synthetic division is a shortcut that can be used to divide a polynomial by a binomial in the form \( x - k \). See Example 3.46, Example 3.47, and Example 3.48.
- Polynomial division can be used to solve application problems, including area and volume. See Example 3.49.

3.6 Zeros of Polynomial Functions

- To find \( f(k) \), determine the remainder of the polynomial \( f(x) \) when it is divided by \( x - k \). See Example 3.50.
- \( k \) is a zero of \( f(x) \) if and only if \( x - k \) is a factor of \( f(x) \). See Example 3.51.
- Each rational zero of a polynomial function with integer coefficients will be equal to a factor of the constant term divided by a factor of the leading coefficient. See Example 3.52 and Example 3.53.
- When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.
- Synthetic division can be used to find the zeros of a polynomial function. See Example 3.54.
- According to the Fundamental Theorem, every polynomial function has at least one complex zero. See Example 3.55.
- Every polynomial function with degree greater than 0 has at least one complex zero.
- Allowing for multiplicities, a polynomial function will have the same number of factors as its degree. Each factor will be in the form \( x - c \), where \( c \) is a complex number. See Example 3.56.
- The number of positive real zeros of a polynomial function is either the number of sign changes of the function or less than the number of sign changes by an even integer.
- The number of negative real zeros of a polynomial function is either the number of sign changes of \( f(-x) \) or less than the number of sign changes by an even integer. See Example 3.57.
- Polynomial equations model many real-world scenarios. Solving the equations is easiest done by synthetic division. See Example 3.58.
3.7 Rational Functions

- We can use arrow notation to describe local behavior and end behavior of the toolkit functions \( f(x) = \frac{1}{x} \) and \( f(x) = \frac{1}{x^2} \). See Example 3.59.

- A function that levels off at a horizontal value has a horizontal asymptote. A function can have more than one vertical asymptote. See Example 3.60.

- Application problems involving rates and concentrations often involve rational functions. See Example 3.61.

- The domain of a rational function includes all real numbers except those that cause the denominator to equal zero. See Example 3.62.

- The vertical asymptotes of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero. See Example 3.63.

- A removable discontinuity might occur in the graph of a rational function if an input causes both numerator and denominator to be zero. See Example 3.64.

- A rational function’s end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions. See Example 3.65, Example 3.66, Example 3.67, and Example 3.68.

- Graph rational functions by finding the intercepts, behavior at the intercepts and asymptotes, and end behavior. See Example 3.69.

- If a rational function has \( x \)-intercepts at \( x = x_1, x_2, \ldots, x_n \), vertical asymptotes at \( x = v_1, v_2, \ldots, v_m \), and no \( x_i = \) any \( v_j \) then the function can be written in the form

\[
 f(x) = \frac{a(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n}}{(x - v_1)^{q_1}(x - v_2)^{q_2} \cdots (x - v_m)^{q_n}}
\]

See Example 3.70.

3.8 Inverses and Radical Functions

- The inverse of a quadratic function is a square root function.

- If \( f^{-1} \) is the inverse of a function \( f \), then \( f \) is the inverse of the function \( f^{-1} \). See Example 3.71.

- While it is not possible to find an inverse of most polynomial functions, some basic polynomials are invertible. See Example 3.72.

- To find the inverse of certain functions, we must restrict the function to a domain on which it will be one-to-one. See Example 3.73 and Example 3.74.

- When finding the inverse of a radical function, we need a restriction on the domain of the answer. See Example 3.75 and Example 3.77.

- Inverse and radical functions can be used to solve application problems. See Example 3.76 and Example 3.78.

3.9 Modeling Using Variation

- A relationship where one quantity is a constant multiplied by another quantity is called direct variation. See Example 3.79.

- Two variables that are directly proportional to one another will have a constant ratio.

- A relationship where one quantity is a constant divided by another quantity is called inverse variation. See Example 3.80.

- Two variables that are inversely proportional to one another will have a constant multiple. See Example 3.81.

- In many problems, a variable varies directly or inversely with multiple variables. We call this type of relationship joint variation. See Example 3.82.
You have reached the end of Chapter 3: Polynomial and Rational Functions. Let’s review some of the Key Terms, Concepts and Equations you have learned.

m10378 (http://legacy.cnx.org/content/m10378/latest/)
Perform the indicated operation with complex numbers.

722. \((4 + 3i) + (-2 - 5i)\)

723. \((6 - 5i) - (10 + 3i)\)

724. \((2 - 3i)(3 + 6i)\)

725. \(\frac{2 - i}{2 + i}\)

Solve the following equations over the complex number system.

726. \(x^2 - 4x + 5 = 0\)

727. \(x^2 + 2x + 10 = 0\)

m10384 (http://legacy.cnx.org/content/m10384/latest/)
For the following exercises, write the quadratic function in standard form. Then, give the vertex and axes intercepts. Finally, graph the function.

728. \(f(x) = x^2 - 4x - 5\)

729. \(f(x) = -2x^2 - 4x\)

For the following problems, find the equation of the quadratic function using the given information.

730. The vertex is \((-2, 3)\) and a point on the graph is \((3, 6)\).

731. The vertex is \((-3, 6.5)\) and a point on the graph is \((2, 6)\).

Answer the following questions.

732. A rectangular plot of land is to be enclosed by fencing. One side is along a river and so needs no fence. If the total fencing available is 600 meters, find the dimensions of the plot to have maximum area.

733. An object projected from the ground at a 45 degree angle with initial velocity of 120 feet per second has height, \(h\), in terms of horizontal distance traveled, \(x\), given by \(h(x) = \frac{-32}{(120)^2}x^2 + x\). Find the maximum height the object attains.

m10387 (http://legacy.cnx.org/content/m10387/latest/)
For the following exercises, determine if the function is a polynomial function and, if so, give the degree and leading coefficient.

734. \(f(x) = 4x^5 - 3x^3 + 2x - 1\)

735. \(f(x) = 5x + 1 - x^2\)

736. \(f(x) = x^2(3 - 6x + x^2)\)

For the following exercises, determine end behavior of the polynomial function.
737. \( f(x) = 2x^4 + 3x^3 - 5x^2 + 7 \)

738. \( f(x) = 4x^3 - 6x^2 + 2 \)

739. \( f(x) = 2x^2(1 + 3x - x^2) \)

m10383 (http://legacy.cnx.org/content/m10383/latest/)

For the following exercises, find all zeros of the polynomial function, noting multiplicities.

740. \( f(x) = (x + 3)^2(2x - 1)(x + 1)^3 \)

741. \( f(x) = x^5 + 4x^4 + 4x^3 \)

742. \( f(x) = x^3 - 4x^2 + x - 4 \)

For the following exercises, based on the given graph, determine the zeros of the function and note multiplicity.

743.

744.

745. Use the Intermediate Value Theorem to show that at least one zero lies between 2 and 3 for the function \( f(x) = x^3 - 5x + 1 \)

m10379 (http://legacy.cnx.org/content/m10379/latest/)

For the following exercises, use long division to find the quotient and remainder.

746. \( \frac{x^3 - 2x^2 + 4x + 4}{x - 2} \)
For the following exercises, use synthetic division to find the quotient. If the divisor is a factor, then write the factored form.

747. \[ \frac{3x^4 - 4x^2 + 4x + 8}{x + 1} \]

For the following exercises, use the Rational Zero Theorem to help you solve the polynomial equation.

752. \[ 2x^3 - 3x^2 - 18x - 8 = 0 \]
753. \[ 3x^3 + 11x^2 + 8x - 4 = 0 \]
754. \[ 2x^4 - 17x^3 + 46x^2 - 43x + 12 = 0 \]
755. \[ 4x^4 + 8x^3 + 19x^2 + 32x + 12 = 0 \]

For the following exercises, use Descartes’ Rule of Signs to find the possible number of positive and negative solutions.

756. \[ x^3 - 3x^2 - 2x + 4 = 0 \]
757. \[ 2x^4 - x^3 + 4x^2 - 5x + 1 = 0 \]

For the following rational functions, find the intercepts and the vertical and horizontal asymptotes, and then use them to sketch a graph.

758. \[ f(x) = \frac{x + 2}{x - 5} \]
759. \[ f(x) = \frac{x^2 + 1}{x^2 - 4} \]
760. \[ f(x) = \frac{3x^2 - 27}{x^2 + x - 2} \]
761. \[ f(x) = \frac{x + 2}{x^2 - 9} \]

For the following exercises, find the slant asymptote.

762. \[ f(x) = \frac{x^2 - 1}{x + 2} \]
For the following exercises, find the inverse of the function with the domain given.

764. \( f(x) = (x - 2)^2, \quad x \geq 2 \)

765. \( f(x) = (x + 4)^2 - 3, \quad x \geq -4 \)

766. \( f(x) = x^2 + 6x - 2, \quad x \geq -3 \)

767. \( f(x) = 2x^3 - 3 \)

768. \( f(x) = \sqrt{4x + 5} - 3 \)

769. \( f(x) = \frac{x - 3}{2x + 1} \)

m10390 (http://legacy.cnx.org/content/m10390/latest/)

For the following exercises, find the unknown value.

770. \( y \) varies directly as the square of \( x \). If when \( x = 3 \), \( y = 36 \), find \( y \) if \( x = 4 \).

771. \( y \) varies inversely as the square root of \( x \). If when \( x = 25 \), \( y = 2 \), find \( y \) if \( x = 4 \).

772. \( y \) varies jointly as the cube of \( x \) and as \( z \). If when \( x = 1 \) and \( z = 2 \), \( y = 6 \), find \( y \) if \( x = 2 \) and \( z = 3 \).

773. \( y \) varies jointly as \( x \) and the square of \( z \) and inversely as the cube of \( w \). If when \( x = 3 \), \( z = 4 \), and \( w = 2 \), \( y = 48 \), find \( y \) if \( x = 4 \), \( z = 5 \), and \( w = 3 \).

For the following exercises, solve the application problem.

774. The weight of an object above the surface of the earth varies inversely with the distance from the center of the earth. If a person weighs 150 pounds when he is on the surface of the earth (3,960 miles from center), find the weight of the person if he is 20 miles above the surface.

775. The volume \( V \) of an ideal gas varies directly with the temperature \( T \) and inversely with the pressure \( P \). A cylinder contains oxygen at a temperature of 310 degrees K and a pressure of 18 atmospheres in a volume of 120 liters. Find the pressure if the volume is decreased to 100 liters and the temperature is increased to 320 degrees K.

CHAPTER 3 PRACTICE TEST

Perform the indicated operation or solve the equation.

723. \((3 - 4i)(4 + 2i)\)

724. \(\frac{1 - 4i}{3 + 4i}\)

725. \(x^2 - 4x + 13 = 0\)

Give the degree and leading coefficient of the following polynomial function.
726. \( f(x) = x^3(3 - 6x^2 - 2x^3) \)

Determine the end behavior of the polynomial function.

727. \( f(x) = 8x^3 - 3x^2 + 2x - 4 \)

728. \( f(x) = -2x^2(4 - 3x - 5x^2) \)

Write the quadratic function in standard form. Determine the vertex and axes intercepts and graph the function.

729. \( f(x) = x^2 + 2x - 8 \)

Given information about the graph of a quadratic function, find its equation.

730. Vertex (2, 0) and point on graph (4, 12).

Solve the following application problem.

731. A rectangular field is to be enclosed by fencing. In addition to the enclosing fence, another fence is to divide the field into two parts, running parallel to two sides. If 1,200 feet of fencing is available, find the maximum area that can be enclosed.

Find all zeros of the following polynomial functions, noting multiplicities.

732. \( f(x) = (x - 3)^3(3x - 1)(x - 1)^2 \)

733. \( f(x) = 2x^6 - 6x^5 + 18x^4 \)

Based on the graph, determine the zeros of the function and multiplicities.

734.

Use long division to find the quotient.

735. \( \frac{2x^3 + 3x - 4}{x + 2} \)

Use synthetic division to find the quotient. If the divisor is a factor, write the factored form.

736. \( \frac{x^4 + 3x^2 - 4}{x - 2} \)

737. \( \frac{2x^3 + 5x^2 - 7x - 12}{x + 3} \)

Use the Rational Zero Theorem to help you find the zeros of the polynomial functions.

738. \( f(x) = 2x^3 + 5x^2 - 6x - 9 \)
739. \( f(x) = 4x^4 + 8x^3 + 21x^2 + 17x + 4 \)

740. \( f(x) = 4x^4 + 16x^3 + 13x^2 - 15x - 18 \)

741. \( f(x) = x^5 + 6x^4 + 13x^3 + 14x^2 + 12x + 8 \)

Given the following information about a polynomial function, find the function.

742. It has a double zero at \( x = 3 \) and zeroes at \( x = 1 \) and \( x = -2 \). It’s \( y \)-intercept is \((0, 12)\).

743. It has a zero of multiplicity 3 at \( x = \frac{1}{2} \) and another zero at \( x = -3 \). It contains the point \((1, 8)\).

Use Descartes’ Rule of Signs to determine the possible number of positive and negative solutions.

744. \( 8x^3 - 21x^2 + 6 = 0 \)

For the following rational functions, find the intercepts and horizontal and vertical asymptotes, and sketch a graph.

745. \( f(x) = \frac{x + 4}{x^2 - 2x - 3} \)

746. \( f(x) = \frac{x^2 + 2x - 3}{x^2 - 4} \)

Find the slant asymptote of the rational function.

747. \( f(x) = \frac{x^2 + 3x - 3}{x - 1} \)

Find the inverse of the function.

748. \( f(x) = \sqrt{x - 2} + 4 \)

749. \( f(x) = 3x^3 - 4 \)

750. \( f(x) = \frac{2x + 3}{3x - 1} \)

Find the unknown value.

751. \( y \) varies inversely as the square of \( x \) and when \( x = 3 \), \( y = 2 \). Find \( y \) if \( x = 1 \).

752. \( y \) varies jointly with \( x \) and the cube root of \( z \). If when \( x = 2 \) and \( z = 27 \), \( y = 12 \), find \( y \) if \( x = 5 \) and \( z = 8 \).

Solve the following application problem.

753. The distance a body falls varies directly as the square of the time it falls. If an object falls 64 feet in 2 seconds, how long will it take to fall 256 feet?
4 | EXPONENTIAL AND LOGARITHMIC FUNCTIONS

4.1 | Introduction to Exponential and Logarithmic Functions

4.2 | Exponential Functions

4.3 | Graphs of Exponential Functions

4.4 | Logarithmic Functions

4.5 | Graphs of Logarithmic Functions

4.6 | Logarithmic Properties

4.7 | Exponential and Logarithmic Equations
4.8 | Exponential and Logarithmic Models

Learning Objectives

4.9 | Fitting Exponentials to Data

Learning Objectives
5 | TRIGONOMETRIC FUNCTIONS

5.1 | Introduction to Trigonometric Functions

5.2 | Angles

5.3 | Unit Circle: Sine and Cosine Functions

5.4 | The Other Trigonometric Functions

5.5 | Right Triangle Trigonometry
Introduction

Each day, the sun rises in an easterly direction, approaches some maximum height relative to the celestial equator, and sets in a westerly direction. The celestial equator is an imaginary line that divides the visible universe into two halves in much the same way Earth’s equator is an imaginary line that divides the planet into two halves. The exact path the sun appears to follow depends on the exact location on Earth, but each location observes a predictable pattern over time.

The pattern of the sun’s motion throughout the course of a year is a periodic function. Creating a visual representation of a periodic function in the form of a graph can help us analyze the properties of the function. In this chapter, we will investigate graphs of sine, cosine, and other trigonometric functions.

6.1 | Graphs of the Sine and Cosine Functions

Learning Objectives

In this section, you will:

6.1.1 Graph variations of $y=\sin(x)$ and $y=\cos(x)$.
6.1.2 Use phase shifts of sine and cosine curves.
White light, such as the light from the sun, is not actually white at all. Instead, it is a composition of all the colors of the rainbow in the form of waves. The individual colors can be seen only when white light passes through an optical prism that separates the waves according to their wavelengths to form a rainbow.

Light waves can be represented graphically by the sine function. In the chapter on **Trigonometric Functions**, we examined trigonometric functions such as the sine function. In this section, we will interpret and create graphs of sine and cosine functions.

### Graphing Sine and Cosine Functions

Recall that the sine and cosine functions relate real number values to the $x$- and $y$-coordinates of a point on the unit circle. So what do they look like on a graph on a coordinate plane? Let’s start with the sine function. We can create a table of values and use them to sketch a graph. **Table 6.1** lists some of the values for the sine function on a unit circle.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>$\frac{\pi}{6}$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{\pi}{3}$</th>
<th>$\frac{\pi}{2}$</th>
<th>$\frac{2\pi}{3}$</th>
<th>$\frac{3\pi}{4}$</th>
<th>$\frac{5\pi}{6}$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin(x)$</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>1</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{\sqrt{2}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
</tr>
</tbody>
</table>

**Table 6.1**

Plotting the points from the table and continuing along the $x$-axis gives the shape of the sine function. See **Figure 6.3**.

Notice how the sine values are positive between 0 and $\pi$, which correspond to the values of the sine function in quadrants I and II on the unit circle, and the sine values are negative between $\pi$ and $2\pi$, which correspond to the values of the sine function in quadrants III and IV on the unit circle. See **Figure 6.4**.

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**Figure 6.2** Light can be separated into colors because of its wavelike properties. (credit: “wonderferret”/Flickr)

**Figure 6.3** The sine function

**Figure 6.4**
Figure 6.4 Plotting values of the sine function

Now let’s take a similar look at the cosine function. Again, we can create a table of values and use them to sketch a graph. **Table 6.2** lists some of the values for the cosine function on a unit circle.

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>(\frac{\pi}{6})</th>
<th>(\frac{\pi}{4})</th>
<th>(\frac{\pi}{3})</th>
<th>(\frac{\pi}{2})</th>
<th>(\frac{2\pi}{3})</th>
<th>(\frac{3\pi}{4})</th>
<th>(\frac{5\pi}{6})</th>
<th>(\pi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\cos(x))</td>
<td>1</td>
<td>(\frac{\sqrt{3}}{2})</td>
<td>(\frac{\sqrt{2}}{2})</td>
<td>(\frac{1}{2})</td>
<td>0</td>
<td>(-\frac{1}{2})</td>
<td>(-\frac{\sqrt{2}}{2})</td>
<td>(-\frac{\sqrt{3}}{2})</td>
<td>-1</td>
</tr>
</tbody>
</table>

**Table 6.2**

As with the sine function, we can plot points to create a graph of the cosine function as in Figure 6.5.

Figure 6.5 The cosine function

Because we can evaluate the sine and cosine of any real number, both of these functions are defined for all real numbers. By thinking of the sine and cosine values as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval \([-1, 1]\).

In both graphs, the shape of the graph repeats after \(2\pi\), which means the functions are periodic with a period of \(2\pi\). A **periodic function** is a function for which a specific horizontal shift, \(P\), results in a function equal to the original function: \(f(x + P) = f(x)\) for all values of \(x\) in the domain of \(f\). When this occurs, we call the smallest such horizontal shift with \(P > 0\) the period of the function. Figure 6.6 shows several periods of the sine and cosine functions.
Looking again at the sine and cosine functions on a domain centered at the y-axis helps reveal symmetries. As we can see in **Figure 6.7**, the sine function is symmetric about the origin. Recall from *The Other Trigonometric Functions* that we determined from the unit circle that the sine function is an odd function because \( \sin(-x) = -\sin x \). Now we can clearly see this property from the graph.

**Figure 6.7**  Odd symmetry of the sine function

**Figure 6.8** shows that the cosine function is symmetric about the y-axis. Again, we determined that the cosine function is an even function. Now we can see from the graph that \( \cos(-x) = \cos x \).

**Figure 6.8**  Even symmetry of the cosine function

### Characteristics of Sine and Cosine Functions

The sine and cosine functions have several distinct characteristics:

- They are periodic functions with a period of \( 2\pi \).
- The domain of each function is \((\infty, \infty)\) and the range is \([-1, 1]\).
- The graph of \( y = \sin x \) is symmetric about the origin, because it is an odd function.
- The graph of \( y = \cos x \) is symmetric about the \( y \)-axis, because it is an even function.
Investigating Sinusoidal Functions

As we can see, sine and cosine functions have a regular period and range. If we watch ocean waves or ripples on a pond, we will see that they resemble the sine or cosine functions. However, they are not necessarily identical. Some are taller or longer than others. A function that has the same general shape as a sine or cosine function is known as a sinusoidal function. The general forms of sinusoidal functions are

\[
y = A \sin(Bx - C) + D
\]

and

\[
y = A \cos(Bx - C) + D
\]

Determining the Period of Sinusoidal Functions

Looking at the forms of sinusoidal functions, we can see that they are transformations of the sine and cosine functions. We can use what we know about transformations to determine the period.

In the general formula, \( B \) is related to the period by \( P = \frac{2\pi}{|B|} \). If \(|B| > 1\), then the period is less than \( 2\pi \) and the function undergoes a horizontal compression, whereas if \(|B| < 1\), then the period is greater than \( 2\pi \) and the function undergoes a horizontal stretch. For example, \( f(x) = \sin(x) \), \( B = 1 \), so the period is \( 2\pi \), which we knew. If \( f(x) = \sin(2x) \), then \( B = 2 \), so the period is \( \pi \) and the graph is compressed. If \( f(x) = \sin\left(\frac{x}{2}\right) \), then \( B = \frac{1}{2} \), so the period is \( 4\pi \) and the graph is stretched. Notice in Figure 6.9 how the period is indirectly related to \(|B|\).

![Figure 6.9](image)

**Figure 6.9**

**Period of Sinusoidal Functions**

If we let \( C = 0 \) and \( D = 0 \) in the general form equations of the sine and cosine functions, we obtain the forms

\[
y = A \sin(Bx)
\]

and

\[
y = A \cos(Bx)
\]

The period is \( \frac{2\pi}{|B|} \).

**Example 6.1**

**Identifying the Period of a Sine or Cosine Function**

Determine the period of the function \( f(x) = \sin\left(\frac{\pi}{6}x\right) \).

**Solution**
Determine the period of the function $g(x) = \cos\left(\frac{x}{3}\right)$.

**Determining Amplitude**

Returning to the general formula for a sinusoidal function, we have analyzed how the variable $B$ relates to the period. Now let’s turn to the variable $A$ so we can analyze how it is related to the amplitude, or greatest distance from rest. $A$ represents the vertical stretch factor, and its absolute value $|A|$ is the amplitude. The local maxima will be a distance $|A|$ above the vertical midline of the graph, which is the line $x = D$; because $D = 0$ in this case, the midline is the $x$-axis. The local minima will be the same distance below the midline. If $|A| > 1$, the function is stretched. For example, the amplitude of $f(x) = 4 \sin x$ is twice the amplitude of $f(x) = 2 \sin x$. If $|A| < 1$, the function is compressed. Figure 6.10 compares several sine functions with different amplitudes.

![Figure 6.10](Image)

**Amplitude of Sinusoidal Functions**

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

$$y = A \sin(Bx) \quad \text{and} \quad y = A \cos(Bx)$$

The **amplitude** is $A$, and the vertical height from the **midline** is $|A|$. In addition, notice in the example that

$$|A| = \text{amplitude} = \frac{1}{2} |\text{maximum} - \text{minimum}|$$

**Example 6.2**

**Identifying the Amplitude of a Sine or Cosine Function**

What is the amplitude of the sinusoidal function $f(x) = -4 \sin(x)$? Is the function stretched or compressed vertically?

**Solution**

The negative value of $A$ results in a reflection across the $x$-axis of the sine function, as shown in Figure 6.10.
6.2 What is the amplitude of the sinusoidal function \( f(x) = \frac{1}{2}\sin(x) \)? Is the function stretched or compressed vertically?

**Analyzing Graphs of Variations of \( y = \sin x \) and \( y = \cos x \)**

Now that we understand how \( A \) and \( B \) relate to the general form equation for the sine and cosine functions, we will explore the variables \( C \) and \( D \). Recall the general form:

\[
y = A\sin(Bx - C) + D \quad \text{and} \quad y = A\cos(Bx - C) + D
\]

The value \( \frac{C}{B} \) for a sinusoidal function is called the **phase shift**, or the horizontal displacement of the basic sine or cosine function. If \( C > 0 \), the graph shifts to the right. If \( C < 0 \), the graph shifts to the left. The greater the value of \(|C|\), the more the graph is shifted. **Figure 6.11** shows that the graph of \( f(x) = \sin(x - \pi) \) shifts to the right by \( \pi \) units, which is more than we see in the graph of \( f(x) = \sin\left(x - \frac{\pi}{4}\right) \) which shifts to the right by \( \frac{\pi}{4} \) units.

![Figure 6.11](image)

While \( C \) relates to the horizontal shift, \( D \) indicates the vertical shift from the midline in the general formula for a sinusoidal function. See **Figure 6.12**. The function \( y = \cos(x) + D \) has its midline at \( y = D \).
Any value of $D$ other than zero shifts the graph up or down. Figure 6.13 compares $f(x) = \sin x$ with $f(x) = \sin x + 2$, which is shifted 2 units up on a graph.

### Variations of Sine and Cosine Functions

Given an equation in the form $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$, $\frac{C}{B}$ is the **phase shift** and $D$ is the vertical shift.

### Example 6.3

**Identifying the Phase Shift of a Function**

Determine the direction and magnitude of the phase shift for $f(x) = \sin\left(x + \frac{\pi}{6}\right) - 2$.

**Solution**

**Analysis**

We must pay attention to the sign in the equation for the general form of a sinusoidal function. The equation shows a minus sign before $C$. Therefore $f(x) = \sin\left(x + \frac{\pi}{6}\right) - 2$ can be rewritten as $f(x) = \sin\left(x - \left(-\frac{\pi}{6}\right)\right) - 2$.

If the value of $C$ is negative, the shift is to the left.

**6.3** Determine the direction and magnitude of the phase shift for $f(x) = 3\cos\left(x - \frac{\pi}{2}\right)$.

### Example 6.4
Identifying the Vertical Shift of a Function

Determine the direction and magnitude of the vertical shift for \( f(x) = \cos(x) - 3 \).

Solution

Given a sinusoidal function in the form \( f(x) = A \sin(Bx - C) + D \), identify the midline, amplitude, period, and phase shift.

1. Determine the amplitude as \( |A| \).
2. Determine the period as \( P = \frac{2\pi}{|B|} \).
3. Determine the phase shift as \( \frac{C}{B} \).
4. Determine the midline as \( y = D \).

Example 6.5

Identifying the Variations of a Sinusoidal Function from an Equation

Determine the midline, amplitude, period, and phase shift of the function \( y = 3 \sin(2x) + 1 \).

Solution

Analysis

Inspecting the graph, we can determine that the period is \( \pi \), the midline is \( y = 1 \), and the amplitude is 3. See Figure 6.13.

Try it

Determine the midline, amplitude, period, and phase shift of the function \( y = \frac{1}{2} \cos\left(\frac{x}{3} - \frac{\pi}{3}\right) \).
Example 6.6

Identifying the Equation for a Sinusoidal Function from a Graph

Determine the formula for the cosine function in Figure 6.14.

Solution

Determine the formula for the sine function in Figure 6.15.

Example 6.7

Identifying the Equation for a Sinusoidal Function from a Graph

Determine the equation for the sinusoidal function in Figure 6.16.
6.7 Write a formula for the function graphed in Figure 6.17.

Graphing Variations of $y = \sin x$ and $y = \cos x$

Throughout this section, we have learned about types of variations of sine and cosine functions and used that information to write equations from graphs. Now we can use the same information to create graphs from equations.

Instead of focusing on the general form equations
we will let $C = 0$ and $D = 0$ and work with a simplified form of the equations in the following examples.

**Given the function** $y = A \sin(Bx)$, sketch its graph.

1. Identify the amplitude, $|A|$.
2. Identify the period, $P = \frac{2\pi}{|B|}$.
3. Start at the origin, with the function increasing to the right if $A$ is positive or decreasing if $A$ is negative.
4. At $x = \frac{\pi}{2|B|}$ there is a local maximum for $A > 0$ or a minimum for $A < 0$, with $y = A$.
5. The curve returns to the $x$-axis at $x = \frac{\pi}{|B|}$.
6. There is a local minimum for $A > 0$ (maximum for $A < 0$) at $x = \frac{3\pi}{2|B|}$ with $y = -A$.
7. The curve returns again to the $x$-axis at $x = \frac{\pi}{2|B|}$.

**Example 6.8**

**Graphing a Function and Identifying the Amplitude and Period**

Sketch a graph of $f(x) = -2 \sin \left( \frac{\pi x}{2} \right)$.

**Solution**

Sketch a graph of $g(x) = -0.8 \cos(2x)$. Determine the midline, amplitude, period, and phase shift.

**Example 6.9**

**Graphing a Transformed Sinusoid**

Sketch a graph of $f(x) = 3 \sin \left( \frac{\pi}{4}x - \frac{\pi}{4} \right)$.
6.9

Draw a graph of \( g(x) = -2\cos\left(\frac{\pi}{3}x + \frac{\pi}{6}\right) \). Determine the midline, amplitude, period, and phase shift.

Example 6.10

Identifying the Properties of a Sinusoidal Function

Given \( y = -2\cos\left(\frac{\pi}{2}x + \pi\right) + 3 \), determine the amplitude, period, phase shift, and horizontal shift. Then graph the function.

Solution

Using Transformations of Sine and Cosine Functions

We can use the transformations of sine and cosine functions in numerous applications. As mentioned at the beginning of the chapter, circular motion can be modeled using either the sine or cosine function.

Example 6.11

Finding the Vertical Component of Circular Motion

A point rotates around a circle of radius 3 centered at the origin. Sketch a graph of the y-coordinate of the point as a function of the angle of rotation.

Solution

Analysis

Notice that the period of the function is still \( 2\pi \); as we travel around the circle, we return to the point \((3, 0)\) for \( x = 2\pi, 4\pi, 6\pi, \ldots \). Because the outputs of the graph will now oscillate between \(-3\) and \(3\), the amplitude of the sine wave is \(3\).

6.10

What is the amplitude of the function \( f(x) = 7\cos(x) \)? Sketch a graph of this function.

Example 6.12

Finding the Vertical Component of Circular Motion
A circle with radius 3 ft is mounted with its center 4 ft off the ground. The point closest to the ground is labeled $P$, as shown in Figure 6.18. Sketch a graph of the height above the ground of the point $P$ as the circle is rotated; then find a function that gives the height in terms of the angle of rotation.

Solution

Consider the weight attached to a spring as shown in Figure 6.19. As the spring oscillates up and down, the position $y$ of the weight relative to the board ranges from $-1$ in. (at time $x = 0$) to $-7$ in. (at time $x = \pi$) below the board. Assume the position of $y$ is given as a sinusoidal function of $x$. Sketch a graph of the function, and then find a cosine function that gives the position $y$ in terms of $x$.

**Example 6.13**

**Determining a Rider’s Height on a Ferris Wheel**

The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider’s height above ground as a function of time in minutes.

Solution
Access these online resources for additional instruction and practice with graphs of sine and cosine functions.

- Amplitude and Period of Sine and Cosine (http://openstaxcollege.org/l/ampperiod)
- Translations of Sine and Cosine (http://openstaxcollege.org/l/translasincos)
- Graphing Sine and Cosine Transformations (http://openstaxcollege.org/l/transformsincos)
- Graphing the Sine Function (http://openstaxcollege.org/l/graphsinefunc)
6.1 EXERCISES

Verbal

1. Why are the sine and cosine functions called periodic functions?

2. How does the graph of \( y = \sin x \) compare with the graph of \( y = \cos x \)? Explain how you could horizontally translate the graph of \( y = \sin x \) to obtain \( y = \cos x \).

3. For the equation \( A \cos(Bx + C) + D \), what constants affect the range of the function and how do they affect the range?

4. How does the range of a translated sine function relate to the equation \( y = A \sin(Bx + C) + D \)?

5. How can the unit circle be used to construct the graph of \( f(t) = \sin t \)?

Graphical

For the following exercises, graph two full periods of each function and state the amplitude, period, and midline. State the maximum and minimum y-values and their corresponding x-values on one period for \( x > 0 \). Round answers to two decimal places if necessary.

6. \( f(x) = 2 \sin x \)

7. \( f(x) = \frac{2}{3} \cos x \)

8. \( f(x) = -3 \sin x \)

9. \( f(x) = 4 \sin x \)

10. \( f(x) = 2 \cos x \)

11. \( f(x) = \cos(2x) \)

12. \( f(x) = 2 \sin\left(\frac{1}{2}x\right) \)

13. \( f(x) = 4 \cos(\pi x) \)

14. \( f(x) = 3 \cos\left(\frac{6}{5}x\right) \)

15. \( y = 3 \sin(8(x + 4)) + 5 \)

16. \( y = 2 \sin(3x - 21) + 4 \)

17. \( y = 5 \sin(5x + 20) - 2 \)

For the following exercises, graph one full period of each function, starting at \( x = 0 \). For each function, state the amplitude, period, and midline. State the maximum and minimum y-values and their corresponding x-values on one period for \( x > 0 \). State the phase shift and vertical translation, if applicable. Round answers to two decimal places if necessary.

18. \( f(t) = 2 \sin(t - \frac{5\pi}{6}) \)

19. \( f(t) = -\cos\left(t + \frac{\pi}{3}\right) + 1 \)

20. \( f(t) = 4 \cos\left(2(t + \frac{\pi}{4})\right) - 3 \)
21. \( f(t) = -\sin\left(\frac{1}{2}t + \frac{5\pi}{3}\right) \)

22. \( f(x) = 4\sin\left(\frac{2}{5}(x - 3)\right) + 7 \)

23. Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in Figure 6.20.

![Figure 6.20](image)

24. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in Figure 6.21.

![Figure 6.21](image)

25. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in Figure 6.22.

![Figure 6.22](image)

26. Determine the amplitude, period, midline, and an equation involving sine for the graph shown in Figure 6.23.
27. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in Figure 6.24.

28. Determine the amplitude, period, midline, and an equation involving sine for the graph shown in Figure 6.25.

29. Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in Figure 6.26.

30. Determine the amplitude, period, midline, and an equation involving sine for the graph shown in Figure 6.27.
For the following exercises, let \( f(x) = \sin x \).

31. On \([0, 2\pi]\), solve \( f(x) = 0 \).

32. On \([0, 2\pi]\), solve \( f(x) = \frac{1}{2} \).

33. Evaluate \( f\left(\frac{\pi}{2}\right) \).

34. On \([0, 2\pi]\), \( f(x) = \frac{\sqrt{2}}{2} \). Find all values of \( x \).

35. On \([0, 2\pi]\), the maximum value(s) of the function occur(s) at what \( x \)-value(s)?

36. On \([0, 2\pi]\), the minimum value(s) of the function occur(s) at what \( x \)-value(s)?

37. Show that \( f(-x) = -f(x) \). This means that \( f(x) = \sin x \) is an odd function and possesses symmetry with respect to ____________________.

For the following exercises, let \( f(x) = \cos x \).

38. On \([0, 2\pi]\), solve the equation \( f(x) = \cos x = 0 \).

39. On \([0, 2\pi]\), solve \( f(x) = \frac{1}{2} \).

40. On \([0, 2\pi]\), find the \( x \)-intercepts of \( f(x) = \cos x \).

41. On \([0, 2\pi]\), find the \( x \)-values at which the function has a maximum or minimum value.

42. On \([0, 2\pi]\), solve the equation \( f(x) = \frac{\sqrt{3}}{2} \).

**Technology**

43. Graph \( h(x) = x + \sin x \) on \([0, 2\pi]\). Explain why the graph appears as it does.

44. Graph \( h(x) = x + \sin x \) on \([-100, 100]\). Did the graph appear as predicted in the previous exercise?

45. Graph \( f(x) = x \sin x \) on \([0, 2\pi]\) and verbalize how the graph varies from the graph of \( f(x) = \sin x \).

46. Graph \( f(x) = x \sin x \) on the window \([-10, 10]\) and explain what the graph shows.

47. Graph \( f(x) = \frac{\sin x}{x} \) on the window \([-5\pi, 5\pi]\) and explain what the graph shows.
Real-World Applications

48. A Ferris wheel is 25 meters in diameter and boarded from a platform that is 1 meter above the ground. The six o’clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. The function \( h(t) \) gives a person’s height in meters above the ground \( t \) minutes after the wheel begins to turn.

a. Find the amplitude, midline, and period of \( h(t) \).

b. Find a formula for the height function \( h(t) \).

c. How high off the ground is a person after 5 minutes?
Learning Objectives

In this section, you will:

6.2.1 Analyze the graph of \( y = \tan x \).
6.2.2 Graph variations of \( y = \tan x \).
6.2.3 Analyze the graphs of \( y = \sec x \) and \( y = \csc x \).
6.2.4 Graph variations of \( y = \sec x \) and \( y = \csc x \).
6.2.5 Analyze the graph of \( y = \cot x \).
6.2.6 Graph variations of \( y = \cot x \).

We know the tangent function can be used to find distances, such as the height of a building, mountain, or flagpole. But what if we want to measure repeated occurrences of distance? Imagine, for example, a police car parked next to a warehouse. The rotating light from the police car would travel across the wall of the warehouse in regular intervals. If the input is time, the output would be the distance the beam of light travels. The beam of light would repeat the distance at regular intervals. The tangent function can be used to approximate this distance. Asymptotes would be needed to illustrate the repeated cycles when the beam runs parallel to the wall because, seemingly, the beam of light could appear to extend forever. The graph of the tangent function would clearly illustrate the repeated intervals. In this section, we will explore the graphs of the tangent and other trigonometric functions.

Analyzing the Graph of \( y = \tan x \)

We will begin with the graph of the tangent function, plotting points as we did for the sine and cosine functions. Recall that

\[
\tan x = \frac{\sin x}{\cos x} \tag{6.8}
\]

The period of the tangent function is \( \pi \) because the graph repeats itself on intervals of \( k\pi \) where \( k \) is a constant. If we graph the tangent function on \( -\frac{\pi}{2} \) to \( \frac{\pi}{2} \), we can see the behavior of the graph on one complete cycle. If we look at any larger interval, we will see that the characteristics of the graph repeat.

We can determine whether tangent is an odd or even function by using the definition of tangent.

\[
\tan(-x) = \frac{\sin(-x)}{\cos(-x)} \quad \text{Definition of tangent.} \tag{6.9}
\]

\[
= -\frac{\sin x}{\cos x} \quad \text{Sine is an odd function, cosine is even.}
\]

\[
= -\frac{\sin x}{\cos x} \quad \text{The quotient of an odd and an even function is odd.}
\]

\[
= -\tan x \quad \text{Definition of tangent.}
\]

Therefore, tangent is an odd function. We can further analyze the graphical behavior of the tangent function by looking at values for some of the special angles, as listed in Table 6.3.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\frac{\pi}{2})</th>
<th>(-\frac{\pi}{3})</th>
<th>(-\frac{\pi}{4})</th>
<th>(-\frac{\pi}{6})</th>
<th>0</th>
<th>(\frac{\pi}{6})</th>
<th>(\frac{\pi}{4})</th>
<th>(\frac{\pi}{3})</th>
<th>(\frac{\pi}{2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tan(x) )</td>
<td>undefined</td>
<td>(-\sqrt{3})</td>
<td>(-1)</td>
<td>(-\sqrt{3})</td>
<td>0</td>
<td>(\sqrt{3})</td>
<td>1</td>
<td>(\sqrt{3})</td>
<td>undefined</td>
</tr>
</tbody>
</table>

Table 6.3

These points will help us draw our graph, but we need to determine how the graph behaves where it is undefined. If we look more closely at values when \( \frac{\pi}{3} < x < \frac{2\pi}{3} \), we can use a table to look for a trend. Because \( \frac{\pi}{3} \approx 1.05 \) and \( \frac{\pi}{2} \approx 1.57 \), we will evaluate \( x \) at radian measures \( 1.05 < x < 1.57 \) as shown in Table 6.4.
As $x$ approaches $\frac{\pi}{2}$, the outputs of the function get larger and larger. Because $y = \tan x$ is an odd function, we see the corresponding table of negative values in Table 6.5.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\tan x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.3</td>
<td>3.6</td>
</tr>
<tr>
<td>1.5</td>
<td>14.1</td>
</tr>
<tr>
<td>1.55</td>
<td>48.1</td>
</tr>
<tr>
<td>1.56</td>
<td>92.6</td>
</tr>
</tbody>
</table>

Table 6.4

We can see that, as $x$ approaches $-\frac{\pi}{2}$, the outputs get smaller and smaller. Remember that there are some values of $x$ for which $\cos x = 0$. For example, $\cos \left( \frac{\pi}{2} \right) = 0$ and $\cos \left( \frac{3\pi}{2} \right) = 0$. At these values, the tangent function is undefined, so the graph of $y = \tan x$ has discontinuities at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. At these values, the graph of the tangent has vertical asymptotes. Figure 6.28 represents the graph of $y = \tan x$. The tangent is positive from 0 to $\frac{\pi}{2}$ and from $\pi$ to $\frac{3\pi}{2}$, corresponding to quadrants I and III of the unit circle.

![Graph of the tangent function](http://legacy.cnx.org/content/col11667/1.2)

**Figure 6.28** Graph of the tangent function

### Graphing Variations of $y = \tan x$

As with the sine and cosine functions, the tangent function can be described by a general equation.

$$y = A \tan(Bx) \hspace{1cm} (6.10)$$

We can identify horizontal and vertical stretches and compressions using values of $A$ and $B$. The horizontal stretch can typically be determined from the period of the graph. With tangent graphs, it is often necessary to determine a vertical stretch using a point on the graph.

Because there are no maximum or minimum values of a tangent function, the term *amplitude* cannot be interpreted as it is for the sine and cosine functions. Instead, we will use the phrase *stretching/compressing factor* when referring to the constant $A$. 

Figure 6.28 represents the graph of $y = \tan x$. The tangent is positive from 0 to $\frac{\pi}{2}$ and from $\pi$ to $\frac{3\pi}{2}$, corresponding to quadrants I and III of the unit circle.

![Graph of the tangent function](http://legacy.cnx.org/content/col11667/1.2)
Features of the Graph of \( y = \text{Atan}(Bx) \)

- The stretching factor is \( |A| \).
- The period is \( P = \frac{\pi}{|B|} \).
- The domain is all real numbers \( x \), where \( x \neq \frac{\pi}{2|B|} + \frac{\pi}{|B|}k \) such that \( k \) is an integer.
- The range is \((-\infty, \infty)\).
- The asymptotes occur at \( x = \frac{\pi}{2|B|} + \frac{\pi}{|B|}k \), where \( k \) is an integer.
- \( y = \text{Atan}(Bx) \) is an odd function.

Graphing One Period of a Stretched or Compressed Tangent Function

We can use what we know about the properties of the tangent function to quickly sketch a graph of any stretched and/or compressed tangent function of the form \( f(x) = \text{Atan}(Bx) \). We focus on a single period of the function including the origin, because the periodic property enables us to extend the graph to the rest of the function’s domain if we wish. Our limited domain is then the interval \( \left(-\frac{P}{2}, \frac{P}{2}\right) \) and the graph has vertical asymptotes at \( \pm \frac{P}{2} \) where \( P = \frac{\pi}{|B|} \). On \( \left(-\frac{P}{2}, \frac{P}{2}\right) \), the graph will come up from the left asymptote at \( x = -\frac{\pi}{2} \), cross through the origin, and continue to increase as it approaches the right asymptote at \( x = \frac{\pi}{2} \). To make the function approach the asymptotes at the correct rate, we also need to set the vertical scale by actually evaluating the function for at least one point that the graph will pass through. For example, we can use

\[
f\left(\frac{P}{4}\right) = \text{Atan}\left(B\frac{P}{4}\right) = \text{Atan}\left(B\frac{\pi}{4B}\right) = A
\]

because \( \tan\left(\frac{\pi}{4}\right) = 1 \).

**How To**: Given the function \( f(x) = \text{Atan}(Bx) \), graph one period.

1. Identify the stretching factor, \( |A| \).
2. Identify \( B \) and determine the period, \( P = \frac{\pi}{|B|} \).
3. Draw vertical asymptotes at \( x = -\frac{P}{2} \) and \( x = \frac{P}{2} \).
4. For \( A > 0 \), the graph approaches the left asymptote at negative output values and the right asymptote at positive output values (reverse for \( A < 0 \)).
5. Plot reference points at \( \left(\frac{P}{4}, A\right) \), \( (0, 0) \), and \( \left(-\frac{P}{4}, -A\right) \), and draw the graph through these points.

**Example 6.14**

**Sketching a Compressed Tangent**

Sketch a graph of one period of the function \( y = 0.5\tan\left(\frac{\pi}{2}x\right) \).
Sketch a graph of \( f(x) = 3 \tan \left( \frac{\pi}{6} x \right) \).

**Graphing One Period of a Shifted Tangent Function**

Now that we can graph a tangent function that is stretched or compressed, we will add a vertical and/or horizontal (or phase) shift. In this case, we add \( C \) and \( D \) to the general form of the tangent function.

\[
f(x) = \text{Atan}(Bx - C) + D \tag{6.12}
\]

The graph of a transformed tangent function is different from the basic tangent function \( \tan x \) in several ways:

**Features of the Graph of \( y = \text{Atan}(Bx - C) + D \)**

- The stretching factor is \(|A|\).
- The period is \( \frac{\pi}{|B|} \).
- The domain is \( x \neq \frac{C}{B} + \frac{\pi}{2|B|} k \), where \( k \) is an integer.
- The range is \((-\infty, -|A|] \cup [|A|, \infty)\).
- The vertical asymptotes occur at \( x = \frac{C}{B} + \frac{\pi}{2|B|} k \), where \( k \) is an odd integer.
- There is no amplitude.
- \( y = \text{Atan}(Bx) \) is an odd function because it is the quotient of odd and even functions (sin and cosine respectively).

**How To:**

1. Express the function given in the form \( y = \text{Atan}(Bx - C) + D \).
2. Identify the stretching/compressing factor, \(|A|\).
3. Identify \( B \) and determine the period, \( P = \frac{\pi}{|B|} \).
4. Identify \( C \) and determine the phase shift \( \frac{C}{B} \).
5. Draw the graph of \( y = \text{Atan}(Bx) \) shifted to the right by \( \frac{C}{B} \) and up by \( D \).
6. Sketch the vertical asymptotes, which occur at \( x = \frac{C}{B} + \frac{\pi}{2|B|} k \), where \( k \) is an odd integer.
7. Plot any three reference points and draw the graph through these points.

**Example 6.15**

**Graphing One Period of a Shifted Tangent Function**

Graph one period of the function \( y = -2\tan(\pi x + \pi) - 1 \).

**Solution**

**Analysis**
Note that this is a decreasing function because $A < 0$.

6.13 How would the graph in Example 6.15 look different if we made $A = 2$ instead of $-2$?

**How To:** Given the graph of a tangent function, identify horizontal and vertical stretches.

1. Find the period $P$ from the spacing between successive vertical asymptotes or $x$-intercepts.
2. Write $f(x) = \tan\left(\frac{\pi}{P}x\right)$.
3. Determine a convenient point $(x, f(x))$ on the given graph and use it to determine $A$.

**Example 6.16**

**Identifying the Graph of a Stretched Tangent**

Find a formula for the function graphed in Figure 6.29.

![A stretched tangent function](image)

**Solution**

6.14 Find a formula for the function in Figure 6.30.

![Figure 6.30](image)
Analyzing the Graphs of $y = \sec x$ and $y = \csc x$

The secant was defined by the reciprocal identity $\sec x = \frac{1}{\cos x}$. Notice that the function is undefined when the cosine is 0, leading to vertical asymptotes at $\frac{\pi}{2}$, $\frac{3\pi}{2}$, etc. Because the cosine is never more than 1 in absolute value, the secant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \sec x$ by observing the graph of the cosine function because these two functions are reciprocals of one another. See Figure 6.31. The graph of the cosine is shown as a dashed orange wave so we can see the relationship. Where the graph of the cosine function decreases, the graph of the secant function increases. Where the graph of the cosine function increases, the graph of the secant function decreases. When the cosine function is zero, the secant is undefined.

The secant graph has vertical asymptotes at each value of $x$ where the cosine graph crosses the $x$-axis; we show these in the graph below with dashed vertical lines, but will not show all the asymptotes explicitly on all later graphs involving the secant and cosecant.

Note that, because cosine is an even function, secant is also an even function. That is, $\sec(-x) = \sec x$.

![Graph of the secant function](image)

As we did for the tangent function, we will again refer to the constant $|A|$ as the stretching factor, not the amplitude.

**Features of the Graph of $y = A\sec(Bx)$**

- The stretching factor is $|A|$.
- The period is $\frac{2\pi}{|B|}$.
- The domain is $x \neq \frac{\pi}{2|B|}k$, where $k$ is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \frac{\pi}{2|B|}k$, where $k$ is an odd integer.
- There is no amplitude.
- $y = A\sec(Bx)$ is an even function because cosine is an even function.

Similar to the secant, the cosecant is defined by the reciprocal identity $\csc x = \frac{1}{\sin x}$. Notice that the function is undefined when the sine is 0, leading to a vertical asymptote in the graph at 0, $\pi$, etc. Since the sine is never more than 1 in absolute value, the cosecant, being the reciprocal, will never be less than 1 in absolute value.
We can graph \( y = \csc x \) by observing the graph of the sine function because these two functions are reciprocals of one another. See Figure 6.32. The graph of sine is shown as a dashed orange wave so we can see the relationship. Where the graph of the sine function decreases, the graph of the cosecant function increases. Where the graph of the sine function increases, the graph of the cosecant function decreases.

The cosecant graph has vertical asymptotes at each value of \( x \) where the sine graph crosses the \( x \)-axis; we show these in the graph below with dashed vertical lines.

Note that, since sine is an odd function, the cosecant function is also an odd function. That is, \( \csc(-x) = -\csc x \).

The graph of cosecant, which is shown in Figure 6.32, is similar to the graph of secant.

Features of the Graph of \( y = A \csc(Bx) \)

- The stretching factor is \( |A| \).
- The period is \( \frac{2\pi}{|B|} \).
- The domain is \( x \neq \frac{\pi}{|B|} k \), where \( k \) is an integer.
- The range is \( (-\infty, -|A|] \cup [|A|, \infty) \).
- The asymptotes occur at \( x = \frac{\pi}{|B|} k \), where \( k \) is an integer.
- \( y = A\csc(Bx) \) is an odd function because sine is an odd function.

Graphing Variations of \( y = \sec x \) and \( y = \csc x \)

For shifted, compressed, and/or stretched versions of the secant and cosecant functions, we can follow similar methods to those we used for tangent and cotangent. That is, we locate the vertical asymptotes and also evaluate the functions for a few points (specifically the local extrema). If we want to graph only a single period, we can choose the interval for the period in more than one way. The procedure for secant is very similar, because the cofunction identity means that the secant graph is the same as the cosecant graph shifted half a period to the left. Vertical and phase shifts may be applied to the cosecant function in the same way as for the secant and other functions. The equations become the following.

\[
\begin{align*}
y &= A\sec(Bx - C) + D \\
y &= A\csc(Bx - C) + D
\end{align*}
\]
Features of the Graph of \( y = \sec(Bx-C)+D \)

- The stretching factor is \(|A|\).
- The period is \( \frac{2\pi}{|B|} \).
- The domain is \( x \neq \frac{C+B}{2|B|}k \), where \( k \) is an odd integer.
- The range is \( (\infty, -|A|] \cup [|A|, \infty) \).
- The vertical asymptotes occur at \( x = \frac{C+B}{2|B|}k \), where \( k \) is an odd integer.
- There is no amplitude.
- \( y = \sec(Bx) \) is an even function because cosine is an even function.

Features of the Graph of \( y = \csc(Bx-C)+D \)

- The stretching factor is \(|A|\).
- The period is \( \frac{2\pi}{|B|} \).
- The domain is \( x \neq \frac{C+B}{2|B|}k \), where \( k \) is an integer.
- The range is \( (\infty, -|A|] \cup [|A|, \infty) \).
- The vertical asymptotes occur at \( x = \frac{C+B}{2|B|}k \), where \( k \) is an integer.
- There is no amplitude.
- \( y = \csc(Bx) \) is an odd function because sine is an odd function.

**How To**

**Graphing a Variation of the Secant Function**

1. Express the function given in the form \( y = \sec(Bx) \).
2. Identify the stretching/compressing factor, \(|A|\).
3. Identify \( B \) and determine the period, \( P = \frac{2\pi}{|B|} \).
4. Sketch the graph of \( y = \cos(Bx) \).
5. Use the reciprocal relationship between \( y = \cos x \) and \( y = \sec x \) to draw the graph of \( y = \sec(Bx) \).
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

**Example 6.17**

Graph one period of \( f(x) = 2.5\sec(0.4x) \).

**Solution**
6.15 Graph one period of $f(x) = -2.5\sec(0.4x)$.

**Do the vertical shift and stretch/compression affect the secant’s range?**

Yes. The range of $f(x) = A\sec(Bx - C) + D$ is $(-\infty, -|A| + D] \cup [|A| + D, \infty)$.

**Given a function of the form $f(x) = A\sec(Bx - C) + D$, graph one period.**

1. Express the function given in the form $y = A\sec(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify $B$ and determine the period, $\frac{2\pi}{|B|}$.
4. Identify $C$ and determine the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\sec(Bx)$, but shift it to the right by $\frac{C}{B}$ and up by $D$.
6. Sketch the vertical asymptotes, which occur at $x = \frac{C}{B} + \frac{\pi}{2|B|}k$, where $k$ is an odd integer.

**Example 6.18**

**Graphing a Variation of the Secant Function**

Graph one period of $y = 4\sec\left(\frac{\pi}{3}x - \frac{\pi}{2}\right) + 1$.

**Solution**

**6.16** Graph one period of $f(x) = -6\sec(4x + 2) - 8$.

**The domain of $\csc x$ was given to be all $x$ such that $x \neq k\pi$ for any integer $k$. Would the domain of $y = A\csc(Bx - C) + D$ be $x \neq \frac{C + k\pi}{B}$?**

Yes. The excluded points of the domain follow the vertical asymptotes. Their locations show the horizontal shift and compression or expansion implied by the transformation to the original function’s input.
Given a function of the form $y = Acsc(Bx)$, graph one period.

1. Express the function given in the form $y = Acsc(Bx)$.
2. $|A|$.
3. Identify $B$ and determine the period, $P = \frac{2\pi}{|B|}$.
4. Draw the graph of $y = Asin(Bx)$.
5. Use the reciprocal relationship between $y = \sin x$ and $y = \csc x$ to draw the graph of $y = Acsc(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

**Example 6.19**

**Graphing a Variation of the Cosecant Function**

Graph one period of $f(x) = -3csc(4x)$.

**Solution**

**Example 6.20**

**Graphing a Vertically Stretched, Horizontally Compressed, and Vertically Shifted Cosecant**

Sketch a graph of $y = 2csc\left(\frac{x}{2}\right) + 1$. What are the domain and range of this function?
Analysis
The vertical asymptotes shown on the graph mark off one period of the function, and the local extrema in this interval are shown by dots. Notice how the graph of the transformed cosecant relates to the graph of $f(x) = 2\sin\left(\frac{\pi}{2}x\right) + 1$, shown as the orange dashed wave.

6.18 Given the graph of $f(x) = 2\cos\left(\frac{\pi}{2}x\right) + 1$ shown in Figure 6.33, sketch the graph of $g(x) = 2\sec\left(\frac{\pi}{2}x\right) + 1$ on the same axes.

![Figure 6.33](image)

Analyzing the Graph of $y = \cot x$

The last trigonometric function we need to explore is cotangent. The cotangent is defined by the reciprocal identity $\cot x = \frac{1}{\tan x}$. Notice that the function is undefined when the tangent function is 0, leading to a vertical asymptote in the graph at 0, \(\pi\), etc. Since the output of the tangent function is all real numbers, the output of the cotangent function is also all real numbers.

We can graph $y = \cot x$ by observing the graph of the tangent function because these two functions are reciprocals of one another. See Figure 6.34. Where the graph of the tangent function decreases, the graph of the cotangent function increases. Where the graph of the tangent function increases, the graph of the cotangent function decreases.

The cotangent graph has vertical asymptotes at each value of $x$ where $\tan x = 0$; we show these in the graph below with dashed lines. Since the cotangent is the reciprocal of the tangent, $\cot x$ has vertical asymptotes at all values of $x$ where $\tan x = 0$, and $\cot x = 0$ at all values of $x$ where $\tan x$ has its vertical asymptotes.
Figure 6.34 The cotangent function

Features of the Graph of $y = A \cot(Bx)$

- The stretching factor is $|A|$.
- The period is $P = \frac{\pi}{|B|}$.
- The domain is $x \neq \frac{\pi}{|B|}k$, where $k$ is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \frac{\pi}{|B|}k$, where $k$ is an integer.
- $y = A \cot(Bx)$ is an odd function.

Graphing Variations of $y = \cot x$

We can transform the graph of the cotangent in much the same way as we did for the tangent. The equation becomes the following.

$$y = A \cot(Bx - C) + D$$ (6.15)

Properties of the Graph of $y = A \cot(Bx-C)+D$

- The stretching factor is $|A|$.
- The period is $\frac{\pi}{|B|}$.
- The domain is $x \neq \frac{C}{B} + \frac{\pi}{|B|}k$, where $k$ is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$, where $k$ is an integer.
- There is no amplitude.
- $y = A \cot(Bx)$ is an odd function because it is the quotient of even and odd functions (cosine and sine, respectively).
Given a modified cotangent function of the form $f(x) = A \cot(Bx)$, graph one period.

1. Express the function in the form $f(x) = A \cot(Bx)$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \frac{\pi}{|B|}$.
4. Draw the graph of $y = \tan(Bx)$.
5. Plot any two reference points.
6. Use the reciprocal relationship between tangent and cotangent to draw the graph of $y = A \cot(Bx)$.
7. Sketch the asymptotes.

Example 6.21

**Graphing Variations of the Cotangent Function**

Determine the stretching factor, period, and phase shift of $y = 3 \cot(4x)$, and then sketch a graph.

**Solution**

Given a modified cotangent function of the form $f(x) = A \cot(Bx - C) + D$, graph one period.

1. Express the function in the form $f(x) = A \cot(Bx - C) + D$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \frac{\pi}{|B|}$.
4. Identify the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = \tan(Bx)$ shifted to the right by $\frac{C}{B}$ and up by $D$.
6. Sketch the asymptotes $x = \frac{C}{B} + \frac{\pi}{|B|}k$, where $k$ is an integer.
7. Plot any three reference points and draw the graph through these points.

Example 6.22

**Graphing a Modified Cotangent**

Sketch a graph of one period of the function $f(x) = 4 \cot\left(\frac{2}{3}x - \frac{\pi}{2}\right) - 2$. 

**Solution**
Using the Graphs of Trigonometric Functions to Solve Real-World Problems

Many real-world scenarios represent periodic functions and may be modeled by trigonometric functions. As an example, let’s return to the scenario from the section opener. Have you ever observed the beam formed by the rotating light on a police car and wondered about the movement of the light beam itself across the wall? The periodic behavior of the distance the light shines as a function of time is obvious, but how do we determine the distance? We can use the tangent function.

Example 6.23

Using Trigonometric Functions to Solve Real-World Scenarios

Suppose the function \( y = 5\tan\left(\frac{\pi}{4}t\right) \) marks the distance in the movement of a light beam from the top of a police car across a wall where \( t \) is the time in seconds and \( y \) is the distance in feet from a point on the wall directly across from the police car.

a. Find and interpret the stretching factor and period.

b. Graph on the interval \([0, 5]\).

c. Evaluate \( f(1) \) and discuss the function’s value at that input.

Solution

Access these online resources for additional instruction and practice with graphs of other trigonometric functions.

- Graphing the Tangent (http://openstaxcollege.org/l/graphtangent)
- Graphing Cosecant and Secant (http://openstaxcollege.org/l/graphcscsec)
- Graphing the Cotangent (http://openstaxcollege.org/l/graphcot)
6.2 EXERCISES

Verbal

49. Explain how the graph of the sine function can be used to graph \( y = \csc \, x \).

50. How can the graph of \( y = \cos \, x \) be used to construct the graph of \( y = \sec \, x \)?

51. Explain why the period of \( \tan \, x \) is equal to \( \pi \).

52. Why are there no intercepts on the graph of \( y = \csc \, x \)?

53. How does the period of \( y = \csc \, x \) compare with the period of \( y = \sin \, x \)?

Algebraic

For the following exercises, match each trigonometric function with one of the graphs in Figure 6.35.

\[ f(x) = \tan \, x \]
\[ f(x) = \sec \, x \]
\[ f(x) = \csc \, x \]
\[ f(x) = \cot \, x \]

For the following exercises, find the period and horizontal shift of each of the functions.

54. \( f(x) = \tan \, x \)

55. \( f(x) = \sec \, x \)

56. \( f(x) = \csc \, x \)

57. \( f(x) = \cot \, x \)

For the following exercises, find the period and horizontal shift of each of the functions.

58. \( f(x) = 2\tan(4x - 32) \)
59. \( h(x) = 2\sec\left(\frac{x}{4} + 1\right) \)

60. \( m(x) = 6\csc\left(\frac{x}{3} + \pi\right) \)

61. If \( \tan x = -1.5 \), find \( \tan(-x) \).

62. If \( \sec x = 2 \), find \( \sec(-x) \).

63. If \( \csc x = -5 \), find \( \csc(-x) \).

64. If \( x\sin x = 2 \), find \( (-x)\sin(-x) \).

For the following exercises, rewrite each expression such that the argument \( x \) is positive.

65. \( \cot(-x)\cos(-x) + \sin(-x) \)

66. \( \cos(-x) + \tan(-x)\sin(-x) \)

Graphical

For the following exercises, sketch two periods of the graph for each of the following functions. Identify the stretching factor, period, and asymptotes.

67. \( f(x) = 2\tan(4x - 32) \)

68. \( h(x) = 2\sec\left(\frac{x}{4} + 1\right) \)

69. \( m(x) = 6\csc\left(\frac{x}{3} + \pi\right) \)

70. \( j(x) = \tan\left(\frac{5x}{2}\right) \)

71. \( p(x) = \tan\left(x - \frac{\pi}{2}\right) \)

72. \( f(x) = 4\tan(x) \)

73. \( f(x) = \tan\left(x + \frac{\pi}{4}\right) \)

74. \( f(x) = \pi\tan(\pi x - \pi) - \pi \)

75. \( f(x) = 2\csc(x) \)

76. \( f(x) = -\frac{1}{4}\csc(x) \)

77. \( f(x) = 4\sec(3x) \)

78. \( f(x) = -3\cot(2x) \)

79. \( f(x) = 7\sec(5x) \)

80. \( f(x) = \frac{9}{10}\csc(\pi x) \)

81. \( f(x) = 2\csc\left(x + \frac{\pi}{4}\right) - 1 \)
82. \( f(x) = -\sec\left(x - \frac{\pi}{4}\right) - 2 \)

83. \( f(x) = \frac{7}{5}\csc\left(x - \frac{\pi}{4}\right) \)

84. \( f(x) = 5\left(\cot\left(x + \frac{\pi}{2}\right) - 3\right) \)

For the following exercises, find and graph two periods of the periodic function with the given stretching factor, \(|A|\), period, and phase shift.

85. A tangent curve, \( A = 1 \), period of \( \frac{\pi}{3} \), and phase shift \((h, k) = \left(\frac{\pi}{4}, 2\right)\)

86. A tangent curve, \( A = -2 \), period of \( \frac{\pi}{4} \), and phase shift \((h, k) = \left(-\frac{\pi}{4}, -2\right)\)

For the following exercises, find an equation for the graph of each function.

87.

88.

89.
For the following exercises, use a graphing calculator to graph two periods of the given function. Note: most graphing calculators do not have a cosecant button; therefore, you will need to input \( \csc x \) as \( \frac{1}{\sin x} \).

93. \( f(x) = |\csc(x)| \)

94. \( f(x) = |\cot(x)| \)

95. \( f(x) = 2\csc(x) \)

96. \( f(x) = \frac{\csc(x)}{\sec(x)} \)

97. Graph \( f(x) = 1 + \sec^2(x) - \tan^2(x) \). What is the function shown in the graph?

98. \( f(x) = \sec(0.001x) \)

99. \( f(x) = \cot(100\pi x) \)

100. \( f(x) = \sin^2x + \cos^2x \)

**Real-World Applications**

102. The function \( f(x) = 20\tan\left(\frac{\pi}{10}x\right) \) marks the distance in the movement of a light beam from a police car across a wall for time \( x \), in seconds, and distance \( f(x) \), in feet.
103. Standing on the shore of a lake, a fisherman sights a boat far in the distance to his left. Let $x$, measured in radians, be the angle formed by the line of sight to the ship and a line due north from his position. Assume due north is 0 and $x$ is measured negative to the left and positive to the right. (See Figure 6.36.) The boat travels from due west to due east and, ignoring the curvature of the Earth, the distance $d(x)$, in kilometers, from the fisherman to the boat is given by the function $d(x) = 1.5 \sec(x)$.

a. What is a reasonable domain for $d(x)$?

b. Graph $d(x)$ on this domain.

c. Find and discuss the meaning of any vertical asymptotes on the graph of $d(x)$.

d. Calculate and interpret $d\left(-\frac{\pi}{3}\right)$. Round to the second decimal place.

e. Calculate and interpret $d\left(\frac{\pi}{6}\right)$. Round to the second decimal place.

f. What is the minimum distance between the fisherman and the boat? When does this occur?

104. A laser rangefinder is locked on a comet approaching Earth. The distance $g(x)$, in kilometers, of the comet after $x$ days, for $x$ in the interval 0 to 30 days, is given by $g(x) = 250,000 \csc\left(\frac{\pi}{30}x\right)$.

a. Graph $g(x)$ on the interval $[0, 35]$.

b. Evaluate $g(5)$ and interpret the information.

c. What is the minimum distance between the comet and Earth? When does this occur? To which constant in the equation does this correspond?

d. Find and discuss the meaning of any vertical asymptotes.

105. A video camera is focused on a rocket on a launching pad 2 miles from the camera. The angle of elevation from the ground to the rocket after $x$ seconds is $\frac{\pi}{120}x$.

a. Write a function expressing the altitude $h(x)$, in miles, of the rocket above the ground after $x$ seconds. Ignore the curvature of the Earth.
b. Graph \( h(x) \) on the interval \((0, 60)\).

c. Evaluate and interpret the values \( h(0) \) and \( h(30) \).

d. What happens to the values of \( h(x) \) as \( x \) approaches 60 seconds? Interpret the meaning of this in terms of the problem.
6.3 | Inverse Trigonometric Functions

Learning Objectives

In this section, you will:

- **6.3.1** Understand and use the inverse sine, cosine, and tangent functions.
- **6.3.2** Find the exact value of expressions involving the inverse sine, cosine, and tangent functions.
- **6.3.3** Use a calculator to evaluate inverse trigonometric functions.
- **6.3.4** Find exact values of composite functions with inverse trigonometric functions.

For any right triangle, given one other angle and the length of one side, we can figure out what the other angles and sides are. But what if we are given only two sides of a right triangle? We need a procedure that leads us from a ratio of sides to an angle. This is where the notion of an inverse to a trigonometric function comes into play. In this section, we will explore the inverse trigonometric functions.

### Understanding and Using the Inverse Sine, Cosine, and Tangent Functions

In order to use inverse trigonometric functions, we need to understand that an inverse trigonometric function “undoes” what the original trigonometric function “does,” as is the case with any other function and its inverse. In other words, the domain of the inverse function is the range of the original function, and vice versa, as summarized in **Figure 6.37**.

<table>
<thead>
<tr>
<th>Trig Functions</th>
<th>Inverse Trig Functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain: Measure of an angle</td>
<td>Domain: Ratio</td>
</tr>
<tr>
<td>Range: Ratio</td>
<td>Range: Measure of an angle</td>
</tr>
</tbody>
</table>

**Figure 6.37**

For example, if \( f(x) = \sin x \), then we would write \( f^{-1}(x) = \sin^{-1} x \). Be aware that \( \sin^{-1} x \) does not mean \( \frac{1}{\sin x} \). The following examples illustrate the inverse trigonometric functions:

- \( \sin \left( \frac{\pi}{4} \right) = \frac{\sqrt{2}}{2} \), then \( \frac{\pi}{4} = \sin^{-1} \left( \frac{\sqrt{2}}{2} \right) \).
- Since \( \cos(\pi) = -1 \), then \( \pi = \cos^{-1} (-1) \).
- Since \( \tan \left( \frac{\pi}{4} \right) = 1 \), then \( \frac{\pi}{4} = \tan^{-1} (1) \).

In previous sections, we evaluated the trigonometric functions at various angles, but at times we need to know what angle would yield a specific sine, cosine, or tangent value. For this, we need inverse functions. Recall that, for a one-to-one function, if \( f(a) = b \), then an inverse function would satisfy \( f^{-1}(b) = a \).

Bear in mind that the sine, cosine, and tangent functions are not one-to-one functions. The graph of each function would fail the horizontal line test. In fact, no periodic function can be one-to-one because each output in its range corresponds to at least one input in every period, and there are an infinite number of periods. As with other functions that are not one-to-one, we will need to restrict the domain of each function to yield a new function that is one-to-one. We choose a domain for each function that includes the number 0. **Figure 6.38** shows the graph of the sine function limited to \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \) and the graph of the cosine function limited to \( [0, \pi] \).
These conventional choices for the restricted domain are somewhat arbitrary, but they have important, helpful characteristics. Each domain includes the origin and some positive values, and most importantly, each results in a one-to-one function that is invertible. The conventional choice for the restricted domain of the tangent function also has the useful property that it extends from one vertical asymptote to the next instead of being divided into two parts by an asymptote.

On these restricted domains, we can define the inverse trigonometric functions.

- The **inverse sine function** \( y = \sin^{-1} x \) means \( x = \sin y \). The inverse sine function is sometimes called the **arcsine** function, and notated \( \arcsin x \).

  \[
  y = \sin^{-1} x \text{ has domain } [-1, 1] \text{ and range } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]
  \]

  \[\text{(6.16)}\]

- The **inverse cosine function** \( y = \cos^{-1} x \) means \( x = \cos y \). The inverse cosine function is sometimes called the **arccosine** function, and notated \( \arccos x \).

  \[
  y = \cos^{-1} x \text{ has domain } [-1, 1] \text{ and range } [0, \pi]
  \]

  \[\text{(6.17)}\]

- The **inverse tangent function** \( y = \tan^{-1} x \) means \( x = \tan y \). The inverse tangent function is sometimes called the **arctangent** function, and notated \( \arctan x \).

  \[
  y = \tan^{-1} x \text{ has domain } (-\infty, \infty) \text{ and range } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)
  \]

  \[\text{(6.18)}\]

The graphs of the inverse functions are shown in **Figure 6.40**, **Figure 6.41**, and **Figure 6.42**. Notice that the output of each of these inverse functions is a **number**, an angle in radian measure. We see that \( \sin^{-1} x \) has domain \([-1, 1] \) and
range \([-\frac{\pi}{2}, \frac{\pi}{2}]\). \(\cos^{-1} x\) has domain \([-1, 1]\) and range \([0, \pi]\), and \(\tan^{-1} x\) has domain of all real numbers and range \((-\frac{\pi}{2}, \frac{\pi}{2})\). To find the domain and range of inverse trigonometric functions, switch the domain and range of the original functions. Each graph of the inverse trigonometric function is a reflection of the graph of the original function about the line \(y = x\).

Figure 6.40  The sine function and inverse sine (or arcsine) function

Figure 6.41  The cosine function and inverse cosine (or arccosine) function
Relations for Inverse Sine, Cosine, and Tangent Functions

For angles in the interval $\left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$, if $\sin y = x$, then $\sin^{-1} x = y$.

For angles in the interval $[0, \pi]$, if $\cos y = x$, then $\cos^{-1} x = y$.

For angles in the interval $\left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$, if $\tan y = x$, then $\tan^{-1} x = y$.

Example 6.24

Writing a Relation for an Inverse Function

Given $\sin \left( \frac{5\pi}{12} \right) \approx 0.96593$, write a relation involving the inverse sine.

Solution

Given $\cos(0.5) \approx 0.8776$, write a relation involving the inverse cosine.

Finding the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions

Now that we can identify inverse functions, we will learn to evaluate them. For most values in their domains, we must evaluate the inverse trigonometric functions by using a calculator, interpolating from a table, or using some other numerical technique. Just as we did with the original trigonometric functions, we can give exact values for the inverse functions when we are using the special angles, specifically $\frac{\pi}{6}$ (30°), $\frac{\pi}{4}$ (45°), and $\frac{\pi}{3}$ (60°), and their reflections into other quadrants.
Given a “special” input value, evaluate an inverse trigonometric function.

1. Find angle \( x \) for which the original trigonometric function has an output equal to the given input for the inverse trigonometric function.

2. If \( x \) is not in the defined range of the inverse, find another angle \( y \) that is in the defined range and has the same sine, cosine, or tangent as \( x \), depending on which corresponds to the given inverse function.

Example 6.25

**Evaluating Inverse Trigonometric Functions for Special Input Values**

Evaluate each of the following.

a. \( \sin^{-1}(\frac{1}{2}) \)

b. \( \sin^{-1}(\frac{-\sqrt{2}}{2}) \)

c. \( \cos^{-1}(\frac{-\sqrt{3}}{2}) \)

d. \( \tan^{-1}(1) \)

**Solution**

Evaluate each of the following.

a. \( \sin^{-1}(-1) \)

b. \( \tan^{-1}(-1) \)

c. \( \cos^{-1}(-1) \)

d. \( \cos^{-1}(\frac{1}{2}) \)

**Using a Calculator to Evaluate Inverse Trigonometric Functions**

To evaluate inverse trigonometric functions that do not involve the special angles discussed previously, we will need to use a calculator or other type of technology. Most scientific calculators and calculator-emulating applications have specific keys or buttons for the inverse sine, cosine, and tangent functions. These may be labeled, for example, SIN-1, ARCSIN, or ASIN.

In the previous chapter, we worked with trigonometry on a right triangle to solve for the sides of a triangle given one side and an additional angle. Using the inverse trigonometric functions, we can solve for the angles of a right triangle given two sides, and we can use a calculator to find the values to several decimal places.

In these examples and exercises, the answers will be interpreted as angles and we will use \( \theta \) as the independent variable. The value displayed on the calculator may be in degrees or radians, so be sure to set the mode appropriate to the application.

Example 6.26

**Evaluating the Inverse Sine on a Calculator**
Evaluate $\sin^{-1}(0.97)$ using a calculator.

**Solution**

6.21 Evaluate $\cos^{-1}(-0.4)$ using a calculator.

**How To:**

Given two sides of a right triangle like the one shown in Figure 6.43, find an angle.

![Figure 6.43](image)

1. If one given side is the hypotenuse of length $h$ and the side of length $a$ adjacent to the desired angle is given, use the equation $\theta = \cos^{-1}\left(\frac{a}{h}\right)$.
2. If one given side is the hypotenuse of length $h$ and the side of length $p$ opposite to the desired angle is given, use the equation $\theta = \sin^{-1}\left(\frac{p}{h}\right)$.
3. If the two legs (the sides adjacent to the right angle) are given, then use the equation $\theta = \tan^{-1}\left(\frac{p}{a}\right)$.

**Example 6.27**

**Applying the Inverse Cosine to a Right Triangle**

Solve the triangle in Figure 6.44 for the angle $\theta$.

![Figure 6.44](image)
Solve the triangle in Figure 6.45 for the angle \( \theta \).

![Figure 6.45](image)

**Finding Exact Values of Composite Functions with Inverse Trigonometric Functions**

There are times when we need to compose a trigonometric function with an inverse trigonometric function. In these cases, we can usually find exact values for the resulting expressions without resorting to a calculator. Even when the input to the composite function is a variable or an expression, we can often find an expression for the output. To help sort out different cases, let \( f(x) \) and \( g(x) \) be two different trigonometric functions belonging to the set \{ \sin(x), \cos(x), \tan(x) \} and let \( f^{-1}(y) \) and \( g^{-1}(y) \) be their inverses.

**Evaluating Compositions of the Form \( f(f^{-1}(y)) \) and \( f^{-1}(f(x)) \)**

For any trigonometric function, \( f(f^{-1}(y)) = y \) for all \( y \) in the proper domain for the given function. This follows from the definition of the inverse and from the fact that the range of \( f \) was defined to be identical to the domain of \( f^{-1} \). However, we have to be a little more careful with expressions of the form \( f^{-1}(f(x)) \).

<table>
<thead>
<tr>
<th>Compositions of a trigonometric function and its inverse</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin(\sin^{-1}x) = x ) for (-1 \leq x \leq 1 )</td>
</tr>
<tr>
<td>( \cos(\cos^{-1}x) = x ) for (-1 \leq x \leq 1 )</td>
</tr>
<tr>
<td>( \tan(\tan^{-1}x) = x ) for (-\infty &lt; x &lt; \infty )</td>
</tr>
</tbody>
</table>

\[ (6.19) \]

| \( \sin^{-1}(\sin x) = x \) only for \(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \) |
| \( \cos^{-1}(\cos x) = x \) only for \(0 \leq x \leq \pi \) |
| \( \tan^{-1}(\tan x) = x \) only for \(-\frac{\pi}{2} < x < \frac{\pi}{2} \) |

\[ (6.20) \]

**Is it correct that \( \sin^{-1}(\sin x) = x \)?**

No. This equation is correct if \( x \) belongs to the restricted domain \( \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \), but sine is defined for all real input values, and for \( x \) outside the restricted interval, the equation is not correct because its inverse always returns a value in \( \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \). The situation is similar for cosine and tangent and their inverses. For example, \( \sin^{-1}(\sin(\frac{3\pi}{4})) = \frac{\pi}{4} \).

**Given an expression of the form \( f^{-1}(f(\theta)) \) where \( f(\theta) = \sin \theta, \cos \theta, \) or \( \tan \theta \), evaluate.**

1. If \( \theta \) is in the restricted domain of \( f \), then \( f^{-1}(f(\theta)) = \theta \).
2. If not, then find an angle \( \phi \) within the restricted domain of \( f \) such that \( f(\phi) = f(\theta) \). Then \( f^{-1}(f(\theta)) = \phi \).
Example 6.28

Using Inverse Trigonometric Functions

Evaluate the following:

1. \( \sin^{-1}(\sin\left(\frac{\pi}{3}\right)) \)
2. \( \sin^{-1}(\sin\left(\frac{2\pi}{3}\right)) \)
3. \( \cos^{-1}(\cos\left(\frac{2\pi}{3}\right)) \)
4. \( \cos^{-1}(\cos\left(-\frac{\pi}{3}\right)) \)

Solution

Evaluate \( \tan^{-1}\left(\tan\left(\frac{\pi}{8}\right)\right) \) and \( \tan^{-1}\left(\tan\left(\frac{11\pi}{9}\right)\right) \).

Evaluating Compositions of the Form \( f^{-1}(g(x)) \)

Now that we can compose a trigonometric function with its inverse, we can explore how to evaluate a composition of a trigonometric function and the inverse of another trigonometric function. We will begin with compositions of the form \( f^{-1}(g(x)) \). For special values of \( x \), we can exactly evaluate the inner function and then the outer, inverse function. However, we can find a more general approach by considering the relation between the two acute angles of a right triangle where one is \( \theta \), making the other \( \frac{\pi}{2} - \theta \). Consider the sine and cosine of each angle of the right triangle in Figure 6.46.

Because \( \cos \theta = \frac{b}{c} = \sin\left(\frac{\pi}{2} - \theta\right) \), we have \( \sin^{-1}(\cos \theta) = \frac{\pi}{2} - \theta \) if \( 0 \leq \theta \leq \pi \). If \( \theta \) is not in this domain, then we need to find another angle that has the same cosine as \( \theta \) and does belong to the restricted domain; we then subtract this angle from \( \frac{\pi}{2} \). Similarly, \( \sin \theta = \frac{a}{c} = \cos\left(\frac{\pi}{2} - \theta\right) \), so \( \cos^{-1}(\sin \theta) = \frac{\pi}{2} - \theta \) if \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \). These are just the function-cofunction relationships presented in another way.
Given functions of the form \( \sin^{-1}(\cos x) \) and \( \cos^{-1}(\sin x) \), evaluate them.

1. If \( x \) is in \([0, \pi]\), then \( \sin^{-1}(\cos x) = \frac{\pi}{2} - x \).
2. If \( x \) is not in \([0, \pi]\), then find another angle \( y \) in \([0, \pi]\) such that \( \cos y = \cos x \).\( \sin^{-1}(\cos x) = \frac{\pi}{2} - y \) \( \quad (6.21) \)
3. If \( x \) is in \([\frac{-\pi}{2}, \frac{\pi}{2}]\), then \( \cos^{-1}(\sin x) = \frac{\pi}{2} - x \).
4. If \( x \) is not in \([\frac{-\pi}{2}, \frac{\pi}{2}]\), then find another angle \( y \) in \([\frac{-\pi}{2}, \frac{\pi}{2}]\) such that \( \sin y = \sin x \).\( \cos^{-1}(\sin x) = \frac{\pi}{2} - y \) \( \quad (6.22) \)

Example 6.29

Evaluating the Composition of an Inverse Sine with a Cosine

Evaluate \( \sin^{-1}(\cos(\frac{13\pi}{6})) \)

a. by direct evaluation.

b. by the method described previously.

Solution

\[ \sin^{-1}(\cos(\frac{13\pi}{6})) = \frac{\pi}{2} - \frac{11\pi}{4} \]

Example 6.30

Evaluating the Composition of a Sine with an Inverse Cosine

Find an exact value for \( \sin(\cos^{-1}(\frac{4}{5})) \).

Solution

\[ \sin(\cos^{-1}(\frac{4}{5})) = \frac{3}{5} \]
Evaluate \( \cos\left(\tan^{-1}\left(\frac{5}{12}\right)\right) \).

**Example 6.31**

**Evaluating the Composition of a Sine with an Inverse Tangent**

Find an exact value for \( \sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right) \).

**Solution**

Evaluate \( \cos\left(\sin^{-1}\left(\frac{7}{9}\right)\right) \).

**Example 6.32**

**Finding the Cosine of the Inverse Sine of an Algebraic Expression**

Find a simplified expression for \( \cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right) \) for \(-3 \leq x \leq 3\).

**Solution**

Find a simplified expression for \( \sin\left(\tan^{-1}\left(4x\right)\right) \) for \(-\frac{1}{4} \leq x \leq \frac{1}{4}\).

Access this online resource for additional instruction and practice with inverse trigonometric functions.

- **Evaluate Expressions Involving Inverse Trigonometric Functions** (http://openstaxcollege.org/l/evalinverstrig)
6.3 EXERCISES

Verbal

106. Why do the functions $f(x) = \sin^{-1}x$ and $g(x) = \cos^{-1}x$ have different ranges?

107. Since the functions $y = \cos x$ and $y = \cos^{-1}x$ are inverse functions, why is $\cos^{-1}(\cos(-\frac{\pi}{6}))$ not equal to $-\frac{\pi}{6}$?

108. Explain the meaning of $\frac{\pi}{6} = \arcsin(0.5)$.

109. Most calculators do not have a key to evaluate $\sec^{-1}(2)$. Explain how this can be done using the cosine function or the inverse cosine function.

110. Why must the domain of the sine function, $\sin x$, be restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ for the inverse sine function to exist?

111. Discuss why this statement is incorrect: $\arccos(\cos x) = x$ for all $x$.

112. Determine whether the following statement is true or false and explain your answer: $\arccos(-x) = \pi - \arccos x$.

Algebraic

For the following exercises, evaluate the expressions.

113. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$

114. $\sin^{-1}\left(-\frac{1}{2}\right)$

115. $\cos^{-1}\left(\frac{1}{2}\right)$

116. $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

117. $\tan^{-1}(1)$

118. $\tan^{-1}\left(-\sqrt{3}\right)$

119. $\tan^{-1}(-1)$

120. $\tan^{-1}\left(\sqrt{3}\right)$

121. $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

For the following exercises, use a calculator to evaluate each expression. Express answers to the nearest hundredth.

122. $\cos^{-1}(-0.4)$

123. $\arcsin(0.23)$

124. $\arccos\left(\frac{3}{5}\right)$

125. $\cos^{-1}(0.8)$
126. \( \tan^{-1}(6) \)

For the following exercises, find the angle \( \theta \) in the given right triangle. Round answers to the nearest hundredth.

127.

\[
\begin{array}{c}
\text{10} \\
\theta \\
\text{7}
\end{array}
\]

128.

\[
\begin{array}{c}
\text{12} \\
\theta \\
\text{19}
\end{array}
\]

For the following exercises, find the exact value, if possible, without a calculator. If it is not possible, explain why.

129. \( \sin^{-1}(\cos(\pi)) \)

130. \( \tan^{-1}(\sin(\pi)) \)

131. \( \cos^{-1}(\sin(\frac{\pi}{3})) \)

132. \( \tan^{-1}(\sin(\frac{\pi}{3})) \)

133. \( \sin^{-1}(\cos(-\frac{\pi}{2})) \)

134. \( \tan^{-1}(\sin(\frac{4\pi}{3})) \)

135. \( \sin^{-1}(\sin(\frac{5\pi}{6})) \)

136. \( \tan^{-1}(\sin(-\frac{5\pi}{2})) \)

137. \( \cos(\sin^{-1}(\frac{4}{3})) \)

138. \( \sin(\cos^{-1}(\frac{3}{2})) \)

139. \( \sin(\tan^{-1}(\frac{4}{3})) \)

140. \( \cos(\tan^{-1}(\frac{12}{5})) \)

141. \( \cos(\sin^{-1}(\frac{3}{2})) \)
For the following exercises, find the exact value of the expression in terms of $x$ with the help of a reference triangle.

142. $\tan(\sin^{-1}(x - 1))$
143. $\sin(\cos^{-1}(1 - x))$
144. $\cos(\sin^{-1}(\frac{1}{x}))$
145. $\cos(\tan^{-1}(3x - 1))$
146. $\tan(\sin^{-1}(x + \frac{1}{2}))$

Extensions

For the following exercises, evaluate the expression without using a calculator. Give the exact value.

147. $\frac{\sin^{-1}(\frac{1}{2}) - \cos^{-1}(\frac{\sqrt{3}}{2}) + \sin^{-1}(\frac{\sqrt{3}}{2}) - \cos^{-1}(1)}{\cos^{-1}(\frac{\sqrt{3}}{2}) - \sin^{-1}(\frac{\sqrt{2}}{2}) + \cos^{-1}(\frac{1}{2}) - \sin^{-1}(0)}$

For the following exercises, find the function if $\sin t = \frac{x}{x + 1}$.

148. $\cos t$
149. $\sec t$
150. $\cot t$
151. $\cos(\sin^{-1}(\frac{x}{\sqrt{x^2 + 1}}))$
152. $\tan^{-1}(\frac{x}{\sqrt{2x + 1}})$

Graphical

153. Graph $y = \sin^{-1}x$ and state the domain and range of the function.
154. Graph $y = \arccos x$ and state the domain and range of the function.
155. Graph one cycle of $y = \tan^{-1}x$ and state the domain and range of the function.
156. For what value of $x$ does $\sin x = \sin^{-1}x$? Use a graphing calculator to approximate the answer.
157. For what value of $x$ does $\cos x = \cos^{-1}x$? Use a graphing calculator to approximate the answer.

Real-World Applications

158. Suppose a 13-foot ladder is leaning against a building, reaching to the bottom of a second-floor window 12 feet above the ground. What angle, in radians, does the ladder make with the building?
159. Suppose you drive 0.6 miles on a road so that the vertical distance changes from 0 to 150 feet. What is the angle of elevation of the road?
160. An isosceles triangle has two congruent sides of length 9 inches. The remaining side has a length of 8 inches. Find the angle that a side of 9 inches makes with the 8-inch side.
161.
Without using a calculator, approximate the value of \( \arctan(10,000) \). Explain why your answer is reasonable.

162. A truss for the roof of a house is constructed from two identical right triangles. Each has a base of 12 feet and height of 4 feet. Find the measure of the acute angle adjacent to the 4-foot side.

163. The line \( y = \frac{3}{5}x \) passes through the origin in the \( x,y \)-plane. What is the measure of the angle that the line makes with the positive \( x \)-axis?

164. The line \( y = -\frac{3}{7}x \) passes through the origin in the \( x,y \)-plane. What is the measure of the angle that the line makes with the negative \( x \)-axis?

165. What percentage grade should a road have if the angle of elevation of the road is 4 degrees? (The percentage grade is defined as the change in the altitude of the road over a 100-foot horizontal distance. For example a 5% grade means that the road rises 5 feet for every 100 feet of horizontal distance.)

166. A 20-foot ladder leans up against the side of a building so that the foot of the ladder is 10 feet from the base of the building. If specifications call for the ladder's angle of elevation to be between 35 and 45 degrees, does the placement of this ladder satisfy safety specifications?

167. Suppose a 15-foot ladder leans against the side of a house so that the angle of elevation of the ladder is 42 degrees. How far is the foot of the ladder from the side of the house?
CHAPTER 6 REVIEW

KEY TERMS

**amplitude:** the vertical height of a function; the constant $A$ appearing in the definition of a sinusoidal function

**arccosine:** another name for the inverse cosine; $\arccos x = \cos^{-1} x$

**arcsine:** another name for the inverse sine; $\arcsin x = \sin^{-1} x$

**arctangent:** another name for the inverse tangent; $\arctan x = \tan^{-1} x$

**inverse cosine function:** the function $\cos^{-1} x$, which is the inverse of the cosine function and the angle that has a cosine equal to a given number

**inverse sine function:** the function $\sin^{-1} x$, which is the inverse of the sine function and the angle that has a sine equal to a given number

**inverse tangent function:** the function $\tan^{-1} x$, which is the inverse of the tangent function and the angle that has a tangent equal to a given number

**midline:** the horizontal line $y = D$, where $D$ appears in the general form of a sinusoidal function

**periodic function:** a function $f(x)$ that satisfies $f(x + P) = f(x)$ for a specific constant $P$ and any value of $x$

**phase shift:** the horizontal displacement of the basic sine or cosine function; the constant $\frac{C}{B}$

**sinusoidal function:** any function that can be expressed in the form $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$

KEY EQUATIONS

<table>
<thead>
<tr>
<th>Sinusoidal functions</th>
<th>$f(x) = A\sin(Bx - C) + D$</th>
<th>$f(x) = A\cos(Bx - C) + D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shifted, compressed, and/or stretched tangent function</td>
<td>$y = A\tan(Bx - C) + D$</td>
<td></td>
</tr>
<tr>
<td>Shifted, compressed, and/or stretched secant function</td>
<td>$y = A\sec(Bx - C) + D$</td>
<td></td>
</tr>
<tr>
<td>Shifted, compressed, and/or stretched cosecant function</td>
<td>$y = A\csc(Bx - C) + D$</td>
<td></td>
</tr>
<tr>
<td>Shifted, compressed, and/or stretched cotangent function</td>
<td>$y = A\cot(Bx - C) + D$</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.6

Table 6.7
KEY CONCEPTS

6.1 Graphs of the Sine and Cosine Functions

- Periodic functions repeat after a given value. The smallest such value is the period. The basic sine and cosine functions have a period of $2\pi$.
- The function $\sin x$ is odd, so its graph is symmetric about the origin. The function $\cos x$ is even, so its graph is symmetric about the $y$-axis.
- The graph of a sinusoidal function has the same general shape as a sine or cosine function.
- In the general formula for a sinusoidal function, the period is $P = \frac{2\pi}{|B|}$. See Example 6.1.
- In the general formula for a sinusoidal function, $|A|$ represents amplitude. If $|A| > 1$, the function is stretched, whereas if $|A| < 1$, the function is compressed. See Example 6.2.
- The value $\frac{C}{B}$ in the general formula for a sinusoidal function indicates the phase shift. See Example 6.3.
- The value $D$ in the general formula for a sinusoidal function indicates the vertical shift from the midline. See Example 6.4.
- Combinations of variations of sinusoidal functions can be detected from an equation. See Example 6.5.
- The equation for a sinusoidal function can be determined from a graph. See Example 6.6 and Example 6.7.
- A function can be graphed by identifying its amplitude and period. See Example 6.8 and Example 6.9.
- A function can also be graphed by identifying its amplitude, period, phase shift, and horizontal shift. See Example 6.10.
- Sinusoidal functions can be used to solve real-world problems. See Example 6.11, Example 6.12, and Example 6.13.

6.2 Graphs of the Other Trigonometric Functions

- The tangent function has period $\pi$.
- $f(x) = \tan(Bx - C) + D$ is a tangent with vertical and/or horizontal stretch/compression and shift. See Example 6.14, Example 6.15, and Example 6.16.
- The secant and cosecant are both periodic functions with a period of $2\pi$. $f(x) = \sec(Bx - C) + D$ gives a shifted, compressed, and/or stretched secant function graph. See Example 6.17 and Example 6.18.
- $f(x) = \csc(Bx - C) + D$ gives a shifted, compressed, and/or stretched cosecant function graph. See Example 6.19 and Example 6.20.
- The cotangent function has period $\pi$ and vertical asymptotes at $0, \pm \pi, \pm 2\pi, ...$
- The range of cotangent is $(-\infty, \infty)$, and the function is decreasing at each point in its range.
- The cotangent is zero at $\pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, ...$
- $f(x) = \cot(Bx - C) + D$ is a cotangent with vertical and/or horizontal stretch/compression and shift. See Example 6.21 and Example 6.22.
- Real-world scenarios can be solved using graphs of trigonometric functions. See Example 6.23.

6.3 Inverse Trigonometric Functions

- An inverse function is one that “undoes” another function. The domain of an inverse function is the range of the original function and the range of an inverse function is the domain of the original function.
- Because the trigonometric functions are not one-to-one on their natural domains, inverse trigonometric functions are defined for restricted domains.
For any trigonometric function \( f(x) \), if \( x = f^{-1}(y) \), then \( f(x) = y \). However, \( f(x) = y \) only implies \( x = f^{-1}(y) \) if \( x \) is in the restricted domain of \( f \). See Example 6.24.

Special angles are the outputs of inverse trigonometric functions for special input values; for example, \( \frac{\pi}{4} = \tan^{-1}(1) \) and \( \frac{\pi}{6} = \sin^{-1}\left(\frac{1}{2}\right) \). See Example 6.25.

A calculator will return an angle within the restricted domain of the original trigonometric function. See Example 6.26.

Inverse functions allow us to find an angle when given two sides of a right triangle. See Example 6.27.

In function composition, if the inside function is an inverse trigonometric function, then there are exact expressions; for example, \( \sin(\cos^{-1}(x)) = \sqrt{1 - x^2} \). See Example 6.28.

If the inside function is a trigonometric function, then the only possible combinations are \( \sin^{-1}(\cos x) = \frac{\pi}{2} - x \) if \( 0 \leq x \leq \pi \) and \( \cos^{-1}(\sin x) = \frac{\pi}{2} - x \) if \( -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \). See Example 6.29 and Example 6.30.

When evaluating the composition of a trigonometric function with an inverse trigonometric function, draw a reference triangle to assist in determining the ratio of sides that represents the output of the trigonometric function. See Example 6.31.

When evaluating the composition of a trigonometric function with an inverse trigonometric function, you may use trig identities to assist in determining the ratio of sides. See Example 6.32.

CHAPTER 6 REVIEW EXERCISES

Section 6.1
For the following exercises, graph the functions for two periods and determine the amplitude or stretching factor, period, midline equation, and asymptotes.

195. \( f(x) = -3\cos x + 3 \)

196. \( f(x) = \frac{1}{4}\sin x \)

197. \( f(x) = 3\cos\left(x + \frac{\pi}{6}\right) \)

198. \( f(x) = -2\sin\left(x - \frac{2\pi}{3}\right) \)

199. \( f(x) = 3\sin\left(x - \frac{\pi}{4}\right) - 4 \)

200. \( f(x) = 2\cos\left(x - \frac{4\pi}{3}\right) + 1 \)

201. \( f(x) = 6\sin\left(3x - \frac{\pi}{6}\right) - 1 \)

202. \( f(x) = -100\sin(50x - 20) \)

Section 6.2
For the following exercises, graph the functions for two periods and determine the amplitude or stretching factor, period, midline equation, and asymptotes.

203. \( f(x) = \tan x - 4 \)
204. \( f(x) = 2\tan\left(x - \frac{\pi}{6}\right) \)

205. \( f(x) = -3\tan(4x) - 2 \)

206. \( f(x) = 0.2\cos(0.1x) + 0.3 \)

For the following exercises, graph two full periods. Identify the period, the phase shift, the amplitude, and asymptotes.

207. \( f(x) = \frac{4}{3}\sec x \)

208. \( f(x) = 3\cot x \)

209. \( f(x) = 4\csc(5x) \)

210. \( f(x) = 8\sec\left(\frac{1}{4}x\right) \)

211. \( f(x) = \frac{2}{3}\csc\left(\frac{1}{2}x\right) \)

212. \( f(x) = -\csc(2x + \pi) \)

For the following exercises, use this scenario: The population of a city has risen and fallen over a 20-year interval. Its population may be modeled by the following function: 
\[ y = 12,000 + 8,000\sin(0.628x) \], where the domain is the years since 1980 and the range is the population of the city.

213. What is the largest and smallest population the city may have?

214. Graph the function on the domain of \([0, 40]\).

215. What are the amplitude, period, and phase shift for the function?

216. Over this domain, when does the population reach 18,000? 13,000?

217. What is the predicted population in 2007? 2010?

For the following exercises, suppose a weight is attached to a spring and bobs up and down, exhibiting symmetry.

218. Suppose the graph of the displacement function is shown in Figure 6.47, where the values on the \(x\)-axis represent the time in seconds and the \(y\)-axis represents the displacement in inches. Give the equation that models the vertical displacement of the weight on the spring.
219. At time = 0, what is the displacement of the weight?

220. At what time does the displacement from the equilibrium point equal zero?

221. What is the time required for the weight to return to its initial height of 5 inches? In other words, what is the period for the displacement function?

Section 6.3

For the following exercises, find the exact value without the aid of a calculator.

222. \( \sin^{-1}(1) \)

223. \( \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \)

224. \( \tan^{-1}(-1) \)

225. \( \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \)

226. \( \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) \)

227. \( \sin^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right) \)

228. \( \cos^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right) \)

229. \( \sin\left(\sec^{-1}\left(\frac{3}{5}\right)\right) \)

230. \( \cot\left(\sin^{-1}\left(\frac{3}{5}\right)\right) \)
231. \( \tan \left( \cos^{-1} \left( \frac{5}{13} \right) \right) \)

232. \( \sin \left( \cos^{-1} \left( \frac{x}{x+1} \right) \right) \)

233. Graph \( f(x) = \cos x \) and \( f(x) = \sec x \) on the interval \([0, 2\pi]\) and explain any observations.

234. Graph \( f(x) = \sin x \) and \( f(x) = \csc x \) and explain any observations.

235. Graph the function \( f(x) = \frac{1}{1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}} \) on the interval \([-1, 1]\) and compare the graph to the graph of \( f(x) = \sin x \) on the same interval. Describe any observations.

### CHAPTER 6 PRACTICE TEST

For the following exercises, sketch the graph of each function for two full periods. Determine the amplitude, the period, and the equation for the midline.

209. \( f(x) = 0.5 \sin x \)

210. \( f(x) = 5 \cos x \)

211. \( f(x) = 5 \sin x \)

212. \( f(x) = \sin(3x) \)

213. \( f(x) = -\cos \left( x + \frac{\pi}{3} \right) + 1 \)

214. \( f(x) = 5 \sin \left( 3 \left( x - \frac{\pi}{6} \right) \right) + 4 \)

215. \( f(x) = 3 \cos \left( \frac{1}{3}x - \frac{5\pi}{6} \right) \)

216. \( f(x) = \tan(4x) \)

217. \( f(x) = -2 \tan \left( x - \frac{7\pi}{6} \right) + 2 \)

218. \( f(x) = \pi \cos(3x + \pi) \)

219. \( f(x) = 5 \csc(3x) \)

220. \( f(x) = \pi \sec \left( \frac{4}{2}x \right) \)

221. \( f(x) = 2 \csc \left( x + \frac{\pi}{4} \right) - 3 \)

For the following exercises, determine the amplitude, period, and midline of the graph, and then find a formula for the function.
222. Give in terms of a sine function.

223. Give in terms of a sine function.

224. Give in terms of a tangent function.
For the following exercises, find the amplitude, period, phase shift, and midline.

225. \( y = \sin\left(\frac{\pi}{6}x + \pi\right) - 3 \)

226. \( y = 8\sin\left(\frac{7\pi}{6}x + \frac{7\pi}{2}\right) + 6 \)

227. The outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the temperature is 68°F at midnight and the high and low temperatures during the day are 80°F and 56°F, respectively. Assuming \( t \) is the number of hours since midnight, find a function for the temperature, \( D \), in terms of \( t \).

228. Water is pumped into a storage bin and empties according to a periodic rate. The depth of the water is 3 feet at its lowest at 2:00 a.m. and 71 feet at its highest, which occurs every 5 hours. Write a cosine function that models the depth of the water as a function of time, and then graph the function for one period.

For the following exercises, find the period and horizontal shift of each function.

229. \( g(x) = 3\tan(6x + 42) \)

230. \( n(x) = 4\csc\left(\frac{5\pi}{3}x - \frac{20\pi}{3}\right) \)

231. Write the equation for the graph in Figure 6.48 in terms of the secant function and give the period and phase shift.

![Figure 6.48](image)

232. If \( \tan x = 3 \), find \( \tan(-x) \).

233. If \( \sec x = 4 \), find \( \sec(-x) \).

For the following exercises, graph the functions on the specified window and answer the questions.

234. Graph \( m(x) = \sin(2x) + \cos(3x) \) on the viewing window \([-10, 10]\) by \([-3, 3]\). Approximate the graph’s period.

235. Graph \( n(x) = 0.02\sin(50\pi x) \) on the following domains in \( x \): \([0, 1]\) and \([0, 3]\). Suppose this function models sound waves. Why would these views look so different?

236. Graph \( f(x) = \frac{\sin x}{x} \) on \([-0.5, 0.5]\) and explain any observations.

For the following exercises, let \( f(x) = \frac{3}{5}\cos(6x) \).
237. What is the largest possible value for \( f(x) \)?

238. What is the smallest possible value for \( f(x) \)?

239. Where is the function increasing on the interval \([0, 2\pi]\)?

For the following exercises, find and graph one period of the periodic function with the given amplitude, period, and phase shift.

240. Sine curve with amplitude 3, period \( \frac{\pi}{3} \) and phase shift \((h, k) = \left(\frac{\pi}{4}, 2\right)\)

241. Cosine curve with amplitude 2, period \( \frac{\pi}{6} \) and phase shift \((h, k) = \left(-\frac{\pi}{4}, 3\right)\)

For the following exercises, graph the function. Describe the graph and, wherever applicable, any periodic behavior, amplitude, asymptotes, or undefined points.

242. \( f(x) = 5\cos(3x) + 4\sin(2x) \)

243. \( f(x) = e^{\sin x} \)

For the following exercises, find the exact value.

244. \( \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \)

245. \( \tan^{-1}(\sqrt{3}) \)

246. \( \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) \)

247. \( \cos^{-1}(\sin(x)) \)

248. \( \cos^{-1}\left(\tan\left(\frac{7\pi}{4}\right)\right) \)

249. \( \cos(\sin^{-1}(1 - 2x)) \)

250. \( \cos^{-1}(-0.4) \)

251. \( \cos(\tan^{-1}(x^2)) \)

For the following exercises, suppose \( \sin t = \frac{x}{x + 1} \).

252. \( \tan t \)

253. \( \csc t \)

254. Given Figure 6.49, find the measure of angle \( \theta \) to three decimal places. Answer in radians.
Figure 6.49

For the following exercises, determine whether the equation is true or false.

255. \( \arcsin \left( \sin \left( \frac{5\pi}{6} \right) \right) = \frac{5\pi}{6} \)

256. \( \arccos \left( \cos \left( \frac{5\pi}{6} \right) \right) = \frac{5\pi}{6} \)

257. The grade of a road is 7%. This means that for every horizontal distance of 100 feet on the road, the vertical rise is 7 feet. Find the angle the road makes with the horizontal in radians.
# 7 | Trigonometric Identities and Equations

## 7.1 | Introduction to Trigonometric Identities and Equations

### Learning Objectives

## 7.2 | Solving Trigonometric Equations with Identities

### Learning Objectives

## 7.3 | Sum and Difference Identities

### Learning Objectives

## 7.4 | Double-Angle, Half-Angle, and Reduction Formulas

### Learning Objectives

## 7.5 | Sum-to-Product and Product-to-Sum Formulas

### Learning Objectives

## 7.6 | Solving Trigonometric Equations

### Learning Objectives

## 7.7 | Modeling with Trigonometric Equations

### Learning Objectives
The world’s largest tree by volume, named General Sherman, stands 274.9 feet tall and resides in Northern California. Just how do scientists know its true height? A common way to measure the height involves determining the angle of

elevation, which is formed by the tree and the ground at a point some distance away from the base of the tree. This method is much more practical than climbing the tree and dropping a very long tape measure.

In this chapter, we will explore applications of trigonometry that will enable us to solve many different kinds of problems, including finding the height of a tree. We extend topics we introduced in Trigonometric Functions and investigate applications more deeply and meaningfully.

8.1 | Non-right Triangles: Law of Sines

Learning Objectives

In this section, you will:

- **8.1.1** Use the Law of Sines to solve oblique triangles.
- **8.1.2** Find the area of an oblique triangle using the sine function.
- **8.1.3** Solve applied problems using the Law of Sines.

Suppose two radar stations located 20 miles apart each detect an aircraft between them. The angle of elevation measured by the first station is 35 degrees, whereas the angle of elevation measured by the second station is 15 degrees. How can we determine the altitude of the aircraft? We see in Figure 8.2 that the triangle formed by the aircraft and the two stations is not a right triangle, so we cannot use what we know about right triangles. In this section, we will find out how to solve problems involving non-right triangles.

![Figure 8.2](image)

Using the Law of Sines to Solve Oblique Triangles

In any triangle, we can draw an **altitude**, a perpendicular line from one vertex to the opposite side, forming two right triangles. It would be preferable, however, to have methods that we can apply directly to non-right triangles without first having to create right triangles.

Any triangle that is not a right triangle is an **oblique triangle**. Solving an oblique triangle means finding the measurements of all three angles and all three sides. To do so, we need to start with at least three of these values, including at least one of the sides. We will investigate three possible oblique triangle problem situations:

1. **ASA (angle-side-angle)** We know the measurements of two angles and the included side. See Figure 8.3.

   ![Figure 8.3](image)

2. **AAS (angle-angle-side)** We know the measurements of two angles and a side that is not between the known angles. See Figure 8.4.

   ![Figure 8.4](image)

3. **SSA (side-side-angle)** We know the measurements of two sides and an angle that is not between the known sides. See Figure 8.5.
Knowing how to approach each of these situations enables us to solve oblique triangles without having to drop a perpendicular to form two right triangles. Instead, we can use the fact that the ratio of the measurement of one of the angles to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. Let’s see how this statement is derived by considering the triangle shown in Figure 8.6.

![Figure 8.6](image)

Using the right triangle relationships, we know that \( \sin \alpha = \frac{h}{b} \) and \( \sin \beta = \frac{h}{a} \). Solving both equations for \( h \) gives two different expressions for \( h \).

\[
\begin{align*}
  h &= b \sin \alpha \quad \text{and} \quad h = a \sin \beta
\end{align*}
\]

We then set the expressions equal to each other.

\[
\begin{align*}
  b \sin \alpha &= a \sin \beta \\
  \left( \frac{1}{ab} \right) b \sin \alpha &= (a \sin \beta) \left( \frac{1}{ab} \right) \\
  \sin \alpha &= \frac{\sin \beta}{b}
\end{align*}
\]

Multiply both sides by \( \frac{1}{ab} \).

\[
\begin{align*}
  \frac{\sin \alpha}{a} &= \frac{\sin \beta}{b} \\
  \frac{\sin \alpha}{a} &= \frac{\sin \gamma}{c}
\end{align*}
\]

Similarly, we can compare the other ratios.

\[
\begin{align*}
  \frac{\sin \alpha}{a} &= \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}
\end{align*}
\]

Collectively, these relationships are called the **Law of Sines**.

\[
\begin{align*}
  \frac{\sin \alpha}{a} &= \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}
\end{align*}
\]

Note the standard way of labeling triangles: angle \( \alpha \) (alpha) is opposite side \( a \); angle \( \beta \) (beta) is opposite side \( b \); and angle \( \gamma \) (gamma) is opposite side \( c \). See Figure 8.7.

![Figure 8.7](image)

While calculating angles and sides, be sure to carry the exact values through to the final answer. Generally, final answers are rounded to the nearest tenth, unless otherwise specified.
Law of Sines

Given a triangle with angles and opposite sides labeled as in Figure 8.7, the ratio of the measurement of an angle to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. All proportions will be equal. The Law of Sines is based on proportions and is presented symbolically two ways.

\[ \frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} \]  \hspace{1cm} (8.5)

\[ \frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} \]  \hspace{1cm} (8.6)

To solve an oblique triangle, use any pair of applicable ratios.

Example 8.1

Solving for Two Unknown Sides and Angle of an AAS Triangle

Solve the triangle shown in Figure 8.8 to the nearest tenth.

Solution

Solve the triangle shown in Figure 8.9 to the nearest tenth.

8.1 Solve the triangle shown in Figure 8.9 to the nearest tenth.

Using The Law of Sines to Solve SSA Triangles

We can use the Law of Sines to solve any oblique triangle, but some solutions may not be straightforward. In some cases, more than one triangle may satisfy the given criteria, which we describe as an ambiguous case. Triangles classified as SSA, those in which we know the lengths of two sides and the measurement of the angle opposite one of the given sides, may result in one or two solutions, or even no solution.
Possible Outcomes for SSA Triangles

Oblique triangles in the category SSA may have four different outcomes. Figure 8.10 illustrates the solutions with the known sides $a$ and $b$ and known angle $\alpha$.

- **No triangle, $a < h$**
- **Right triangle, $a = h$**
- **Two triangles, $a > h, a < b$**
- **One triangle, $a \geq b$**

Figure 8.10

**Example 8.2**

**Solving an Oblique SSA Triangle**

Solve the triangle in Figure 8.11 for the missing side and find the missing angle measures to the nearest tenth.

```
Solution
```

8.2 Given $\alpha = 80^\circ$, $a = 120$, and $b = 121$, find the missing side and angles. If there is more than one possible solution, show both.
Solving for the Unknown Sides and Angles of a SSA Triangle

In the triangle shown in Figure 8.12, solve for the unknown side and angles. Round your answers to the nearest tenth.

![Image of a triangle with sides labeled a, b, and c and angles labeled α, β, and γ.]

Solution

Given \( \alpha = 80^\circ \), \( a = 100 \), \( b = 10 \), find the missing side and angles. If there is more than one possible solution, show both. Round your answers to the nearest tenth.

Example 8.4

Finding the Triangles That Meet the Given Criteria

Find all possible triangles if one side has length 4 opposite an angle of 50°, and a second side has length 10.

Solution

Determine the number of triangles possible given \( a = 31 \), \( b = 26 \), \( \beta = 48^\circ \).

Finding the Area of an Oblique Triangle Using the Sine Function

Now that we can solve a triangle for missing values, we can use some of those values and the sine function to find the area of an oblique triangle. Recall that the area formula for a triangle is given as \( \text{Area} = \frac{1}{2}bh \), where \( b \) is base and \( h \) is height. For oblique triangles, we must find \( h \) before we can use the area formula. Observing the two triangles in Figure 8.13, one acute and one obtuse, we can drop a perpendicular to represent the height and then apply the trigonometric property \( \sin \alpha = \frac{\text{opposite}}{\text{hypotenuse}} \) to write an equation for area in oblique triangles. In the acute triangle, we have \( \sin \alpha = \frac{h}{c} \) or \( c \sin \alpha = h \). However, in the obtuse triangle, we drop the perpendicular outside the triangle and extend the base \( b \) to form a right triangle. The angle used in calculation is \( \alpha' \), or \( 180^\circ - \alpha \).
Thus,

\[ \text{Area} = \frac{1}{2} \text{(base)} \times \text{height} = \frac{1}{2} b (c \sin \alpha) \]  

\[ \text{(8.7)} \]

Similarly,

\[ \text{Area} = \frac{1}{2} a (b \sin \gamma) = \frac{1}{2} b (c \sin \beta) \]  

\[ \text{(8.8)} \]

**Area of an Oblique Triangle**

The formula for the area of an oblique triangle is given by

\[ \text{Area} = \frac{1}{2} bc \sin \alpha \]

\[ = \frac{1}{2} ac \sin \beta \]

\[ = \frac{1}{2} ab \sin \gamma \]

\[ \text{(8.9)} \]

This is equivalent to one-half of the product of two sides and the sine of their included angle.

**Example 8.5**

**Finding the Area of an Oblique Triangle**

Find the area of a triangle with sides \( a = 90 \), \( b = 52 \), and angle \( \gamma = 102^\circ \). Round the area to the nearest integer.

**Solution**

Find the area of the triangle given \( \beta = 42^\circ \), \( a = 7.2 \) ft, \( c = 3.4 \) ft. Round the area to the nearest tenth.

**Try It**

Find the area of the triangle given \( \beta = 42^\circ \), \( a = 7.2 \) ft, \( c = 3.4 \) ft. Round the area to the nearest tenth.

**Solving Applied Problems Using the Law of Sines**

The more we study trigonometric applications, the more we discover that the applications are countless. Some are flat, diagram-type situations, but many applications in calculus, engineering, and physics involve three dimensions and motion.
Finding an Altitude

Find the altitude of the aircraft in the problem introduced at the beginning of this section, shown in Figure 8.14. Round the altitude to the nearest tenth of a mile.

Solution

The diagram shown in Figure 8.15 represents the height of a blimp flying over a football stadium. Find the height of the blimp if the angle of elevation at the southern end zone, point A, is 70°, the angle of elevation from the northern end zone, point B, is 62°, and the distance between the viewing points of the two end zones is 145 yards.

Figure 8.15

Access these online resources for additional instruction and practice with trigonometric applications.

8.1 EXERCISES

Verbal
1. Describe the altitude of a triangle.
2. Compare right triangles and oblique triangles.
3. When can you use the Law of Sines to find a missing angle?
4. In the Law of Sines, what is the relationship between the angle in the numerator and the side in the denominator?
5. What type of triangle results in an ambiguous case?

Algebraic
For the following exercises, assume \( \alpha \) is opposite side \( a \), \( \beta \) is opposite side \( b \), and \( \gamma \) is opposite side \( c \). Solve each triangle, if possible. Round each answer to the nearest tenth.

6. \( \alpha = 43^\circ, \gamma = 69^\circ, a = 20 \)
7. \( \alpha = 35^\circ, \gamma = 73^\circ, c = 20 \)
8. \( \alpha = 60^\circ, \beta = 60^\circ, \gamma = 60^\circ \)
9. \( a = 4, \alpha = 60^\circ, \beta = 100^\circ \)
10. \( b = 10, \beta = 95^\circ, \gamma = 30^\circ \)

For the following exercises, use the Law of Sines to solve for the missing side for each oblique triangle. Round each answer to the nearest hundredth. Assume that angle \( A \) is opposite side \( a \), angle \( B \) is opposite side \( b \), and angle \( C \) is opposite side \( c \).

11. Find side \( b \) when \( A = 37^\circ, B = 49^\circ, c = 5 \).
12. Find side \( a \) when \( A = 132^\circ, C = 23^\circ, b = 10 \).
13. Find side \( c \) when \( B = 37^\circ, C = 21, b = 23 \).

For the following exercises, assume \( \alpha \) is opposite side \( a \), \( \beta \) is opposite side \( b \), and \( \gamma \) is opposite side \( c \). Determine whether there is no triangle, one triangle, or two triangles. Then solve each triangle, if possible. Round each answer to the nearest tenth.

14. \( \alpha = 119^\circ, a = 14, b = 26 \)
15. \( \gamma = 113^\circ, b = 10, c = 32 \)
16. \( b = 3.5, c = 5.3, \gamma = 80^\circ \)
17. \( a = 12, c = 17, \alpha = 35^\circ \)
18. \( a = 20.5, b = 35.0, \beta = 25^\circ \)
19. \( a = 7, c = 9, \alpha = 43^\circ \)
20. \( a = 7, b = 3, \beta = 24^\circ \)
21. \( b = 13, c = 5, \gamma = 10^\circ \)
22. \( a = 2.3, c = 1.8, \gamma = 28^\circ \)
23. $\beta = 119^\circ$, $b = 8.2$, $a = 11.3$

For the following exercises, use the Law of Sines to solve, if possible, the missing side or angle for each triangle or triangles in the ambiguous case. Round each answer to the nearest tenth.

24. Find angle $A$ when $a = 24$, $b = 5$, $B = 22^\circ$.

25. Find angle $A$ when $a = 13$, $b = 6$, $B = 20^\circ$.

26. Find angle $B$ when $A = 12^\circ$, $a = 2$, $b = 9$.

For the following exercises, find the area of the triangle with the given measurements. Round each answer to the nearest tenth.

27. $a = 5$, $c = 6$, $\beta = 35^\circ$

28. $b = 11$, $c = 8$, $\alpha = 28^\circ$

29. $a = 32$, $b = 24$, $\gamma = 75^\circ$

30. $a = 7.2$, $b = 4.5$, $\gamma = 43^\circ$

**Graphical**

For the following exercises, find the length of side $x$. Round to the nearest tenth.

31.

```
10
```

```
70^\circ
```

```
50^\circ
```

32.

```
6
```

```
25^\circ
```

```
120^\circ
```

33.

```
15
```

```
45^\circ
```

```
75^\circ
```

34.
For the following exercises, find the measure of angle $x$, if possible. Round to the nearest tenth.

35.

36.

For the following exercises, find the measure of angle $x$, if possible. Round to the nearest tenth.

37.

38.

39.

40.
41. Notice that $x$ is an obtuse angle.

42.

For the following exercises, solve the triangle. Round each answer to the nearest tenth.

43.

For the following exercises, find the area of each triangle. Round each answer to the nearest tenth.

44.
45. 

![Triangle with sides 18, 15, and 25° angles] 

46. 

![Triangle with sides 4.5, 2.9, and 3.5] 

47. 

![Triangle with sides 9, 11, and 51° angles] 

48. 

![Triangle with sides 25, 18, and 40° angles] 

49. 

![Triangle with sides 30, 115°, and 30° angles] 

Extensions

50. Find the radius of the circle in Figure 8.16. Round to the nearest tenth.
51. Find the diameter of the circle in Figure 8.17. Round to the nearest tenth.

52. Find \( m \angle ADC \) in Figure 8.18. Round to the nearest tenth.

53. Find \( AD \) in Figure 8.19. Round to the nearest tenth.

54. Solve both triangles in Figure 8.20. Round each answer to the nearest tenth.
55. Find $AB$ in the parallelogram shown in Figure 8.21.

56. Solve the triangle in Figure 8.22. (Hint: Draw a perpendicular from $H$ to $JK$). Round each answer to the nearest tenth.

57. Solve the triangle in Figure 8.23. (Hint: Draw a perpendicular from $N$ to $LM$). Round each answer to the nearest tenth.

58. In Figure 8.24, $ABCD$ is not a parallelogram. $\angle m$ is obtuse. Solve both triangles. Round each answer to the nearest tenth.
Real-World Applications

59. A pole leans away from the sun at an angle of \(7^\circ\) to the vertical, as shown in Figure 8.25. When the elevation of the sun is \(55^\circ\), the pole casts a shadow 42 feet long on the level ground. How long is the pole? Round the answer to the nearest tenth.

60. To determine how far a boat is from shore, two radar stations 500 feet apart find the angles out to the boat, as shown in Figure 8.26. Determine the distance of the boat from station \(A\) and the distance of the boat from shore. Round your answers to the nearest whole foot.

61. Figure 8.27 shows a satellite orbiting Earth. The satellite passes directly over two tracking stations \(A\) and \(B\), which are 69 miles apart. When the satellite is on one side of the two stations, the angles of elevation at \(A\) and \(B\) are measured to be \(86.2^\circ\) and \(83.9^\circ\), respectively. How far is the satellite from station \(A\) and how high is the satellite above the ground? Round answers to the nearest whole mile.
62. A communications tower is located at the top of a steep hill, as shown in Figure 8.28. The angle of inclination of the hill is $67^\circ$. A guy wire is to be attached to the top of the tower and to the ground, 165 meters downhill from the base of the tower. The angle formed by the guy wire and the hill is $16^\circ$. Find the length of the cable required for the guy wire to the nearest whole meter.

63. The roof of a house is at a $20^\circ$ angle. An 8-foot solar panel is to be mounted on the roof and should be angled $38^\circ$ relative to the horizontal for optimal results. (See Figure 8.29). How long does the vertical support holding up the back of the panel need to be? Round to the nearest tenth.

64. Similar to an angle of elevation, an angle of depression is the acute angle formed by a horizontal line and an observer’s line of sight to an object below the horizontal. A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 6.6 km apart, to be $37^\circ$ and $44^\circ$, as shown in Figure 8.30. Find the distance of the plane from point $A$ to the nearest tenth of a kilometer.
65. A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 4.3 km apart, to be 32° and 56°, as shown in Figure 8.31. Find the distance of the plane from point A to the nearest tenth of a kilometer.

66. In order to estimate the height of a building, two students stand at a certain distance from the building at street level. From this point, they find the angle of elevation from the street to the top of the building to be 39°. They then move 300 feet closer to the building and find the angle of elevation to be 50°. Assuming that the street is level, estimate the height of the building to the nearest foot.

67. In order to estimate the height of a building, two students stand at a certain distance from the building at street level. From this point, they find the angle of elevation from the street to the top of the building to be 35°. They then move 250 feet closer to the building and find the angle of elevation to be 53°. Assuming that the street is level, estimate the height of the building to the nearest foot.

68. Points A and B are on opposite sides of a lake. Point C is 97 meters from A. The measure of angle BAC is determined to be 101°, and the measure of angle ACB is determined to be 53°. What is the distance from A to B, rounded to the nearest whole meter?

69. A man and a woman standing 3 1/2 miles apart spot a hot air balloon at the same time. If the angle of elevation from the man to the balloon is 27°, and the angle of elevation from the woman to the balloon is 41°, find the altitude of the balloon to the nearest foot.

70. Two search teams spot a stranded climber on a mountain. The first search team is 0.5 miles from the second search team, and both teams are at an altitude of 1 mile. The angle of elevation from the first search team to the stranded climber is 15°. The angle of elevation from the second search team to the climber is 22°. What is the altitude of the climber? Round to the nearest tenth of a mile.

71. A street light is mounted on a pole. A 6-foot-tall man is standing on the street a short distance from the pole, casting a shadow. The angle of elevation from the tip of the man’s shadow to the top of his head is 28°. A 6-foot-tall woman is standing on the same street on the opposite side of the pole from the man. The angle of elevation from the tip of her shadow to the top of her head is 28°. If the man and woman are 20 feet apart, how far is the street light from the tip of the shadow of each person? Round the distance to the nearest tenth of a foot.

72. Three cities, A, B, and C, are located so that city A is due east of city B. If city C is located 35° west of north from city B and is 100 miles from city A and 70 miles from city B, how far is city A from city B? Round the distance to the nearest tenth of a mile.

73. Two streets meet at an 80° angle. At the corner, a park is being built in the shape of a triangle. Find the area of the park if, along one road, the park measures 180 feet, and along the other road, the park measures 215 feet.
74. Brian’s house is on a corner lot. Find the area of the front yard if the edges measure 40 and 56 feet, as shown in Figure 8.32.

Figure 8.32

75. The Bermuda triangle is a region of the Atlantic Ocean that connects Bermuda, Florida, and Puerto Rico. Find the area of the Bermuda triangle if the distance from Florida to Bermuda is 1030 miles, the distance from Puerto Rico to Bermuda is 980 miles, and the angle created by the two distances is 62°.

76. A yield sign measures 30 inches on all three sides. What is the area of the sign?

77. Naomi bought a modern dining table whose top is in the shape of a triangle. Find the area of the table top if two of the sides measure 4 feet and 4.5 feet, and the smaller angles measure 32° and 42°, as shown in Figure 8.33.
Suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles as shown in Figure 8.34. How far from port is the boat?

Unfortunately, while the Law of Sines enables us to address many non-right triangle cases, it does not help us with triangles where the known angle is between two known sides, a SAS (side-angle-side) triangle, or when all three sides are known, but no angles are known, a SSS (side-side-side) triangle. In this section, we will investigate another tool for solving oblique triangles described by these last two cases.

**Using the Law of Cosines to Solve Oblique Triangles**

The tool we need to solve the problem of the boat’s distance from the port is the Law of Cosines, which defines the relationship among angle measurements and side lengths in oblique triangles. Three formulas make up the Law of Cosines. At first glance, the formulas may appear complicated because they include many variables. However, once the pattern is understood, the Law of Cosines is easier to work with than most formulas at this mathematical level.

Understanding how the Law of Cosines is derived will be helpful in using the formulas. The derivation begins with the Generalized Pythagorean Theorem, which is an extension of the Pythagorean Theorem to non-right triangles. Here is how it works: An arbitrary non-right triangle $ABC$ is placed in the coordinate plane with vertex $A$ at the origin, side $c$ drawn along the x-axis, and vertex $C$ located at some point $(x, y)$ in the plane, as illustrated in Figure 8.35. Generally, triangles exist anywhere in the plane, but for this explanation we will place the triangle as noted.
We can drop a perpendicular from $C$ to the $x$-axis (this is the altitude or height). Recalling the basic trigonometric identities, we know that

$$\cos \theta = \frac{x\text{(adjacent)}}{b\text{(hypotenuse)}} \quad \text{and} \quad \sin \theta = \frac{y\text{(opposite)}}{b\text{(hypotenuse)}} \quad (8.10)$$

In terms of $\theta$, $x = b\cos \theta$ and $y = b\sin \theta$. The $(x, y)$ point located at $C$ has coordinates $(b\cos \theta, b\sin \theta)$. Using the side $(x - c)$ as one leg of a right triangle and $y$ as the second leg, we can find the length of hypotenuse $a$ using the Pythagorean Theorem. Thus,

$$a^2 = (x - c)^2 + y^2 \quad (8.11)$$

$$= (b\cos \theta - c)^2 + (b\sin \theta)^2 \quad \text{Substitute } (b\cos \theta) \text{ for } x \text{ and } (b\sin \theta) \text{ for } y.$$  

$$= b^2\cos^2 \theta - 2bc\cos \theta + c^2 + b^2\sin^2 \theta \quad \text{Expand the perfect square.}$$  

$$= b^2(\cos^2 \theta + \sin^2 \theta) + c^2 - 2bc\cos \theta \quad \text{Group terms noting that } \cos^2 \theta + \sin^2 \theta = 1.$$  

$$= b^2 + c^2 - 2bc\cos \theta \quad \text{Factor out } b^2.$$  

The formula derived is one of the three equations of the Law of Cosines. The other equations are found in a similar fashion. Keep in mind that it is always helpful to sketch the triangle when solving for angles or sides. In a real-world scenario, try to draw a diagram of the situation. As more information emerges, the diagram may have to be altered. Make those alterations to the diagram and, in the end, the problem will be easier to solve.

**Law of Cosines**

The **Law of Cosines** states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle. For triangles labeled as in **Figure 8.36**, with angles $\alpha$, $\beta$, and $\gamma$, and opposite corresponding sides $a$, $b$, and $c$, respectively, the Law of Cosines is given as three equations.

$$a^2 = b^2 + c^2 - 2bc \cos \alpha \quad (8.12)$$

$$b^2 = a^2 + c^2 - 2ac \cos \beta$$

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$
To solve for a missing side measurement, the corresponding opposite angle measure is needed. When solving for an angle, the corresponding opposite side measure is needed. We can use another version of the Law of Cosines to solve for an angle.

\[
\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}
\]

\[
\cos \beta = \frac{a^2 + c^2 - b^2}{2ac}
\]

\[
\cos \gamma = \frac{a^2 + b^2 - c^2}{2ab}
\]

**How To**

Given two sides and the angle between them (SAS), find the measures of the remaining side and angles of a triangle.

1. Sketch the triangle. Identify the measures of the known sides and angles. Use variables to represent the measures of the unknown sides and angles.
2. Apply the Law of Cosines to find the length of the unknown side or angle.
3. Apply the Law of Sines or Cosines to find the measure of a second angle.
4. Compute the measure of the remaining angle.

**Example 8.7**

**Finding the Unknown Side and Angles of a SAS Triangle**

Find the unknown side and angles of the triangle in Figure 8.37.

\[\alpha = 30°, \quad b = 12, \quad c = 24.\]

**Example 8.8**

**Solving for an Angle of a SSS Triangle**

Find the angle \(\alpha\) for the given triangle if side \(a = 20\), side \(b = 25\), and side \(c = 18\).
Solution

Analysis
Because the inverse cosine can return any angle between 0 and 180 degrees, there will not be any ambiguous cases using this method.

Given \( a = 5 \), \( b = 7 \), and \( c = 10 \), find the missing angles.

Solving Applied Problems Using the Law of Cosines

Just as the Law of Sines provided the appropriate equations to solve a number of applications, the Law of Cosines is applicable to situations in which the given data fits the cosine models. We may see these in the fields of navigation, surveying, astronomy, and geometry, just to name a few.

Example 8.9

Using the Law of Cosines to Solve a Communication Problem

On many cell phones with GPS, an approximate location can be given before the GPS signal is received. This is accomplished through a process called triangulation, which works by using the distances from two known points. Suppose there are two cell phone towers within range of a cell phone. The two towers are located 6000 feet apart along a straight highway, running east to west, and the cell phone is north of the highway. Based on the signal delay, it can be determined that the signal is 5050 feet from the first tower and 2420 feet from the second tower. Determine the position of the cell phone north and east of the first tower, and determine how far it is from the highway.

Solution

Example 8.10

Calculating Distance Traveled Using a SAS Triangle

Returning to our problem at the beginning of this section, suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles. How far from port is the boat? The diagram is repeated here in Figure 8.38.
Using Heron’s Formula to Find the Area of a Triangle

We already learned how to find the area of an oblique triangle when we know two sides and an angle. We also know the formula to find the area of a triangle using the base and the height. When we know the three sides, however, we can use Heron’s formula instead of finding the height. Heron of Alexandria was a geometer who lived during the first century A.D. He discovered a formula for finding the area of oblique triangles when three sides are known.

Heron’s Formula

Heron’s formula finds the area of oblique triangles in which sides \(a\), \(b\), and \(c\) are known.

\[
\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} \tag{8.14}
\]

where \(s = \frac{(a + b + c)}{2}\) is one half of the perimeter of the triangle, sometimes called the semi-perimeter.

**Example 8.11**

**Using Heron’s Formula to Find the Area of a Given Triangle**

Find the area of the triangle in Figure 8.39 using Heron’s formula.
Use Heron’s formula to find the area of a triangle with sides of lengths \( a = 29.7 \text{ ft} \), \( b = 42.3 \text{ ft} \), and \( c = 38.4 \text{ ft} \).

**Example 8.12**

**Applying Heron’s Formula to a Real-World Problem**

A Chicago city developer wants to construct a building consisting of artist’s lofts on a triangular lot bordered by Rush Street, Wabash Avenue, and Pearson Street. The frontage along Rush Street is approximately 62.4 meters, along Wabash Avenue it is approximately 43.5 meters, and along Pearson Street it is approximately 34.1 meters. How many square meters are available to the developer? See Figure 8.40 for a view of the city property.

**Try it**

**8.10** Find the area of a triangle given \( a = 4.38 \text{ ft} \), \( b = 3.79 \text{ ft} \), and \( c = 5.22 \text{ ft} \).
Access these online resources for additional instruction and practice with the Law of Cosines.

- Law of Cosines (http://openstaxcollege.org/l/lawcosines)
- Law of Cosines: Applications (http://openstaxcollege.org/l/cosineapp)
- Law of Cosines: Applications 2 (http://openstaxcollege.org/l/cosineapp2)
8.2 EXERCISES

Verbal

78. If you are looking for a missing side of a triangle, what do you need to know when using the Law of Cosines?

79. If you are looking for a missing angle of a triangle, what do you need to know when using the Law of Cosines?

80. Explain what $s$ represents in Heron’s formula.

81. Explain the relationship between the Pythagorean Theorem and the Law of Cosines.

82. When must you use the Law of Cosines instead of the Pythagorean Theorem?

Algebraic

For the following exercises, assume $\alpha$ is opposite side $a$, $\beta$ is opposite side $b$, and $\gamma$ is opposite side $c$. If possible, solve each triangle for the unknown side. Round to the nearest tenth.

83. $\gamma = 41.2^\circ$, $a = 2.49$, $b = 3.13$

84. $\alpha = 120^\circ$, $b = 6$, $c = 7$

85. $\beta = 58.7^\circ$, $a = 10.6$, $c = 15.7$

86. $\gamma = 115^\circ$, $a = 18$, $b = 23$

87. $\alpha = 119^\circ$, $a = 26$, $b = 14$

88. $\gamma = 113^\circ$, $b = 10$, $c = 32$

89. $\beta = 67^\circ$, $a = 49$, $b = 38$

90. $\alpha = 43.1^\circ$, $a = 184.2$, $b = 242.8$

91. $\alpha = 36.6^\circ$, $a = 186.2$, $b = 242.2$

92. $\beta = 50^\circ$, $a = 105$, $b = 45$

For the following exercises, use the Law of Cosines to solve for the missing angle of the oblique triangle. Round to the nearest tenth.

93. $a = 42$, $b = 19$, $c = 30$; find angle $A$.

94. $a = 14$, $b = 13$, $c = 20$; find angle $C$.

95. $a = 16$, $b = 31$, $c = 20$; find angle $B$.

96. $a = 13$, $b = 22$, $c = 28$; find angle $A$.

97. $a = 108$, $b = 132$, $c = 160$; find angle $C$.

For the following exercises, solve the triangle. Round to the nearest tenth.

98. $A = 35^\circ$, $b = 8$, $c = 11$

99. $B = 88^\circ$, $a = 4.4$, $c = 5.2$

100. $C = 121^\circ$, $a = 21$, $b = 37$

101. $a = 13$, $b = 11$, $c = 15$
102. \(a = 3.1, \ b = 3.5, \ c = 5\)

103. \(a = 51, \ b = 25, \ c = 29\)

For the following exercises, use Heron’s formula to find the area of the triangle. Round to the nearest hundredth.

104. Find the area of a triangle with sides of length 18 in, 21 in, and 32 in. Round to the nearest tenth.

105. Find the area of a triangle with sides of length 20 cm, 26 cm, and 37 cm. Round to the nearest tenth.

106. \(a = \frac{1}{2} \text{ m}, \ b = \frac{1}{3} \text{ m}, \ c = \frac{1}{4} \text{ m}\)

107. \(a = 12.4 \text{ ft}, \ b = 13.7 \text{ ft}, \ c = 20.2 \text{ ft}\)

108. \(a = 1.6 \text{ yd}, \ b = 2.6 \text{ yd}, \ c = 4.1 \text{ yd}\)

**Graphical**

For the following exercises, find the length of side \(x\). Round to the nearest tenth.

109.

110.

111.

112.
For the following exercises, find the measurement of angle $A$.

113. 

114. 

115. 

116.
117. Find the measure of each angle in the triangle shown in Figure 8.41. Round to the nearest tenth.

118. For the following exercises, solve for the unknown side. Round to the nearest tenth.

119. Find the measure of each angle in the triangle shown in Figure 8.41. Round to the nearest tenth.

Figure 8.41

For the following exercises, solve for the unknown side. Round to the nearest tenth.

120.
For the following exercises, find the area of the triangle. Round to the nearest hundredth.

121.

122.

123.

124.

125.
Extensions

126. A parallelogram has sides of length 16 units and 10 units. The shorter diagonal is 12 units. Find the measure of the longer diagonal.

127. The sides of a parallelogram are 11 feet and 17 feet. The longer diagonal is 22 feet. Find the length of the shorter diagonal.

128. The sides of a parallelogram are 28 centimeters and 40 centimeters. The measure of the larger angle is 100°. Find the length of the shorter diagonal.

132. A regular octagon is inscribed in a circle with a radius of 8 inches. (See Figure 8.42.) Find the perimeter of the octagon.
Figure 8.42

A regular pentagon is inscribed in a circle of radius 12 cm. (See Figure 8.43.) Find the perimeter of the pentagon. Round to the nearest tenth of a centimeter.

Figure 8.43

For the following exercises, suppose that \( x^2 = 25 + 36 - 60\cos(52) \) represents the relationship of three sides of a triangle and the cosine of an angle.

134. Draw the triangle.
135. Find the length of the third side.

For the following exercises, find the area of the triangle.
136.

137.

138.
Real-World Applications

139. A surveyor has taken the measurements shown in Figure 8.44. Find the distance across the lake. Round answers to the nearest tenth.

140. A satellite calculates the distances and angle shown in Figure 8.45 (not to scale). Find the distance between the two cities. Round answers to the nearest tenth.

141. An airplane flies 220 miles with a heading of 40°, and then flies 180 miles with a heading of 170°. How far is the plane from its starting point, and at what heading? Round answers to the nearest tenth.

142. A 113-foot tower is located on a hill that is inclined 34° to the horizontal, as shown in Figure 8.46. A guy-wire is to be attached to the top of the tower and anchored at a point 98 feet uphill from the base of the tower. Find the length of wire needed.
143. Two ships left a port at the same time. One ship traveled at a speed of 18 miles per hour at a heading of 320°. The other ship traveled at a speed of 22 miles per hour at a heading of 194°. Find the distance between the two ships after 10 hours of travel.

144. The graph in Figure 8.47 represents two boats departing at the same time from the same dock. The first boat is traveling at 18 miles per hour at a heading of 327° and the second boat is traveling at 4 miles per hour at a heading of 60°. Find the distance between the two boats after 2 hours.

145. A triangular swimming pool measures 40 feet on one side and 65 feet on another side. These sides form an angle that measures 50°. How long is the third side (to the nearest tenth)?

146. A pilot flies in a straight path for 1 hour 30 min. She then makes a course correction, heading 10° to the right of her original course, and flies 2 hours in the new direction. If she maintains a constant speed of 680 miles per hour, how far is she from her starting position?

147. Los Angeles is 1,744 miles from Chicago, Chicago is 714 miles from New York, and New York is 2,451 miles from Los Angeles. Draw a triangle connecting these three cities, and find the angles in the triangle.

148. Philadelphia is 140 miles from Washington, D.C., Washington, D.C. is 442 miles from Boston, and Boston is 315 miles from Philadelphia. Draw a triangle connecting these three cities and find the angles in the triangle.

149. Two planes leave the same airport at the same time. One flies at 20° east of north at 500 miles per hour. The second flies at 30° east of south at 600 miles per hour. How far apart are the planes after 2 hours?

150. Two airplanes take off in different directions. One travels 300 mph due west and the other travels 25° north of west at 420 mph. After 90 minutes, how far apart are they, assuming they are flying at the same altitude?

151. A parallelogram has sides of length 15.4 units and 9.8 units. Its area is 72.9 square units. Find the measure of the longer diagonal.

152.
The four sequential sides of a quadrilateral have lengths 4.5 cm, 7.9 cm, 9.4 cm, and 12.9 cm. The angle between the two smallest sides is 117°. What is the area of this quadrilateral?

153. The four sequential sides of a quadrilateral have lengths 5.7 cm, 7.2 cm, 9.4 cm, and 12.8 cm. The angle between the two smallest sides is 106°. What is the area of this quadrilateral?

154. Find the area of a triangular piece of land that measures 30 feet on one side and 42 feet on another; the included angle measures 132°. Round to the nearest whole square foot.

155. Find the area of a triangular piece of land that measures 110 feet on one side and 250 feet on another; the included angle measures 85°. Round to the nearest whole square foot.
8.3 | Polar Coordinates

Learning Objectives

In this section, you will:

8.3.1 Plot points using polar coordinates.
8.3.2 Convert from polar coordinates to rectangular coordinates.
8.3.3 Convert from rectangular coordinates to polar coordinates.
8.3.4 Transform equations between polar and rectangular forms.
8.3.5 Identify and graph polar equations by converting to rectangular equations.

Over 12 kilometers from port, a sailboat encounters rough weather and is blown off course by a 16-knot wind (see Figure 8.48). How can the sailor indicate his location to the Coast Guard? In this section, we will investigate a method of representing location that is different from a standard coordinate grid.

Plotting Points Using Polar Coordinates

When we think about plotting points in the plane, we usually think of rectangular coordinates \((x, y)\) in the Cartesian coordinate plane. However, there are other ways of writing a coordinate pair and other types of grid systems. In this section, we introduce to polar coordinates, which are points labeled \((r, \theta)\) and plotted on a polar grid. The polar grid is represented as a series of concentric circles radiating out from the pole, or the origin of the coordinate plane.

The polar grid is scaled as the unit circle with the positive \(x\)-axis now viewed as the polar axis and the origin as the pole. The first coordinate \(r\) is the radius or length of the directed line segment from the pole. The angle \(\theta\), measured in radians, indicates the direction of \(r\). We move counterclockwise from the polar axis by an angle of \(\theta\), and measure a directed line segment the length of \(r\) in the direction of \(\theta\). Even though we measure \(\theta\) first and then \(r\), the polar point is written with the \(r\)-coordinate first. For example, to plot the point \((2, \frac{\pi}{4})\), we would move \(\frac{\pi}{4}\) units in the counterclockwise direction and then a length of 2 from the pole. This point is plotted on the grid in Figure 8.49.
Example 8.13

Plotting a Point on the Polar Grid

Plot the point \((3, \frac{\pi}{2})\) on the polar grid.

Solution

Example 8.14

Plotting a Point in the Polar Coordinate System with a Negative Component

Plot the point \((-2, \frac{\pi}{6})\) on the polar grid.

Solution

Try It

8.11 Plot the point \((2, \frac{\pi}{3})\) in the polar plane.

8.12 Plot the points \((3, -\frac{\pi}{6})\) and \((2, \frac{9\pi}{4})\) on the same polar grid.

Converting from Polar Coordinates to Rectangular Coordinates

When given a set of polar coordinates, we may need to convert them to rectangular coordinates. To do so, we can recall the relationships that exist among the variables \(x, y, r,\) and \(\theta\).
\[ \cos \theta = \frac{x}{r} \rightarrow x = r \cos \theta \quad \text{(8.15)} \]
\[ \sin \theta = \frac{y}{r} \rightarrow y = r \sin \theta \]

Dropping a perpendicular from the point in the plane to the x-axis forms a right triangle, as illustrated in Figure 8.50. An easy way to remember the equations above is to think of \( \cos \theta \) as the adjacent side over the hypotenuse and \( \sin \theta \) as the opposite side over the hypotenuse.

**Figure 8.50**

---

**Converting from Polar Coordinates to Rectangular Coordinates**

To convert polar coordinates \((r, \theta)\) to rectangular coordinates \((x, y)\), let
\[ \cos \theta = \frac{x}{r} \rightarrow x = r \cos \theta \quad \text{(8.16)} \]
\[ \sin \theta = \frac{y}{r} \rightarrow y = r \sin \theta \quad \text{(8.17)} \]

**How To:**

1. Given the polar coordinate \((r, \theta)\), write \(x = r \cos \theta\) and \(y = r \sin \theta\).
2. Evaluate \(\cos \theta\) and \(\sin \theta\).
3. Multiply \(\cos \theta\) by \(r\) to find the \(x\)-coordinate of the rectangular form.
4. Multiply \(\sin \theta\) by \(r\) to find the \(y\)-coordinate of the rectangular form.

---

**Example 8.15**

**Writing Polar Coordinates as Rectangular Coordinates**

Write the polar coordinates \(\left(3, \frac{\pi}{2}\right)\) as rectangular coordinates.

**Solution**

---

**Example 8.16**

**Writing Polar Coordinates as Rectangular Coordinates**
Write the polar coordinates \((-2, 0)\) as rectangular coordinates.

Solution

8.13 Write the polar coordinates \((-1, \frac{2\pi}{3})\) as rectangular coordinates.

Converting from Rectangular Coordinates to Polar Coordinates

To convert rectangular coordinates to polar coordinates, we will use two other familiar relationships. With this conversion, however, we need to be aware that a set of rectangular coordinates will yield more than one polar point.

Converting from Rectangular Coordinates to Polar Coordinates

Converting from rectangular coordinates to polar coordinates requires the use of one or more of the relationships illustrated in Figure 8.51.

\[
\begin{align*}
\cos \theta &= \frac{x}{r} \quad \text{or} \quad x = r\cos \theta \\
\sin \theta &= \frac{y}{r} \quad \text{or} \quad y = r\sin \theta \\
r^2 &= x^2 + y^2 \\
\tan \theta &= \frac{y}{x}
\end{align*}
\]  

Figure 8.51

Example 8.17

Writing Rectangular Coordinates as Polar Coordinates

Convert the rectangular coordinates \((3, 3)\) to polar coordinates.

Solution

Analysis

There are other sets of polar coordinates that will be the same as our first solution. For example, the points \((-3\sqrt{2}, \frac{5\pi}{4})\) and \((3\sqrt{2}, -\frac{7\pi}{4})\) will coincide with the original solution of \((3\sqrt{2}, \frac{\pi}{4})\). The point \((-3\sqrt{2}, \frac{5\pi}{4})\)
indicates a move further counterclockwise by $\pi$, which is directly opposite $\frac{\pi}{4}$. The radius is expressed as $-\frac{\sqrt{2}}{2}$. However, the angle $\frac{5\pi}{4}$ is located in the third quadrant and, as $r$ is negative, we extend the directed line segment in the opposite direction, into the first quadrant. This is the same point as $\left(3\sqrt{2}, \frac{\pi}{4}\right)$. The point $\left(3\sqrt{2}, -\frac{7\pi}{4}\right)$ is a move further clockwise by $-\frac{7\pi}{4}$ from $\frac{\pi}{4}$. The radius, $3\sqrt{2}$, is the same.

**Transforming Equations between Polar and Rectangular Forms**

We can now convert coordinates between polar and rectangular form. Converting equations can be more difficult, but it can be beneficial to be able to convert between the two forms. Since there are a number of polar equations that cannot be expressed clearly in Cartesian form, and vice versa, we can use the same procedures we used to convert points between the coordinate systems. We can then use a graphing calculator to graph either the rectangular form or the polar form of the equation.

**Given an equation in polar form, graph it using a graphing calculator.**

1. Change the **MODE** to **POL**, representing polar form.
2. Press the **Y=** button to bring up a screen allowing the input of six equations: $r_1$, $r_2$, . . . , $r_6$.
3. Enter the polar equation, set equal to $r$.
4. Press **GRAPH**.

**Example 8.18**

**Writing a Cartesian Equation in Polar Form**

Write the Cartesian equation $x^2 + y^2 = 9$ in polar form.

**Solution**

**Example 8.19**

**Rewriting a Cartesian Equation as a Polar Equation**

Rewrite the Cartesian equation $x^2 + y^2 = 6y$ as a polar equation.

**Solution**

**Example 8.20**

**Rewriting a Cartesian Equation in Polar Form**
Rewrite the Cartesian equation \( y = 3x + 2 \) as a polar equation.

Solution

8.14 Rewrite the Cartesian equation \( y^2 = 3 - x^2 \) in polar form.

Identify and Graph Polar Equations by Converting to Rectangular Equations

We have learned how to convert rectangular coordinates to polar coordinates, and we have seen that the points are indeed the same. We have also transformed polar equations to rectangular equations and vice versa. Now we will demonstrate that their graphs, while drawn on different grids, are identical.

Example 8.21

**Graphing a Polar Equation by Converting to a Rectangular Equation**

Convert the polar equation \( r = 2\sec \theta \) to a rectangular equation, and draw its corresponding graph.

Solution

Example 8.22

**Rewriting a Polar Equation in Cartesian Form**

Rewrite the polar equation \( r = \frac{3}{1 - \frac{2}{3}\cos \theta} \) as a Cartesian equation.

Solution

**Analysis**

In this example, the right side of the equation can be expanded and the equation simplified further, as shown above. However, the equation cannot be written as a single function in Cartesian form. We may wish to write the rectangular equation in the hyperbola’s standard form. To do this, we can start with the initial equation.
\[
\begin{align*}
\quad & x^2 + y^2 = (3 + 2x)^2 \\
\quad & x^2 + y^2 - (3 + 2x)^2 = 0 \\
\quad & x^2 + y^2 - (9 + 12x + 4x^2) = 0 \\
\quad & x^2 + y^2 - 9 - 12x - 4x^2 = 0 \\
\quad & -3x^2 - 12x + y^2 = 9 \quad \text{Multiply through by } -1. \\
\quad & 3x^2 + 12x - y^2 = -9 \\
\quad & 3(x^2 + 4x + \quad ) - y^2 = -9 \quad \text{Organize terms to complete the square for } x. \\
\quad & 3(x^2 + 4x + 4) - y^2 = -9 + 12 \\
\quad & 3(x + 2)^2 - y^2 = 3 \\
\quad & (x + 2)^2 - \frac{y^2}{3} = 1
\end{align*}
\]

**Example 8.23**

**Rewriting a Polar Equation in Cartesian Form**

Rewrite the polar equation \( r = \sin(2\theta) \) in Cartesian form.

**Solution**

Access these online resources for additional instruction and practice with polar coordinates.

- [Introduction to Polar Coordinates](http://openstaxcollege.org/l/intropolar)
- [Comparing Polar and Rectangular Coordinates](http://openstaxcollege.org/l/polarrect)
8.3 EXERCISES

Verbal

156. How are polar coordinates different from rectangular coordinates?

157. How are the polar axes different from the \( x \)- and \( y \)-axes of the Cartesian plane?

158. Explain how polar coordinates are graphed.

159. How are the points \( \left( 3, \frac{\pi}{2} \right) \) and \( \left( -3, \frac{\pi}{2} \right) \) related?

160. Explain why the points \( \left( -3, \frac{\pi}{2} \right) \) and \( \left( 3, -\frac{\pi}{2} \right) \) are the same.

Algebraic

For the following exercises, convert the given polar coordinates to Cartesian coordinates with \( r > 0 \) and \( 0 \leq \theta \leq 2\pi \). Remember to consider the quadrant in which the given point is located when determining \( \theta \) for the point.

161. \( (7, \frac{7\pi}{6}) \)

162. \( (5, \pi) \)

163. \( (6, -\frac{\pi}{4}) \)

164. \( (-3, \frac{\pi}{6}) \)

165. \( (4, \frac{7\pi}{4}) \)

For the following exercises, convert the given Cartesian coordinates to polar coordinates with \( r > 0 \), \( 0 \leq \theta < 2\pi \). Remember to consider the quadrant in which the given point is located.

166. \( (4, 2) \)

167. \( (-4, 6) \)

168. \( (3, -5) \)

169. \( (-10, -13) \)

170. \( (8, 8) \)

For the following exercises, convert the given Cartesian equation to a polar equation.

171. \( x = 3 \)

172. \( y = 4 \)

173. \( y = 4x^2 \)

174. \( y = 2x^4 \)

175. \( x^2 + y^2 = 4y \)

176. \( x^2 + y^2 = 3x \)
177. \( x^2 - y^2 = x \)
178. \( x^2 - y^2 = 3y \)
179. \( x^2 + y^2 = 9 \)
180. \( x^2 = 9y \)
181. \( y^2 = 9x \)
182. \( 9xy = 1 \)

For the following exercises, convert the given polar equation to a Cartesian equation. Write in the standard form of a conic if possible, and identify the conic section represented.

183. \( r = 3\sin \theta \)
184. \( r = 4\cos \theta \)
185. \( r = \frac{4}{\sin \theta + 7\cos \theta} \)
186. \( r = \frac{6}{\cos \theta + 3\sin \theta} \)
187. \( r = 2\sec \theta \)
188. \( r = 3\csc \theta \)
189. \( r = \frac{1}{r \cos \theta + 2} \)
190. \( r^2 = 4\sec \theta \csc \theta \)
191. \( r = 4 \)
192. \( r^2 = 4 \)
193. \( r = \frac{1}{4\cos \theta - 3\sin \theta} \)
194. \( r = \frac{3}{\cos \theta - 5\sin \theta} \)

**Graphical**

For the following exercises, find the polar coordinates of the point.

195. 
196.

197.

198.
For the following exercises, plot the points.

200. \((-2, \frac{\pi}{3})\)

201. \((-1, -\frac{\pi}{2})\)

202. \((3.5, \frac{7\pi}{4})\)

203. \((-4, \frac{\pi}{3})\)

204. \((5, \frac{\pi}{2})\)

205. \((4, -\frac{5\pi}{4})\)

206. \((3, \frac{5\pi}{6})\)

207. \((-1.5, \frac{7\pi}{6})\)

208. \((-2, \frac{\pi}{4})\)

209.
For the following exercises, convert the equation from rectangular to polar form and graph on the polar axis.

210. \( 5x - y = 6 \)
211. \( 2x + 7y = -3 \)
212. \( x^2 + (y - 1)^2 = 1 \)
213. \( (x + 2)^2 + (y + 3)^2 = 13 \)
214. \( x = 2 \)
215. \( x^2 + y^2 = 5y \)
216. \( x^2 + y^2 = 3x \)

For the following exercises, convert the equation from polar to rectangular form and graph on the rectangular plane.

217. \( r = 6 \)
218. \( r = -4 \)
219. \( \theta = -\frac{2\pi}{3} \)
220. \( \theta = \frac{\pi}{4} \)
221. \( r = \sec \theta \)
222. \( r = -10\sin \theta \)
223. \( r = 3\cos \theta \)

**Technology**

224. Use a graphing calculator to find the rectangular coordinates of \( \left(2, -\frac{\pi}{5}\right) \). Round to the nearest thousandth.

225. Use a graphing calculator to find the rectangular coordinates of \( \left(-3, \frac{3\pi}{5}\right) \). Round to the nearest thousandth.

226. Use a graphing calculator to find the polar coordinates of \((-7, 8)\) in degrees. Round to the nearest thousandth.

227. Use a graphing calculator to find the polar coordinates of \((3, -4)\) in degrees. Round to the nearest hundredth.

228. Use a graphing calculator to find the polar coordinates of \((-2, 0)\) in radians. Round to the nearest hundredth.

**Extensions**

229. Describe the graph of \( r = a\sec \theta; \ a > 0 \).

230. Describe the graph of \( r = a\sec \theta; \ a < 0 \).

231. Describe the graph of \( r = a\csc \theta; \ a > 0 \).

232. Describe the graph of \( r = a\csc \theta; \ a < 0 \).
233. What polar equations will give an oblique line?

For the following exercise, graph the polar inequality.

234. \( r < 4 \)

235. \( 0 \leq \theta \leq \frac{\pi}{4} \)

236. \( \theta = \frac{\pi}{4}, \ r \geq 2 \)

237. \( \theta = \frac{\pi}{4}, \ r \geq -3 \)

238. \( 0 \leq \theta \leq \frac{\pi}{3}, \ r < 2 \)

239. \( -\frac{\pi}{6} < \theta \leq \frac{\pi}{3}, \ -3 < r < 2 \)
8.4 | Polar Coordinates: Graphs

Learning Objectives

In this section you will:

8.4.1 Test polar equations for symmetry.
8.4.2 Graph polar equations by plotting points.

The planets move through space in elliptical, periodic orbits about the sun, as shown in Figure 8.52. They are in constant motion, so fixing an exact position of any planet is valid only for a moment. In other words, we can fix only a planet’s instantaneous position. This is one application of polar coordinates, represented as \((r, \theta)\). We interpret \(r\) as the distance from the sun and \(\theta\) as the planet’s angular bearing, or its direction from a fixed point on the sun. In this section, we will focus on the polar system and the graphs that are generated directly from polar coordinates.

Figure 8.52  Planets follow elliptical paths as they orbit around the Sun. (credit: modification of work by NASA/JPL-Caltech)

Testing Polar Equations for Symmetry

Just as a rectangular equation such as \(y = x^2\) describes the relationship between \(x\) and \(y\) on a Cartesian grid, a polar equation describes a relationship between \(r\) and \(\theta\) on a polar grid. Recall that the coordinate pair \((r, \theta)\) indicates that we move counterclockwise from the polar axis (positive \(x\)-axis) by an angle of \(\theta\), and extend a ray from the pole (origin) \(r\) units in the direction of \(\theta\). All points that satisfy the polar equation are on the graph.

Symmetry is a property that helps us recognize and plot the graph of any equation. If an equation has a graph that is symmetric with respect to an axis, it means that if we folded the graph in half over that axis, the portion of the graph on one side would coincide with the portion on the other side. By performing three tests, we will see how to apply the properties of symmetry to polar equations. Further, we will use symmetry (in addition to plotting key points, zeros, and maximums of \(r\)) to determine the graph of a polar equation.

In the first test, we consider symmetry with respect to the line \(\theta = \frac{\pi}{2}\) (y-axis). We replace \((r, \theta)\) with \((-r, -\theta)\) to determine if the new equation is equivalent to the original equation. For example, suppose we are given the equation \(r = 2\sin\theta\);

\[
\begin{align*}
\text{Original:} & & r &= 2\sin\theta \\
\text{Replace:} & & -r &= 2\sin(-\theta) \\
\text{Identity:} & & -r &= -2\sin\theta \\
\text{Multiply:} & & r &= 2\sin\theta
\end{align*}
\]

This content is available for free at http://legacy.cnx.org/content/col11667/1.2
This equation exhibits symmetry with respect to the line \( \theta = \frac{\pi}{2} \).

In the second test, we consider symmetry with respect to the polar axis (x-axis). We replace \((r, \theta)\) with \((-r, \theta)\) or \((-r, \pi - \theta)\) to determine equivalency between the tested equation and the original. For example, suppose we are given the equation \( r = 1 - 2\cos \theta \).

\[
\begin{align*}
    r &= 1 - 2\cos \theta \\
    r &= 1 - 2\cos(-\theta) \quad \text{Replace \((r, \theta)\) with \((-r, \theta)\).} \\
    r &= 1 - 2\cos \theta \quad \text{Even/Odd identity}
\end{align*}
\]

The graph of this equation exhibits symmetry with respect to the polar axis.

In the third test, we consider symmetry with respect to the pole (origin). We replace \((r, \theta)\) with \((-r, \theta)\) to determine if the tested equation is equivalent to the original equation. For example, suppose we are given the equation \( r = 2\sin(3\theta) \).

\[
\begin{align*}
    r &= 2\sin(3\theta) \\
    -r &= 2\sin(3\theta)
\end{align*}
\]

The equation has failed the symmetry test, but that does not mean that it is not symmetric with respect to the pole. Passing one or more of the symmetry tests verifies that symmetry will be exhibited in a graph. However, failing the symmetry tests does not necessarily indicate that a graph will not be symmetric about the line \( \theta = \frac{\pi}{2} \), the polar axis, or the pole. In these instances, we can confirm that symmetry exists by plotting reflecting points across the apparent axis of symmetry or the pole. Testing for symmetry is a technique that simplifies the graphing of polar equations, but its application is not perfect.

**Symmetry Tests**

A **polar equation** describes a curve on the polar grid. The graph of a polar equation can be evaluated for three types of symmetry, as shown in **Figure 8.53**.

**Figure 8.53** (a) A graph is symmetric with respect to the line \( \theta = \frac{\pi}{2} \) (y-axis) if replacing \((r, \theta)\) with \((-r, -\theta)\) yields an equivalent equation. (b) A graph is symmetric with respect to the polar axis (x-axis) if replacing \((r, \theta)\) with \((r, -\theta)\) or \((-r, \pi - \theta)\) yields an equivalent equation. (c) A graph is symmetric with respect to the pole (origin) if replacing \((r, \theta)\) with \((-r, \theta)\) yields an equivalent equation.

**How To:**

1. Substitute the appropriate combination of components for \((r, \theta)\): \((-r, -\theta)\) for \( \theta = \frac{\pi}{2} \) symmetry; \((r, -\theta)\) for polar axis symmetry; and \((-r, \theta)\) for symmetry with respect to the pole.
2. If the resulting equations are equivalent in one or more of the tests, the graph produces the expected symmetry.

**Example 8.24**
Testing a Polar Equation for Symmetry

Test the equation $r = 2\sin \theta$ for symmetry.

Solution

Analysis

Using a graphing calculator, we can see that the equation $r = 2\sin \theta$ is a circle centered at $(0, 1)$ with radius $r = 1$ and is indeed symmetric to the line $\theta = \frac{\pi}{2}$. We can also see that the graph is not symmetric with the polar axis or the pole. See Figure 8.53.

Graphing Polar Equations by Plotting Points

To graph in the rectangular coordinate system we construct a table of $x$ and $y$ values. To graph in the polar coordinate system we construct a table of $\theta$ and $r$ values. We enter values of $\theta$ into a polar equation and calculate $r$. However, using the properties of symmetry and finding key values of $\theta$ and $r$ means fewer calculations will be needed.

Finding Zeros and Maxima

To find the zeros of a polar equation, we solve for the values of $\theta$ that result in $r = 0$. Recall that, to find the zeros of polynomial functions, we set the equation equal to zero and then solve for $x$. We use the same process for polar equations. Set $r = 0$, and solve for $\theta$.

For many of the forms we will encounter, the maximum value of a polar equation is found by substituting those values of $\theta$ into the equation that result in the maximum value of the trigonometric functions. Consider $r = 5\cos \theta$; the maximum distance between the curve and the pole is 5 units. The maximum value of the cosine function is 1 when $\theta = 0$, so our polar equation is $5\cos \theta$, and the value $\theta = 0$ will yield the maximum $|r|$.

Similarly, the maximum value of the sine function is 1 when $\theta = \frac{\pi}{2}$, and if our polar equation is $r = 5\sin \theta$, the value $\theta = \frac{\pi}{2}$ will yield the maximum $|r|$. We may find additional information by calculating values of $r$ when $\theta = 0$. These points would be polar axis intercepts, which may be helpful in drawing the graph and identifying the curve of a polar equation.
Example 8.25
Finding Zeros and Maximum Values for a Polar Equation

Using the equation in Example 8.24, find the zeros and maximum $|r|$ and, if necessary, the polar axis intercepts of $r = 2\sin \theta$.

Solution

Analysis

The point $\left(2, \frac{\pi}{2}\right)$ will be the maximum value on the graph. Let’s plot a few more points to verify the graph of a circle. See Table 8.0 and Figure 8.53.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$r = 2\sin \theta$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$r = 2\sin(0) = 0$</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{\pi}{6}$</td>
<td>$r = 2\sin\left(\frac{\pi}{6}\right) = 1$</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{\pi}{3}$</td>
<td>$r = 2\sin\left(\frac{\pi}{3}\right) \approx 1.73$</td>
<td>1.73</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>$r = 2\sin\left(\frac{\pi}{2}\right) = 2$</td>
<td>2</td>
</tr>
<tr>
<td>$\frac{2\pi}{3}$</td>
<td>$r = 2\sin\left(\frac{2\pi}{3}\right) \approx 1.73$</td>
<td>1.73</td>
</tr>
<tr>
<td>$\frac{5\pi}{6}$</td>
<td>$r = 2\sin\left(\frac{5\pi}{6}\right) = 1$</td>
<td>1</td>
</tr>
<tr>
<td>$\pi$</td>
<td>$r = 2\sin(\pi) = 0$</td>
<td>0</td>
</tr>
</tbody>
</table>
Without converting to Cartesian coordinates, test the given equation for symmetry and find the zeros and maximum values of \(|r|\): 

\[ r = 3 \cos \theta. \]

**Investigating Circles**

Now we have seen the equation of a circle in the polar coordinate system. In the last two examples, the same equation was used to illustrate the properties of symmetry and demonstrate how to find the zeros, maximum values, and plotted points that produced the graphs. However, the circle is only one of many shapes in the set of polar curves.

There are five classic polar curves: **cardioids, limaçons, lemniscates, rose curves, and Archimedes’ spirals**. We will briefly touch on the polar formulas for the circle before moving on to the classic curves and their variations.

**Formulas for the Equation of a Circle**

Some of the formulas that produce the graph of a circle in polar coordinates are given by \( r = a \cos \theta \) and \( r = a \sin \theta \), where \( a \) is the diameter of the circle or the distance from the pole to the farthest point on the circumference. The radius is \( \frac{|a|}{2} \), or one-half the diameter. For \( r = a \cos \theta \), the center is \( \left( \frac{a}{2}, 0 \right) \). For \( r = a \sin \theta \), the center is \( \left( \frac{a}{2}, \pi \right) \). Figure 8.54 shows the graphs of these four circles.

**Figure 8.54**

\[ r = a \cos \theta, \ a > 0 \]  
\[ r = a \cos \theta, \ a < 0 \]  
\[ r = a \sin \theta, \ a > 0 \]  
\[ r = a \sin \theta, \ a < 0 \]  

**Example 8.26**
Sketching the Graph of a Polar Equation for a Circle

Sketch the graph of \( r = 4 \cos \theta \).

Investigating Cardioids

While translating from polar coordinates to Cartesian coordinates may seem simpler in some instances, graphing the classic curves is actually less complicated in the polar system. The next curve is called a cardioid, as it resembles a heart. This shape is often included with the family of curves called limaçons, but here we will discuss the cardioid on its own.

Formulas for a Cardioid

The formulas that produce the graphs of a cardioid are given by \( r = a \pm b \cos \theta \) and \( r = a \pm b \sin \theta \) where \( a > 0 \), \( b > 0 \), and \( \frac{a}{b} = 1 \). The cardioid graph passes through the pole, as we can see in Figure 8.55.

![Figure 8.55](image)

**How To:**

1. Check equation for the three types of symmetry.
2. Find the zeros. Set \( r = 0 \).
3. Find the maximum value of the equation according to the maximum value of the trigonometric expression.
4. Make a table of values for \( r \) and \( \theta \).
5. Plot the points and sketch the graph.

Example 8.27

Sketching the Graph of a Cardioid

Sketch the graph of \( r = 2 + 2 \cos \theta \).
Investigating Limaçons

The word *limaçon* is Old French for “snail,” a name that describes the shape of the graph. As mentioned earlier, the cardioid is a member of the limaçon family, and we can see the similarities in the graphs. The other images in this category include the one-loop limaçon and the two-loop (or inner-loop) limaçon. One-loop limaçons are sometimes referred to as dimpled limaçons when \( 1 < \frac{a}{b} < 2 \) and convex limaçons when \( \frac{a}{b} \geq 2 \).

### Formulas for One-Loop Limaçons

The formulas that produce the graph of a dimpled one-loop limaçon are given by

\[
r = a \pm b \cos \theta \quad \text{and} \quad r = a \pm b \sin \theta
\]

where \( a > 0, \ b > 0, \) and \( 1 < \frac{a}{b} < 2 \). All four graphs are shown in Figure 8.56.

![Figure 8.56 Dimpled limaçons](image)

#### How to

Given a polar equation for a one-loop limaçon, sketch the graph.

1. Test the equation for symmetry. Remember that failing a symmetry test does not mean that the shape will not exhibit symmetry. Often the symmetry may reveal itself when the points are plotted.
2. Find the zeros.
3. Find the maximum values according to the trigonometric expression.
4. Make a table.
5. Plot the points and sketch the graph.

### Example 8.28

**Sketching the Graph of a One-Loop Limaçon**

Graph the equation \( r = 4 - 3 \sin \theta \).

**Solution**

**Analysis**

This is an example of a curve for which making a table of values is critical to producing an accurate graph. The symmetry tests fail; the zero is undefined. While it may be apparent that an equation involving \( \sin \theta \) is likely symmetric with respect to the line \( \theta = \frac{\pi}{2} \), evaluating more points helps to verify that the graph is correct.

8.18 Sketch the graph of \( r = 3 - 2 \cos \theta \).
Another type of limaçon, the **inner-loop limaçon**, is named for the loop formed inside the general limaçon shape. It was discovered by the German artist Albrecht Dürer (1471-1528), who revealed a method for drawing the inner-loop limaçon in his 1525 book *Underwysung der Messing*. A century later, the father of mathematician Blaise Pascal, Étienne Pascal (1588-1651), rediscovered it.

### Formulas for Inner-Loop Limaçons

The formulas that generate the **inner-loop limaçons** are given by $r = a \pm b \cos \theta$ and $r = a \pm b \sin \theta$ where $a > 0$, $b > 0$, and $a < b$. The graph of the inner-loop limaçon passes through the pole twice: once for the outer loop, and once for the inner loop. See **Figure 8.57** for the graphs.

<table>
<thead>
<tr>
<th>$r - a + b \cos \theta$, $a &lt; b$</th>
<th>$r - a - b \cos \theta$, $a &lt; b$</th>
<th>$r - a + b \sin \theta$, $a &lt; b$</th>
<th>$r - a - b \sin \theta$, $a &lt; b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>(b)</td>
<td>(c)</td>
<td>(d)</td>
</tr>
</tbody>
</table>

**Figure 8.57**

### Example 8.29

**Sketching the Graph of an Inner-Loop Limaçon**

Sketch the graph of $r = 2 + 5 \cos \theta$.

#### Solution

### Investigating Lemniscates

The lemniscate is a polar curve resembling the infinity symbol ∞ or a figure 8. Centered at the pole, a lemniscate is symmetrical by definition.

### Formulas for Lemniscates

The formulas that generate the graph of a **lemniscate** are given by $r^2 = a^2 \cos 2\theta$ and $r^2 = a^2 \sin 2\theta$ where $a \neq 0$. The formula $r^2 = a^2 \sin 2\theta$ is symmetric with respect to the pole. The formula $r^2 = a^2 \cos 2\theta$ is symmetric with respect to the pole, the line $\theta = \frac{\pi}{2}$, and the polar axis. See **Figure 8.58** for the graphs.
Example 8.30

Sketching the Graph of a Lemniscate

Sketch the graph of \( r^2 = 4 \cos 2\theta \).

Solution

Analysis

Making a substitution such as \( u = 2\theta \) is a common practice in mathematics because it can make calculations simpler. However, we must not forget to replace the substitution term with the original term at the end, and then solve for the unknown.

Some of the points on this graph may not show up using the Trace function on the TI-84 graphing calculator, and the calculator table may show an error for these same points of \( r \). This is because there are no real square roots for these values of \( \theta \). In other words, the corresponding \( r \)-values of \( \sqrt{4\cos(2\theta)} \) are complex numbers because there is a negative number under the radical.

Investigating Rose Curves

The next type of polar equation produces a petal-like shape called a rose curve. Although the graphs look complex, a simple polar equation generates the pattern.

Rose Curves

The formulas that generate the graph of a rose curve are given by \( r = a \cos n\theta \) and \( r = a \sin n\theta \) where \( a \neq 0 \). If \( n \) is even, the curve has \( 2n \) petals. If \( n \) is odd, the curve has \( n \) petals. See Figure 8.59.
Example 8.31

**Sketching the Graph of a Rose Curve \((n \text{ Even})\)**

Sketch the graph of \( r = 2\cos 4\theta \).

**Solution**

**Analysis**

When these curves are drawn, it is best to plot the points in order, as in the Table 0.0. This allows us to see how the graph hits a maximum (the tip of a petal), loops back crossing the pole, hits the opposite maximum, and loops back to the pole. The action is continuous until all the petals are drawn.

8.19 Sketch the graph of \( r = 4\sin(2\theta) \).

Example 8.32

**Sketching the Graph of a Rose Curve \((n \text{ Odd})\)**

Sketch the graph of \( r = 2\sin(5\theta) \).

**Solution**

8.20 Sketch the graph of \( r = 3\cos(3\theta) \).
Investigating the Archimedes' Spiral

The final polar equation we will discuss is the Archimedes’ spiral, named for its discoverer, the Greek mathematician Archimedes (c. 287 BCE - c. 212 BCE), who is credited with numerous discoveries in the fields of geometry and mechanics.

Archimedes’ Spiral

The formula that generates the graph of the Archimedes’ spiral is given by \( r = \theta \) for \( \theta \geq 0 \). As \( \theta \) increases, \( r \) increases at a constant rate in an ever-widening, never-ending, spiraling path. See Figure 8.60.

\[
\begin{align*}
\text{(a)} & \quad r = \theta, [0, 2\pi] \\
\text{(b)} & \quad r = \theta, [0, 4\pi]
\end{align*}
\]

Figure 8.60

**How To:** Given an Archimedes’ spiral over \([0, 2\pi]\), sketch the graph.

1. Make a table of values for \( r \) and \( \theta \) over the given domain.
2. Plot the points and sketch the graph.

**Example 8.33**

**Sketching the Graph of an Archimedes’ Spiral**

Sketch the graph of \( r = \theta \) over \([0, 2\pi]\).

**Solution**

**Analysis**

The domain of this polar curve is \([0, 2\pi]\). In general, however, the domain of this function is \((\infty, \infty)\). Graphing the equation of the Archimedes’ spiral is rather simple, although the image makes it seem like it would be complex.

**Try 8.21** Sketch the graph of \( r = -\theta \) over the interval \([0, 4\pi]\).

**Summary of Curves**

We have explored a number of seemingly complex polar curves in this section. Figure 8.61 and Figure 8.62 summarize the graphs and equations for each of these curves.
Access these online resources for additional instruction and practice with graphs of polar coordinates.

- Graphing Polar Equations Part 1 (http://openstaxcollege.org/l/polargraph1)
- Graphing Polar Equations Part 2 (http://openstaxcollege.org/l/polargraph2)
- Animation: The Graphs of Polar Equations (http://openstaxcollege.org/l/polaranim)
- Graphing Polar Equations on the TI-84 (http://openstaxcollege.org/l/polarTI84)
8.4 EXERCISES

Verbal

240. Describe the three types of symmetry in polar graphs, and compare them to the symmetry of the Cartesian plane.

241. Which of the three types of symmetries for polar graphs correspond to the symmetries with respect to the x-axis, y-axis, and origin?

242. What are the steps to follow when graphing polar equations?

243. Describe the shapes of the graphs of cardioids, limaçons, and lemniscates.

244. What part of the equation determines the shape of the graph of a polar equation?

Graphical

For the following exercises, test the equation for symmetry.

245. \( r = 5\cos 3\theta \)

246. \( r = 3 - 3\cos \theta \)

247. \( r = 3 + 2\sin \theta \)

248. \( r = 3\sin 2\theta \)

249. \( r = 4 \)

250. \( r = 2\theta \)

251. \( r = 4\cos \frac{\theta}{2} \)

252. \( r = \frac{2}{\theta} \)

253. \( r = 3\sqrt{1 - \cos^2 \theta} \)

254. \( r = \sqrt{3\sin 2\theta} \)

For the following exercises, graph the polar equation. Identify the name of the shape.

255. \( r = 3\cos \theta \)

256. \( r = 4\sin \theta \)

257. \( r = 2 + 2\cos \theta \)

258. \( r = 2 - 2\cos \theta \)

259. \( r = 5 - 5\sin \theta \)

260. \( r = 3 + 3\sin \theta \)

261. \( r = 3 + 2\sin \theta \)

262. \( r = 7 + 4\sin \theta \)

263. \( r = 4 + 3\cos \theta \)

264. \( r = 5 + 4\cos \theta \)

265.
266. \( r = 10 + 9\cos \theta \)
267. \( r = 1 + 3\sin \theta \)
268. \( r = 2 + 5\sin \theta \)
269. \( r = 5 + 7\sin \theta \)
270. \( r = 2 + 4\cos \theta \)
271. \( r^2 = 36\cos(2\theta) \)
272. \( r^2 = 10\cos(2\theta) \)
273. \( r^2 = 4\sin(2\theta) \)
274. \( r^2 = 10\sin(2\theta) \)
275. \( r = 3\sin(2\theta) \)
276. \( r = 3\cos(2\theta) \)
277. \( r = 5\sin(3\theta) \)
278. \( r = 4\sin(4\theta) \)
279. \( r = 4\sin(5\theta) \)
280. \( r = -\theta \)
281. \( r = 2\theta \)
282. \( r = -3\theta \)

**Technology**

For the following exercises, use a graphing calculator to sketch the graph of the polar equation.

283. \( r = \frac{1}{\theta} \)
284. \( r = \frac{1}{\sqrt{\theta}} \)
285. \( r = 2\sin \theta \tan \theta \), a cissoid
286. \( r = 2\sqrt{1 - \sin^2 \theta} \), a hippopede
287. \( r = 5 + \cos(4\theta) \)
288. \( r = 2 - \sin(2\theta) \)
289. \( r = \theta^2 \)
290. \( r = \theta + 1 \)
291. \( r = \theta \sin \theta \)
292. \( r = \theta \cos \theta \)

For the following exercises, use a graphing utility to graph each pair of polar equations on a domain of \([0, 4\pi]\) and then explain the differences shown in the graphs.

293. \( r = \theta, \quad r = -\theta \)

294. \( r = \theta, \quad r = \theta + \sin \theta \)

295. \( r = \sin \theta + \theta, \quad r = \sin \theta - \theta \)

296. \( r = 2\sin\left(\frac{\theta}{2}\right), \quad r = \theta \sin\left(\frac{\theta}{2}\right) \)

297. \( r = \sin(\cos(3\theta)), \quad r = \sin(3\theta) \)

298. On a graphing utility, graph \( r = \sin\left(\frac{16\theta}{3}\right) \) on \([0, 4\pi], [0, 8\pi], [0, 12\pi], \) and \([0, 16\pi]\). Describe the effect of increasing the width of the domain.

299. On a graphing utility, graph and sketch \( r = \sin \theta + \left(\sin\left(\frac{5\theta}{2}\right)\right)^3 \) on \([0, 4\pi]\).

300. On a graphing utility, graph each polar equation. Explain the similarities and differences you observe in the graphs.

\[
\begin{align*}
  r_1 &= 3\sin(3\theta) \\
  r_2 &= 2\sin(3\theta) \\
  r_3 &= \sin(3\theta)
\end{align*}
\]

(8.22)

301. On a graphing utility, graph each polar equation. Explain the similarities and differences you observe in the graphs.

\[
\begin{align*}
  r_1 &= 3 + 3\cos \theta \\
  r_2 &= 2 + 2\cos \theta \\
  r_3 &= 1 + \cos \theta
\end{align*}
\]

(8.23)

302. On a graphing utility, graph each polar equation. Explain the similarities and differences you observe in the graphs.

\[
\begin{align*}
  r_1 &= 3\theta \\
  r_2 &= 2\theta \\
  r_3 &= \theta
\end{align*}
\]

(8.24)

### Extensions

For the following exercises, draw each polar equation on the same set of polar axes, and find the points of intersection.

303. \( r_1 = 3 + 2\sin \theta, \quad r_2 = 2 \)

304. \( r_1 = 6 - 4\cos \theta, \quad r_2 = 4 \)

305. \( r_1 = 1 + \sin \theta, \quad r_2 = 3\sin \theta \)

306. \( r_1 = 1 + \cos \theta, \quad r_2 = 3\cos \theta \)

307. \( r_1 = \cos(2\theta), \quad r_2 = \sin(2\theta) \)

308. \( r_1 = \sin^2(2\theta), \quad r_2 = 1 - \cos(4\theta) \)

309. \( r_1 = \sqrt{3}, \quad r_2 = 2\sin(\theta) \)
310. \( r_1^2 = \sin \theta, \ r_2^2 = \cos \theta \)

311. \( r_1 = 1 + \cos \theta, \ r_2 = 1 - \sin \theta \)
8.5 | Polar Form of Complex Numbers

### Learning Objectives

In this section, you will:

- **8.5.1** Plot complex numbers in the complex plane.
- **8.5.2** Find the absolute value of a complex number.
- **8.5.3** Write complex numbers in polar form.
- **8.5.4** Convert a complex number from polar to rectangular form.
- **8.5.5** Find products of complex numbers in polar form.
- **8.5.6** Find quotients of complex numbers in polar form.
- **8.5.7** Find powers of complex numbers in polar form.
- **8.5.8** Find roots of complex numbers in polar form.

“God made the integers; all else is the work of man.” This rather famous quote by nineteenth-century German mathematician Leopold Kronecker sets the stage for this section on the polar form of a complex number. Complex numbers were invented by people and represent over a thousand years of continuous investigation and struggle by mathematicians such as Pythagoras, Descartes, De Moivre, Euler, Gauss, and others. Complex numbers answered questions that for centuries had puzzled the greatest minds in science.

We first encountered complex numbers in **Complex Numbers**. In this section, we will focus on the mechanics of working with complex numbers: translation of complex numbers from polar form to rectangular form and vice versa, interpretation of complex numbers in the scheme of applications, and application of De Moivre’s Theorem.

### Plotting Complex Numbers in the Complex Plane

Plotting a complex number $a + bi$ is similar to plotting a real number, except that the horizontal axis represents the real part of the number, $a$, and the vertical axis represents the imaginary part of the number, $bi$.

**How To:**

Given a complex number $a + bi$, plot it in the complex plane.

1. Label the horizontal axis as the *real* axis and the vertical axis as the *imaginary* axis.
2. Plot the point in the complex plane by moving $a$ units in the horizontal direction and $b$ units in the vertical direction.

### Example 8.34

**Plotting a Complex Number in the Complex Plane**

Plot the complex number $2 – 3i$ in the complex plane.

**Solution**

---

**8.22** Plot the point $1 + 5i$ in the complex plane.

### Finding the Absolute Value of a Complex Number

The first step toward working with a complex number in polar form is to find the absolute value. The absolute value of a complex number is the same as its magnitude, or $|z|$. It measures the distance from the origin to a point in the plane. For example, the graph of $z = 2 + 4i$, in Figure 8.63, shows $|z|$.
Absolute Value of a Complex Number

Given \( z = x + yi \), a complex number, the absolute value of \( z \) is defined as

\[ |z| = \sqrt{x^2 + y^2} \]  \hspace{1cm} (8.25)

It is the distance from the origin to the point \((x, y)\).

Notice that the absolute value of a real number gives the distance of the number from 0, while the absolute value of a complex number gives the distance of the number from the origin, \((0, 0)\).

Example 8.35

Finding the Absolute Value of a Complex Number with a Radical

Find the absolute value of \( z = \sqrt{5} - i \).

Solution

\( z = \sqrt{5} - i \)

Example 8.36

Finding the Absolute Value of a Complex Number

Given \( z = 3 - 4i \), find \( |z| \).

Solution

\( z = 3 - 4i \)
8.24 Given $z = 1 - 7i$, find $|z|$.

**Writing Complex Numbers in Polar Form**

The **polar form of a complex number** expresses a number in terms of an angle $\theta$ and its distance from the origin $r$. Given a complex number in rectangular form expressed as $z = x + yi$, we use the same conversion formulas as we do to write the number in trigonometric form:

$$
\begin{align*}
x &= r\cos \theta \\
y &= r\sin \theta \\
r &= \sqrt{x^2 + y^2}
\end{align*}
$$

We review these relationships in **Figure 8.64**.

![Figure 8.64](image)

We use the term **modulus** to represent the absolute value of a complex number, or the distance from the origin to the point $(x, y)$. The modulus, then, is the same as $r$, the radius in polar form. We use $\theta$ to indicate the angle of direction (just as with polar coordinates). Substituting, we have

$$
\begin{align*}
z &= x + yi \\
z &= r\cos \theta + (r\sin \theta)i \\
z &= r(\cos \theta + i\sin \theta)
\end{align*}
$$

**Polar Form of a Complex Number**

Writing a complex number in polar form involves the following conversion formulas:

$$
\begin{align*}
x &= r\cos \theta \\
y &= r\sin \theta \\
r &= \sqrt{x^2 + y^2}
\end{align*}
$$

Making a direct substitution, we have

$$
\begin{align*}
z &= x + yi \\
z &= (r\cos \theta) + i(r\sin \theta) \\
z &= r(\cos \theta + i\sin \theta)
\end{align*}
$$

where $r$ is the **modulus** and $\theta$ is the **argument**. We often use the abbreviation $r\text{cis} \theta$ to represent $r(\cos \theta + i\sin \theta)$.
Expressing a Complex Number Using Polar Coordinates

Express the complex number $4i$ using polar coordinates.

Solution

Example 8.38

Finding the Polar Form of a Complex Number

Find the polar form of $-4 + 4i$.

Solution

Example 8.39

Converting from Polar to Rectangular Form

Convert the polar form of the given complex number to rectangular form:

$$z = 12\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$$

Solution

Example 8.40

Finding the Rectangular Form of a Complex Number
Find the rectangular form of the complex number given \( r = 13 \) and \( \tan \theta = \frac{5}{12} \).

**Solution**

Convert the complex number to rectangular form:

\[
z = 4 \left( \cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6} \right)
\]  

(8.30)

**Finding Products of Complex Numbers in Polar Form**

Now that we can convert complex numbers to polar form we will learn how to perform operations on complex numbers in polar form. For the rest of this section, we will work with formulas developed by French mathematician Abraham de Moivre (1667-1754). These formulas have made working with products, quotients, powers, and roots of complex numbers much simpler than they appear. The rules are based on multiplying the moduli and adding the arguments.

**Products of Complex Numbers in Polar Form**

If \( z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \) and \( z_2 = r_2(\cos \theta_2 + i \sin \theta_2) \), then the product of these numbers is given as:

\[
z_1z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]
\]

(8.31)

\[
z_1z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)
\]

Notice that the product calls for multiplying the moduli and adding the angles.

**Example 8.41**

**Finding the Product of Two Complex Numbers in Polar Form**

Find the product of \( z_1z_2 \), given \( z_1 = 4(\cos(80^\circ) + i \sin(80^\circ)) \) and \( z_2 = 2(\cos(145^\circ) + i \sin(145^\circ)) \).

**Solution**

**Finding Quotients of Complex Numbers in Polar Form**

The quotient of two complex numbers in polar form is the quotient of the two moduli and the difference of the two arguments.

**Quotients of Complex Numbers in Polar Form**

If \( z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \) and \( z_2 = r_2(\cos \theta_2 + i \sin \theta_2) \), then the quotient of these numbers is

\[
\frac{z_1}{z_2} = \frac{r_1}{r_2} \left[ \cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2) \right], \quad z_2 \neq 0
\]

(8.32)

\[
\frac{z_1}{z_2} = \frac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2), \quad z_2 \neq 0
\]

Notice that the moduli are divided, and the angles are subtracted.
Given two complex numbers in polar form, find the quotient.

1. Divide $\frac{r_1}{r_2}$.
2. Find $\theta_1 - \theta_2$.
3. Substitute the results into the formula: $z = r(\cos \theta + i\sin \theta)$. Replace $r$ with $\frac{r_1}{r_2}$, and replace $\theta$ with $\theta_1 - \theta_2$.
4. Calculate the new trigonometric expressions and multiply through by $r$.

Example 8.42

Finding the Quotient of Two Complex Numbers

Find the quotient of $z_1 = 2(\cos(213^\circ) + i\sin(213^\circ))$ and $z_2 = 4(\cos(33^\circ) + i\sin(33^\circ))$.

Solution

Find the product and the quotient of $z_1 = 2\sqrt{3}(\cos(150^\circ) + i\sin(150^\circ))$ and $z_2 = 2(\cos(30^\circ) + i\sin(30^\circ))$.

Finding Powers of Complex Numbers in Polar Form

Finding powers of complex numbers is greatly simplified using De Moivre’s Theorem. It states that, for a positive integer $n$, $z^n$ is found by raising the modulus to the $n$th power and multiplying the argument by $n$. It is the standard method used in modern mathematics.

De Moivre’s Theorem

If $z = r(\cos \theta + i\sin \theta)$ is a complex number, then

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)] \quad (8.33)$$

$$z^n = r^n \text{cis}(n\theta)$$

where $n$ is a positive integer.

Example 8.43

Evaluating an Expression Using De Moivre’s Theorem

Evaluate the expression $(1 + i)^5$ using De Moivre’s Theorem.

Solution
Finding Roots of Complex Numbers in Polar Form

To find the $n$th root of a complex number in polar form, we use the $n$th Root Theorem or De Moivre’s Theorem and raise the complex number to a power with a rational exponent. There are several ways to represent a formula for finding $n$th roots of complex numbers in polar form.

**The $n$th Root Theorem**

To find the $n$th root of a complex number in polar form, use the formula given as

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \left[ \cos \left( \frac{\theta}{n} + \frac{2k\pi}{n} \right) + i \sin \left( \frac{\theta}{n} + \frac{2k\pi}{n} \right) \right]$$  \hspace{1cm} (8.34)

where $k = 0, 1, 2, 3, \ldots, n - 1$. We add $\frac{2k\pi}{n}$ to $\frac{\theta}{n}$ in order to obtain the periodic roots.

**Example 8.44**

**Finding the $n$th Root of a Complex Number**

Evaluate the cube roots of $z = 8 \left( \cos \left( \frac{2\pi}{3} \right) + i \sin \left( \frac{2\pi}{3} \right) \right)$.

**Solution**

Find the four fourth roots of $16(\cos(120^\circ) + i \sin(120^\circ))$.

Access these online resources for additional instruction and practice with polar forms of complex numbers.

- The Product and Quotient of Complex Numbers in Trigonometric Form (http://openstaxcollege.org/l/prodquocomplex)
- De Moivre’s Theorem (http://openstaxcollege.org/l/demoivre)
8.5 EXERCISES

Verbal

312. A complex number is \( a + bi \). Explain each part.

313. What does the absolute value of a complex number represent?

314. How is a complex number converted to polar form?

315. How do we find the product of two complex numbers?

316. What is De Moivre’s Theorem and what is it used for?

Algebraic

For the following exercises, find the absolute value of the given complex number.

317. \( 5 + 3i \)

318. \( -7 + i \)

319. \( -3 - 3i \)

320. \( \sqrt{2} - 6i \)

321. \( 2i \)

322. \( 2.2 - 3.1i \)

For the following exercises, write the complex number in polar form.

323. \( 2 + 2i \)

324. \( 8 - 4i \)

325. \( -\frac{1}{2} - \frac{1}{2}i \)

326. \( \sqrt{3} + i \)

327. \( 3i \)

For the following exercises, convert the complex number from polar to rectangular form.

328. \( z = 7\text{cis}\left(\frac{\pi}{6}\right) \)

329. \( z = 2\text{cis}\left(\frac{\pi}{3}\right) \)

330. \( z = 4\text{cis}\left(\frac{7\pi}{6}\right) \)

331. \( z = 7\text{cis}(25^\circ) \)

332. \( z = 3\text{cis}(240^\circ) \)

333. \( z = \sqrt{2}\text{cis}(100^\circ) \)

For the following exercises, find \( z_1z_2 \) in polar form.

334. \( z_1 = 2\sqrt{3}\text{cis}(116^\circ); \quad z_2 = 2\text{cis}(82^\circ) \)
335. $z_1 = \sqrt[3]{2} \text{cis}(205^\circ); \ z_2 = 2\sqrt[3]{2} \text{cis}(118^\circ)$

336. $z_1 = 3 \text{cis}(120^\circ); \ z_2 = \frac{1}{4} \text{cis}(60^\circ)$

337. $z_1 = 3 \text{cis}\left(\frac{\pi}{4}\right); \ z_2 = 5 \text{cis}\left(\frac{\pi}{6}\right)$

338. $z_1 = \sqrt[3]{2} \text{cis}\left(\frac{5\pi}{8}\right); \ z_2 = \sqrt[3]{5} \text{cis}\left(\frac{\pi}{12}\right)$

339. $z_1 = 4 \text{cis}\left(\frac{\pi}{2}\right); \ z_2 = 2 \text{cis}\left(\frac{\pi}{4}\right)$

For the following exercises, find $\frac{z_1}{z_2}$ in polar form.

340. $z_1 = 21 \text{cis}(135^\circ); \ z_2 = 3 \text{cis}(65^\circ)$

341. $z_1 = \sqrt[3]{2} \text{cis}(90^\circ); \ z_2 = 2 \text{cis}(60^\circ)$

342. $z_1 = 15 \text{cis}(120^\circ); \ z_2 = 3 \text{cis}(40^\circ)$

343. $z_1 = 6 \text{cis}\left(\frac{\pi}{3}\right); \ z_2 = 2 \text{cis}\left(\frac{\pi}{4}\right)$

344. $z_1 = 5 \sqrt[3]{2} \text{cis}(\pi); \ z_2 = \sqrt[3]{2} \text{cis}\left(\frac{2\pi}{3}\right)$

345. $z_1 = 2 \text{cis}\left(\frac{3\pi}{4}\right); \ z_2 = 3 \text{cis}\left(\frac{\pi}{4}\right)$

For the following exercises, find the powers of each complex number in polar form.

346. Find $z^3$ when $z = 5 \text{cis}(45^\circ)$.

347. Find $z^4$ when $z = 2 \text{cis}(70^\circ)$.

348. Find $z^2$ when $z = 3 \text{cis}(120^\circ)$.

349. Find $z^2$ when $z = 4 \text{cis}\left(\frac{\pi}{4}\right)$.

350. Find $z^4$ when $z = \text{cis}\left(\frac{3\pi}{16}\right)$.

351. Find $z^3$ when $z = 3 \text{cis}\left(\frac{5\pi}{3}\right)$.

For the following exercises, evaluate each root.

352. Evaluate the cube root of $z$ when $z = 27 \text{cis}(240^\circ)$.

353. Evaluate the square root of $z$ when $z = 16 \text{cis}(100^\circ)$.

354. Evaluate the cube root of $z$ when $z = 32 \text{cis}\left(\frac{2\pi}{3}\right)$.

355. Evaluate the square root of $z$ when $z = 32 \text{cis}(\pi)$.

356.
Evaluate the cube root of $z$ when $z = 8 \text{cis} \left( \frac{7\pi}{4} \right)$.

**Graphical**

For the following exercises, plot the complex number in the complex plane.

357. $2 + 4i$
358. $-3 - 3i$
359. $5 - 4i$
360. $-1 - 5i$
361. $3 + 2i$
362. $2i$
363. $-4$
364. $6 - 2i$
365. $-2 + i$
366. $1 - 4i$

**Technology**

For the following exercises, find all answers rounded to the nearest hundredth.

367. Use the rectangular to polar feature on the graphing calculator to change $5 + 5i$ to polar form.
368. Use the rectangular to polar feature on the graphing calculator to change $3 - 2i$ to polar form.
369. Use the rectangular to polar feature on the graphing calculator to change $-3 - 8i$ to polar form.
370. Use the polar to rectangular feature on the graphing calculator to change $4 \text{cis}(120^\circ)$ to rectangular form.
371. Use the polar to rectangular feature on the graphing calculator to change $2 \text{cis}(45^\circ)$ to rectangular form.
372. Use the polar to rectangular feature on the graphing calculator to change $5 \text{cis}(210^\circ)$ to rectangular form.
Consider the path a moon follows as it orbits a planet, which simultaneously rotates around the sun, as seen in Figure 8.65. At any moment, the moon is located at a particular spot relative to the planet. But how do we write and solve the equation for the position of the moon when the distance from the planet, the speed of the moon’s orbit around the planet, and the speed of rotation around the sun are all unknowns? We can solve only for one variable at a time.

In this section, we will consider sets of equations given by \( x(t) \) and \( y(t) \) where \( t \) is the independent variable of time. We can use these parametric equations in a number of applications when we are looking for not only a particular position but also the direction of the movement. As we trace out successive values of \( t \), the orientation of the curve becomes clear. This is one of the primary advantages of using parametric equations: we are able to trace the movement of an object along a path according to time. We begin this section with a look at the basic components of parametric equations and what it means to parameterize a curve. Then we will learn how to eliminate the parameter, translate the equations of a curve defined parametrically into rectangular equations, and find the parametric equations for curves defined by rectangular equations.

### Parameterizing a Curve

When an object moves along a curve—or curvilinear path—in a given direction and in a given amount of time, the position of the object in the plane is given by the \( x \)-coordinate and the \( y \)-coordinate. However, both \( x \) and \( y \) vary over time and so are functions of time. For this reason, we add another variable, the parameter, upon which both \( x \) and \( y \) are dependent functions. In the example in the section opener, the parameter is time, \( t \). The \( x \) position of the moon at time, \( t \), is represented as the function \( x(t) \), and the \( y \) position of the moon at time, \( t \), is represented as the function \( y(t) \). Together, \( x(t) \) and \( y(t) \) are called parametric equations, and generate an ordered pair \( (x(t), y(t)) \). Parametric equations primarily describe motion and direction.

When we parameterize a curve, we are translating a single equation in two variables, such as \( x \) and \( y \), into an equivalent pair of equations in three variables, \( x \), \( y \), and \( t \). One of the reasons we parameterize a curve is because the parametric equations yield more information: specifically, the direction of the object’s motion over time.

When we graph parametric equations, we can observe the individual behaviors of \( x \) and \( y \). There are a number of shapes that cannot be represented in the form \( y = f(x) \), meaning that they are not functions. For example, consider
the graph of a circle, given as \( r^2 = x^2 + y^2 \). Solving for \( y \) gives \( y = \pm \sqrt{r^2 - x^2} \), or two equations: \( y_1 = \sqrt{r^2 - x^2} \) and \( y_2 = -\sqrt{r^2 - x^2} \). If we graph \( y_1 \) and \( y_2 \) together, the graph will not pass the vertical line test, as shown in Figure 8.66. Thus, the equation for the graph of a circle is not a function.

![Figure 8.66](image)

However, if we were to graph each equation on its own, each one would pass the vertical line test and therefore would represent a function. In some instances, the concept of breaking up the equation for a circle into two functions is similar to the concept of creating parametric equations, as we use two functions to produce a non-function. This will become clearer as we move forward.

**Parametric Equations**

Suppose \( t \) is a number on an interval, \( I \). The set of ordered pairs, \((x(t), y(t))\), where \( x = f(t) \) and \( y = g(t) \), forms a plane curve based on the parameter \( t \). The equations \( x = f(t) \) and \( y = g(t) \) are the parametric equations.

**Example 8.45**

**Parameterizing a Curve**

Parameterize the curve \( y = x^2 - 1 \) letting \( x(t) = t \). Graph both equations.

**Solution**

**Analysis**

The arrows indicate the direction in which the curve is generated. Notice the curve is identical to the curve of \( y = x^2 - 1 \).

**Try It**

Construct a table of values and plot the parametric equations: \( x(t) = t - 3, \quad y(t) = 2t + 4; \quad -1 \leq t \leq 2. \)

**Example 8.46**

**Finding a Pair of Parametric Equations**

Find a pair of parametric equations that models the graph of \( y = 1 - x^2 \), using the parameter \( x(t) = t \). Plot some points and sketch the graph.
8.31 Parameterize the curve given by \( x = y^3 - 2y \).

Example 8.47

**Finding Parametric Equations That Model Given Criteria**

An object travels at a steady rate along a straight path \((-5, 3)\) to \((3, -1)\) in the same plane in four seconds. The coordinates are measured in meters. Find parametric equations for the position of the object.

**Solution**

**Analysis**

Again, we see that, in Figure \(0.0(c)\), when the parameter represents time, we can indicate the movement of the object along the path with arrows.

Eliminating the Parameter

In many cases, we may have a pair of parametric equations but find that it is simpler to draw a curve if the equation involves only two variables, such as \( x \) and \( y \). Eliminating the parameter is a method that may make graphing some curves easier. However, if we are concerned with the mapping of the equation according to time, then it will be necessary to indicate the orientation of the curve as well. There are various methods for eliminating the parameter \( t \) from a set of parametric equations; not every method works for every type of equation. Here we will review the methods for the most common types of equations.

**Eliminating the Parameter from Polynomial, Exponential, and Logarithmic Equations**

For polynomial, exponential, or logarithmic equations expressed as two parametric equations, we choose the equation that is most easily manipulated and solve for \( t \). We substitute the resulting expression for \( t \) into the second equation. This gives one equation in \( x \) and \( y \).

Example 8.48

**Eliminating the Parameter in Polynomials**

Given \( x(t) = t^2 + 1 \) and \( y(t) = 2 + t \), eliminate the parameter, and write the parametric equations as a Cartesian equation.

**Solution**

**Analysis**

This is an equation for a parabola in which, in rectangular terms, \( x \) is dependent on \( y \). From the curve’s vertex at \((1, 2)\), the graph sweeps out to the right. See Figure \(8.66\). In this section, we consider sets of equations given
by the functions $x(t)$ and $y(t)$, where $t$ is the independent variable of time. Notice, both $x$ and $y$ are functions of time; so in general $y$ is not a function of $x$.

Given the equations below, eliminate the parameter and write as a rectangular equation for $y$ as a function of $x$.

$$x(t) = 2t^2 + 6$$
$$y(t) = 5 - t$$

Example 8.49

**Eliminating the Parameter in Exponential Equations**

Eliminate the parameter and write as a Cartesian equation: $x(t) = e^{-t}$ and $y(t) = 3e^t$, $t > 0$.

**Solution**

**Analysis**

The graph of the parametric equation is shown in Figure 8.66(a). The domain is restricted to $t > 0$. The Cartesian equation, $y = \frac{3}{x}$ is shown in Figure 8.66(b) and has only one restriction on the domain, $x \neq 0$. 
Example 8.50

Eliminating the Parameter in Logarithmic Equations

Eliminate the parameter and write as a Cartesian equation: \( x(t) = \sqrt{t} + 2 \) and \( y(t) = \log(t) \).

Solution

Analysis

To be sure that the parametric equations are equivalent to the Cartesian equation, check the domains. The parametric equations restrict the domain on \( x = \sqrt{t} + 2 \) to \( t > 0 \); we restrict the domain on \( x \) to \( x > 2 \). The domain for the parametric equation \( y = \log(t) \) is restricted to \( t > 0 \); we limit the domain on \( y = \log(x - 2)^2 \) to \( x > 2 \).

Try It 8.33

Eliminate the parameter and write as a rectangular equation.

\[
\begin{align*}
x(t) &= t^2 \\
y(t) &= \ln t \quad t > 0
\end{align*}
\]  

Eliminating the Parameter from Trigonometric Equations

Eliminating the parameter from trigonometric equations is a straightforward substitution. We can use a few of the familiar trigonometric identities and the Pythagorean Theorem.

First, we use the identities:
\[ x(t) = a \cos t \]
\[ y(t) = b \sin t \]  \hfill (8.37)

Solving for \( \cos t \) and \( \sin t \), we have
\[ \frac{x}{a} = \cos t \]  \hfill (8.38)
\[ \frac{y}{b} = \sin t \]

Then, use the Pythagorean Theorem:
\[ \cos^2 t + \sin^2 t = 1 \]  \hfill (8.39)

Substituting gives
\[ \cos^2 t + \sin^2 t = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \]  \hfill (8.40)

---

**Example 8.51**

**Eliminating the Parameter from a Pair of Trigonometric Parametric Equations**

Eliminate the parameter from the given pair of trigonometric equations where \( 0 \leq t \leq 2\pi \) and sketch the graph.
\[ x(t) = 4 \cos t \]
\[ y(t) = 3 \sin t \]

**Solution**

**Analysis**
Applying the general equations for conic sections (introduced in Analytic Geometry), we can identify \( \frac{x^2}{16} + \frac{y^2}{9} = 1 \) as an ellipse centered at \((0, 0)\). Notice that when \( t = 0 \) the coordinates are \((4, 0)\), and when \( t = \frac{\pi}{2} \) the coordinates are \((0, 3)\). This shows the orientation of the curve with increasing values of \( t \).

---

**Finding Cartesian Equations from Curves Defined Parametrically**

When we are given a set of parametric equations and need to find an equivalent Cartesian equation, we are essentially “eliminating the parameter.” However, there are various methods we can use to rewrite a set of parametric equations as a Cartesian equation. The simplest method is to set one equation equal to the parameter, such as \( x(t) = t \). In this case, \( y(t) \) can be any expression. For example, consider the following pair of equations.
\[ x(t) = t \]
\[ y(t) = t^2 - 3 \]  \hfill (8.41)

Rewriting this set of parametric equations is a matter of substituting \( x \) for \( t \). Thus, the Cartesian equation is \( y = x^2 - 3 \).

---

**Example 8.52**

**Finding a Cartesian Equation Using Alternate Methods**
Use two different methods to find the Cartesian equation equivalent to the given set of parametric equations.

\[ x(t) = 3t - 2 \]
\[ y(t) = t + 1 \]

**Solution**

**Example 8.53**

**Finding a Set of Parametric Equations for Curves Defined by Rectangular Equations**

Find a set of equivalent parametric equations for \( y = (x + 3)^2 + 1 \).

**Solution**

**Try It**

Write the given parametric equations as a Cartesian equation: \( x(t) = t^3 \) and \( y(t) = t^6 \).

**Finding Parametric Equations for Curves Defined by Rectangular Equations**

Although we have just shown that there is only one way to interpret a set of parametric equations as a rectangular equation, there are multiple ways to interpret a rectangular equation as a set of parametric equations. Any strategy we may use to find the parametric equations is valid if it produces equivalency. In other words, if we choose an expression to represent \( x \), and then substitute it into the \( y \) equation, and it produces the same graph over the same domain as the rectangular equation, then the set of parametric equations is valid. If the domain becomes restricted in the set of parametric equations, and the function does not allow the same values for \( x \) as the domain of the rectangular equation, then the graphs will be different.

Access these online resources for additional instruction and practice with parametric equations.

- Introduction to Parametric Equations (http://openstaxcollege.org/l/introparametric)
- Converting Parametric Equations to Rectangular Form (http://openstaxcollege.org/l/convertpara)
8.6 EXERCISES

Verbal

373. What is a system of parametric equations?

374. Some examples of a third parameter are time, length, speed, and scale. Explain when time is used as a parameter.

375. Explain how to eliminate a parameter given a set of parametric equations.

376. What is a benefit of writing a system of parametric equations as a Cartesian equation?

377. What is a benefit of using parametric equations?

378. Why are there many sets of parametric equations to represent on Cartesian function?

Algebraic

For the following exercises, eliminate the parameter \( t \) to rewrite the parametric equation as a Cartesian equation.

379. \[
\begin{align*}
x(t) &= 5 - t \\
y(t) &= 8 - 2t
\end{align*}
\]

380. \[
\begin{align*}
x(t) &= 6 - 3t \\
y(t) &= 10 - t
\end{align*}
\]

381. \[
\begin{align*}
x(t) &= 2t + 1 \\
y(t) &= 3\sqrt{t}
\end{align*}
\]

382. \[
\begin{align*}
x(t) &= 3t - 1 \\
y(t) &= 2t^2
\end{align*}
\]

383. \[
\begin{align*}
x(t) &= 2e^t \\
y(t) &= 1 - 5t
\end{align*}
\]

384. \[
\begin{align*}
x(t) &= e^{-2t} \\
y(t) &= 2e^{-t}
\end{align*}
\]

385. \[
\begin{align*}
x(t) &= 4\log(t) \\
y(t) &= 3 + 2t
\end{align*}
\]

386. \[
\begin{align*}
x(t) &= \log(2t) \\
y(t) &= \sqrt{t - 1}
\end{align*}
\]

387. \[
\begin{align*}
x(t) &= t^3 - t \\
y(t) &= 2t
\end{align*}
\]

388. \[
\begin{align*}
x(t) &= t - t^4 \\
y(t) &= t + 2
\end{align*}
\]

389. \[
\begin{align*}
x(t) &= e^{2t} \\
y(t) &= e^{6t}
\end{align*}
\]

390. \[
\begin{align*}
x(t) &= t^5 \\
y(t) &= t^{10}
\end{align*}
\]

391.
\[\begin{align*}
\{x(t) &= 4\cos t \\
y(t) &= 5\sin t
\end{align*}\]

392. \[\begin{align*}
\{x(t) &= 3\sin t \\
y(t) &= 6\cos t
\end{align*}\]

393. \[\begin{align*}
\{x(t) &= 2\cos^2 t \\
y(t) &= -\sin t
\end{align*}\]

394. \[\begin{align*}
\{x(t) &= \cos t + 4 \\
y(t) &= 2\sin^2 t
\end{align*}\]

395. \[\begin{align*}
\{x(t) &= t - 1 \\
y(t) &= t^2
\end{align*}\]

396. \[\begin{align*}
\{x(t) &= -t \\
y(t) &= t^3 + 1
\end{align*}\]

397. \[\begin{align*}
\{x(t) &= 2t - 1 \\
y(t) &= t^3 - 2
\end{align*}\]

For the following exercises, rewrite the parametric equation as a Cartesian equation by building an \(x-y\) table.

398. \[\begin{align*}
\{x(t) &= 2t - 1 \\
y(t) &= t + 4
\end{align*}\]

399. \[\begin{align*}
\{x(t) &= 4 - t \\
y(t) &= 3t + 2
\end{align*}\]

400. \[\begin{align*}
\{x(t) &= 2t - 1 \\
y(t) &= 5t
\end{align*}\]

401. \[\begin{align*}
\{x(t) &= 4t - 1 \\
y(t) &= 4t + 2
\end{align*}\]

For the following exercises, parameterize (write parametric equations for) each Cartesian equation by setting \(x(t) = t\) or by setting \(y(t) = t\).

402. \(y(x) = 3x^2 + 3\)

403. \(y(x) = 2\sin x + 1\)

404. \(x(y) = 3\log(y) + y\)

405. \(x(y) = y^2 + 2y\)

For the following exercises, parameterize (write parametric equations for) each Cartesian equation by using \(x(t) = a\cos t\) and \(y(t) = b\sin t\). Identify the curve.

406. \[\frac{x^2}{4} + \frac{y^2}{9} = 1\]

407. \[\frac{x^2}{16} + \frac{y^2}{36} = 1\]
408. \( x^2 + y^2 = 16 \)

409. \( x^2 + y^2 = 10 \)

410. Parameterize the line from \((3, 0)\) to \((-2, -5)\) so that the line is at \((3, 0)\) at \(t = 0\), and at \((-2, -5)\) at \(t = 1\).

411. Parameterize the line from \((-1, 0)\) to \((3, -2)\) so that the line is at \((-1, 0)\) at \(t = 0\), and at \((3, -2)\) at \(t = 1\).

412. Parameterize the line from \((-1, 5)\) to \((2, 3)\) so that the line is at \((-1, 5)\) at \(t = 0\), and at \((2, 3)\) at \(t = 1\).

413. Parameterize the line from \((4, 1)\) to \((6, -2)\) so that the line is at \((4, 1)\) at \(t = 0\), and at \((6, -2)\) at \(t = 1\).

**Technology**

For the following exercises, use the table feature in the graphing calculator to determine whether the graphs intersect.

414. \[ \begin{align*} x_1(t) &= 3t \\ y_1(t) &= 2t - 1 \end{align*} \quad \text{and} \quad \begin{align*} x_2(t) &= t + 3 \\ y_2(t) &= 4t - 4 \end{align*} \]

415. \[ \begin{align*} x_1(t) &= t^2 \\ y_1(t) &= 2t - 1 \end{align*} \quad \text{and} \quad \begin{align*} x_2(t) &= -t + 6 \\ y_2(t) &= t + 1 \end{align*} \]

For the following exercises, use a graphing calculator to complete the table of values for each set of parametric equations.

416. \( \begin{align*} x_1(t) &= 3t^2 - 3t + 7 \\ y_1(t) &= 2t + 3 \end{align*} \)

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*Table 8.1*

417. \( \begin{align*} x_1(t) &= t^2 - 4 \\ y_1(t) &= 2t^2 - 1 \end{align*} \)

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</table>

*Table 8.2*
\[
\begin{align*}
    x_1(t) &= t^4 \\
    y_1(t) &= t^3 + 4
\end{align*}
\]

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Table 8.3

Extensions

419. Find two different sets of parametric equations for \( y = (x + 1)^2 \).

420. Find two different sets of parametric equations for \( y = 3x - 2 \).

421. Find two different sets of parametric equations for \( y = x^2 - 4x + 4 \).
8.7 | Parametric Equations: Graphs

Learning Objectives

In this section you will:

- 8.7.1 Graph plane curves described by parametric equations by plotting points.
- 8.7.2 Graph parametric equations.

It is the bottom of the ninth inning, with two outs and two men on base. The home team is losing by two runs. The batter swings and hits the baseball at 140 feet per second and at an angle of approximately 45° to the horizontal. How far will the ball travel? Will it clear the fence for a game-winning home run? The outcome may depend partly on other factors (for example, the wind), but mathematicians can model the path of a projectile and predict approximately how far it will travel using parametric equations. In this section, we’ll discuss parametric equations and some common applications, such as projectile motion problems.

![Figure 8.67](credit: Paul Kreher, Flickr)

**Figure 8.67** Parametric equations can model the path of a projectile.

### Graphing Parametric Equations by Plotting Points

In lieu of a graphing calculator or a computer graphing program, plotting points to represent the graph of an equation is the standard method. As long as we are careful in calculating the values, point-plotting is highly dependable.

**Given a pair of parametric equations, sketch a graph by plotting points.**

1. Construct a table with three columns: \( t \), \( x(t) \), and \( y(t) \).
2. Evaluate \( x \) and \( y \) for values of \( t \) over the interval for which the functions are defined.
3. Plot the resulting pairs \((x, y)\).

**Example 8.54**

**Sketching the Graph of a Pair of Parametric Equations by Plotting Points**

Sketch the graph of the parametric equations \( x(t) = t^2 + 1 \), \( y(t) = 2 + t \).
As values for $t$ progress in a positive direction from 0 to 5, the plotted points trace out the top half of the parabola. As values of $t$ become negative, they trace out the lower half of the parabola. There are no restrictions on the domain. The arrows indicate direction according to increasing values of $t$. The graph does not represent a function, as it will fail the vertical line test. The graph is drawn in two parts: the positive values for $t$, and the negative values for $t$.

Example 8.55

Sketching the Graph of Trigonometric Parametric Equations

Construct a table of values for the given parametric equations and sketch the graph:

$$x = 2\cos t$$
$$y = 4\sin t$$

Solution

Analysis

We have seen that parametric equations can be graphed by plotting points. However, a graphing calculator will save some time and reveal nuances in a graph that may be too tedious to discover using only hand calculations.

Make sure to change the mode on the calculator to parametric (PAR). To confirm, the $Y =$ window should show $X_1T =$ $Y_1T =$ instead of $Y_1 =$ .

Example 8.56

Graphing Parametric Equations and Rectangular Form Together

Graph the parametric equations $x = 5\cos t$ and $y = 3\sin t$. First, construct the graph using data points generated from the parametric form. Then graph the rectangular form of the equation. Compare the two graphs.

Solution

Analysis

In Figure 8.67, the data from the parametric equations and the rectangular equation are plotted together. The parametric equations are plotted in blue; the graph for the rectangular equation is drawn on top of the parametric in a dashed style colored red. Clearly, both forms produce the same graph.
Graphing Parametric Equations and Rectangular Equations on the Coordinate System

Graph the parametric equations \( x = t + 1 \) and \( y = \sqrt{t} \), \( t \geq 0 \), and the rectangular equivalent \( y = \sqrt{x-1} \) on the same coordinate system.

Solution

Analysis

With the domain on \( t \) restricted, we only plot positive values of \( t \). The parametric data is graphed in blue and the graph of the rectangular equation is dashed in red. Once again, we see that the two forms overlap.

Sketch the graph of the parametric equations \( x = 2\cos \theta \) and \( y = 4\sin \theta \), along with the rectangular equation on the same grid.

Applications of Parametric Equations

Many of the advantages of parametric equations become obvious when applied to solving real-world problems. Although rectangular equations in \( x \) and \( y \) give an overall picture of an object’s path, they do not reveal the position of an object at a specific time. Parametric equations, however, illustrate how the values of \( x \) and \( y \) change depending on \( t \), as the location of a moving object at a particular time.

A common application of parametric equations is solving problems involving projectile motion. In this type of motion, an object is propelled forward in an upward direction forming an angle of \( \theta \) to the horizontal, with an initial speed of \( v_0 \), and at a height \( h \) above the horizontal.

The path of an object propelled at an inclination of \( \theta \) to the horizontal, with initial speed \( v_0 \), and at a height \( h \) above the horizontal, is given by

\[
\begin{align*}
  x &= (v_0 \cos \theta) t \\
  y &= -\frac{1}{2}gt^2 + (v_0 \sin \theta) t + h
\end{align*}
\]  

(8.42)

where \( g \) accounts for the effects of gravity and \( h \) is the initial height of the object. Depending on the units involved in the problem, use \( g = 32 \text{ ft/s}^2 \) or \( g = 9.8 \text{ m/s}^2 \). The equation for \( x \) gives horizontal distance, and the equation for \( y \) gives the vertical distance.
Given a projectile motion problem, use parametric equations to solve.

1. The horizontal distance is given by \( x = (v_0 \cos \theta)t \). Substitute the initial speed of the object for \( v_0 \).

2. The expression \( \cos \theta \) indicates the angle at which the object is propelled. Substitute that angle in degrees for \( \cos \theta \).

3. The vertical distance is given by the formula \( y = -\frac{1}{2}gt^2 + (v_0 \sin \theta)t + h \). The term \( -\frac{1}{2}gt^2 \) represents the effect of gravity. Depending on units involved, use \( g = 32 \text{ ft/s}^2 \) or \( g = 9.8 \text{ m/s}^2 \). Again, substitute the initial speed for \( v_0 \), and the height at which the object was propelled for \( h \).

4. Proceed by calculating each term to solve for \( t \).

Example 8.58

**Finding the Parametric Equations to Describe the Motion of a Baseball**

Solve the problem presented at the beginning of this section. Does the batter hit the game-winning home run? Assume that the ball is hit with an initial velocity of 140 feet per second at an angle of 45° to the horizontal, making contact 3 feet above the ground.

a. Find the parametric equations to model the path of the baseball.

b. Where is the ball after 2 seconds?

c. How long is the ball in the air?

d. Is it a home run?

**Solution**

Access the following online resource for additional instruction and practice with graphs of parametric equations.

• **Graphing Parametric Equations on the TI-84** (http://openstaxcollege.org/l/graphpara84)
8.7 EXERCISES

Verbal

422. What are two methods used to graph parametric equations?

423. What is one difference in point-plotting parametric equations compared to Cartesian equations?

424. Why are some graphs drawn with arrows?

425. Name a few common types of graphs of parametric equations.

426. Why are parametric graphs important in understanding projectile motion?

Graphical

For the following exercises, graph each set of parametric equations by making a table of values. Include the orientation on the graph.

427. \[ \begin{align*} x(t) &= t \\ y(t) &= t^2 - 1 \end{align*} \]

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Table 8.4

428. \[ \begin{align*} x(t) &= t - 1 \\ y(t) &= t^2 \end{align*} \]

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Table 8.5

429. \[ \begin{align*} x(t) &= 2 + t \\ y(t) &= 3 - 2t \end{align*} \]
Table 8.6

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430. \[
\begin{align*}
  x(t) &= -2 - 2t \\
  y(t) &= 3 + t 
\end{align*}
\]

431. \[
\begin{align*}
  x(t) &= t^3 \\
  y(t) &= t + 2 
\end{align*}
\]

432. \[
\begin{align*}
  x(t) &= t^2 \\
  y(t) &= t + 3 
\end{align*}
\]

For the following exercises, sketch the curve and include the orientation.
For the following exercises, graph the equation and include the orientation. Then, write the Cartesian equation.

433. \[ \begin{align*}
    x(t) &= t \\
    y(t) &= \sqrt{t}
\end{align*} \]

434. \[ \begin{align*}
    x(t) &= -\sqrt{t} \\
    y(t) &= t
\end{align*} \]

435. \[ \begin{align*}
    x(t) &= 5 - |t| \\
    y(t) &= t + 2
\end{align*} \]

436. \[ \begin{align*}
    x(t) &= -t + 2 \\
    y(t) &= 5 - |t|
\end{align*} \]

437. \[ \begin{align*}
    x(t) &= 4\sin t \\
    y(t) &= 2\cos t
\end{align*} \]

438. \[ \begin{align*}
    x(t) &= 2\sin t \\
    y(t) &= 4\cos t
\end{align*} \]

439. \[ \begin{align*}
    x(t) &= 3\cos^2 t \\
    y(t) &= -3\sin t
\end{align*} \]

440. \[ \begin{align*}
    x(t) &= 3\cos^2 t \\
    y(t) &= -3\sin^2 t
\end{align*} \]

441. \[ \begin{align*}
    x(t) &= \sec t \\
    y(t) &= \tan t
\end{align*} \]

442. \[ \begin{align*}
    x(t) &= \sec t \\
    y(t) &= \tan^2 t
\end{align*} \]

443. \[ \begin{align*}
    x(t) &= -\frac{1}{e^{2t}} \\
    y(t) &= e^{-t}
\end{align*} \]

For the following exercises, graph the equation and include the orientation. Then, write the Cartesian equation.

444. \[ \begin{align*}
    x(t) &= t - 1 \\
    y(t) &= -t^2
\end{align*} \]

445. \[ \begin{align*}
    x(t) &= t^3 \\
    y(t) &= t + 3
\end{align*} \]

446. \[ \begin{align*}
    x(t) &= 2\cos t \\
    y(t) &= -\sin t
\end{align*} \]

447. \[ \begin{align*}
    x(t) &= 7\cos t \\
    y(t) &= 7\sin t
\end{align*} \]

448. \[ \begin{align*}
    x(t) &= e^{2t} \\
    y(t) &= -e^t
\end{align*} \]

For the following exercises, graph the equation and include the orientation.

449. \[ x = t^2, \quad y = 3t, \quad 0 \leq t \leq 5 \]
450. \[ x = 2t, \quad y = t^2, \quad -5 \leq t \leq 5 \]

451. \[ x = t, \quad y = \sqrt{25 - t^2}, \quad 0 < t \leq 5 \]

452. \[ x(t) = -t, \quad y(t) = \sqrt{7}, \quad t \geq 0 \]

453. \[ x = -2\cos t, \quad y = 6\sin t, \quad 0 \leq t \leq \pi \]

454. \[ x = -\sec t, \quad y = \tan t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2} \]

For the following exercises, use the parametric equations for integers \( a \) and \( b \):

\[
\begin{align*}
  x(t) &= a\cos((a + b)t) \\
  y(t) &= a\cos((a - b)t)
\end{align*}
\]

(8.43)

455. Graph on the domain \([-\pi, 0]\), where \(a = 2\) and \(b = 1\), and include the orientation.

456. Graph on the domain \([-\pi, 0]\), where \(a = 3\) and \(b = 2\), and include the orientation.

457. Graph on the domain \([-\pi, 0]\), where \(a = 4\) and \(b = 3\), and include the orientation.

458. Graph on the domain \([-\pi, 0]\), where \(a = 5\) and \(b = 4\), and include the orientation.

459. If \(a\) is 1 more than \(b\), describe the effect the values of \(a\) and \(b\) have on the graph of the parametric equations.

460. Describe the graph if \(a = 100\) and \(b = 99\).

461. What happens if \(b\) is 1 more than \(a\)? Describe the graph.

462. If the parametric equations \(x(t) = t^2\) and \(y(t) = 6 - 3t\) have the graph of a horizontal parabola opening to the right, what would change the direction of the curve?

For the following exercises, describe the graph of the set of parametric equations.

463. \(x(t) = -t^2\) and \(y(t)\) is linear

464. \(y(t) = t^2\) and \(x(t)\) is linear

465. \(y(t) = -t^2\) and \(x(t)\) is linear

466. Write the parametric equations of a circle with center \((0, 0)\), radius 5, and a counterclockwise orientation.

467. Write the parametric equations of an ellipse with center \((0, 0)\), major axis of length 10, minor axis of length 6, and a counterclockwise orientation.

For the following exercises, use a graphing utility to graph on the window \([-3, 3]\) by \([-3, 3]\) on the domain \([0, 2\pi]\) for the following values of \(a\) and \(b\), and include the orientation.

\[
\begin{align*}
  \begin{cases}
    x(t) = \sin(at) \\
    y(t) = \sin(bt)
  \end{cases}
\end{align*}
\]

(8.44)

468. \(a = 1, \quad b = 2\)

469. \(a = 2, \quad b = 1\)

470. \(a = 3, \quad b = 3\)

471.
472. \( a = 2, \; b = 5 \)

473. \( a = 5, \; b = 2 \)

**Technology**

For the following exercises, look at the graphs that were created by parametric equations of the form \( \begin{cases} x(t) = a \cos(bt) \\ y(t) = c \sin(dt) \end{cases} \). Use the parametric mode on the graphing calculator to find the values of \( a, \; b, \; c, \; \text{ and } d \) to achieve each graph.

474.

475.

476.
For the following exercises, use a graphing utility to graph the given parametric equations.

a. \[
\begin{align*}
    x(t) &= \cos t - 1 \\
    y(t) &= \sin t + t
\end{align*}
\]

b. \[
\begin{align*}
    x(t) &= \cos t + t \\
    y(t) &= \sin t - 1
\end{align*}
\]

c. \[
\begin{align*}
    x(t) &= t - \sin t \\
    y(t) &= \cos t - 1
\end{align*}
\]

478. Graph all three sets of parametric equations on the domain \([0, 2\pi]\).

479. Graph all three sets of parametric equations on the domain \([0, 4\pi]\).

480. Graph all three sets of parametric equations on the domain \([-4\pi, 6\pi]\).

481. The graph of each set of parametric equations appears to “creep” along one of the axes. What controls which axis the graph creeps along?
482. Explain the effect on the graph of the parametric equation when we switched \( \sin t \) and \( \cos t \).

483. Explain the effect on the graph of the parametric equation when we changed the domain.

**Extensions**

484. An object is thrown in the air with vertical velocity of 20 ft/s and horizontal velocity of 15 ft/s. The object’s height can be described by the equation \( y(t) = -16t^2 + 20t \), while the object moves horizontally with constant velocity 15 ft/s. Write parametric equations for the object’s position, and then eliminate time to write height as a function of horizontal position.

485. A skateboarder riding on a level surface at a constant speed of 9 ft/s throws a ball in the air, the height of which can be described by the equation \( y(t) = -16t^2 + 10t + 5 \). Write parametric equations for the ball’s position, and then eliminate time to write height as a function of horizontal position.

For the following exercises, use this scenario: A dart is thrown upward with an initial velocity of 65 ft/s at an angle of elevation of 52°. Consider the position of the dart at any time \( t \). Neglect air resistance.

486. Find parametric equations that model the problem situation.

487. Find all possible values of \( x \) that represent the situation.

488. When will the dart hit the ground?

489. Find the maximum height of the dart.

490. At what time will the dart reach maximum height?

For the following exercises, look at the graphs of each of the four parametric equations. Although they look unusual and beautiful, they are so common that they have names, as indicated in each exercise. Use a graphing utility to graph each on the indicated domain.

491. An epicycloid: \[
\begin{align*}
x(t) &= 14\cos t - \cos(14t) \\
y(t) &= 14\sin t + \sin(14t)
\end{align*}
\] on the domain \([0, 2\pi]\).

492. A hypocycloid: \[
\begin{align*}
x(t) &= 6\sin t + 2\sin(6t) \\
y(t) &= 6\cos t - 2\cos(6t)
\end{align*}
\] on the domain \([0, 2\pi]\).

493. A hypotrochoid: \[
\begin{align*}
x(t) &= 2\sin t + 5\cos(6t) \\
y(t) &= 5\cos t - 2\sin(6t)
\end{align*}
\] on the domain \([0, 2\pi]\).

494. A rose: \[
\begin{align*}
x(t) &= 5\sin(2t)\sin t \\
y(t) &= 5\sin(2t)\cos t
\end{align*}
\] on the domain \([0, 2\pi]\).
8.8 | Vectors

Learning Objectives

In this section you will:

8.8.1 View vectors geometrically.
8.8.2 Find magnitude and direction.
8.8.3 Perform vector addition and scalar multiplication.
8.8.4 Find the component form of a vector.
8.8.5 Find the unit vector in the direction of \( \mathbf{v} \).
8.8.6 Perform operations with vectors in terms of \( \mathbf{i} \) and \( \mathbf{j} \).
8.8.7 Find the dot product of two vectors.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140°. A north wind (from north to south) is blowing at 16.2 miles per hour, as shown in Figure 8.68. What are the ground speed and actual bearing of the plane?

![Figure 8.68](image)

Ground speed refers to the speed of a plane relative to the ground. Airspeed refers to the speed a plane can travel relative to its surrounding air mass. These two quantities are not the same because of the effect of wind. In an earlier section, we used triangles to solve a similar problem involving the movement of boats. Later in this section, we will find the airplane’s groundspeed and bearing, while investigating another approach to problems of this type. First, however, let’s examine the basics of vectors.

A Geometric View of Vectors

A vector is a specific quantity drawn as a line segment with an arrowhead at one end. It has an initial point, where it begins, and a terminal point, where it ends. A vector is defined by its magnitude, or the length of the line, and its direction, indicated by an arrowhead at the terminal point. Thus, a vector is a directed line segment. There are various symbols that distinguish vectors from other quantities:

- Lower case, boldfaced type, with or without an arrow on top such as \( \mathbf{v}, \mathbf{u}, \mathbf{w}, \mathbf{v}, \mathbf{u}, \mathbf{w} \).
- Given initial point \( P \) and terminal point \( Q \), a vector can be represented as \( \overrightarrow{PQ} \). The arrowhead on top is what indicates that it is not just a line, but a directed line segment.
- Given an initial point of \((0, 0)\) and terminal point \((a, b)\), a vector may be represented as \((a, b)\).
This last symbol \( \langle a, b \rangle \) has special significance. It is called the **standard position**. The position vector has an initial point \((0, 0)\) and a terminal point \(\langle a, b \rangle\). To change any vector into the position vector, we think about the change in the \(x\)-coordinates and the change in the \(y\)-coordinates. Thus, if the initial point of a vector \( \vec{CD} \) is \( C(x_1, y_1) \) and the terminal point is \( D(x_2, y_2) \), then the position vector is found by calculating

\[
\vec{AB} = \langle x_2 - x_1, y_2 - y_1 \rangle = \langle a, b \rangle
\]

(8.45)

In **Figure 8.69**, we see the original vector \( \vec{CD} \) and the position vector \( \vec{AB} \).

![Figure 8.69]

**Properties of Vectors**

A vector is a directed line segment with an initial point and a terminal point. Vectors are identified by magnitude, or the length of the line, and direction, represented by the arrowhead pointing toward the terminal point. The position vector has an initial point at \((0, 0)\) and is identified by its terminal point \(\langle a, b \rangle\).

**Example 8.59**

**Find the Position Vector**

Consider the vector whose initial point is \( P(2, 3) \) and terminal point is \( Q(6, 4) \). Find the position vector.

**Solution**

**Example 8.60**

**Drawing a Vector with the Given Criteria and Its Equivalent Position Vector**

Find the position vector given that vector \( \vec{v} \) has an initial point at \((-3, 2)\) and a terminal point at \((4, 5)\), then graph both vectors in the same plane.

**Solution**
8.39 Draw a vector \( \mathbf{v} \) that connects from the origin to the point \((3, 5)\).

Finding Magnitude and Direction

To work with a vector, we need to be able to find its magnitude and its direction. We find its magnitude using the Pythagorean Theorem or the distance formula, and we find its direction using the inverse tangent function.

Magnitude and Direction of a Vector

Given a position vector \( \mathbf{v} = \langle a, b \rangle \), the magnitude is found by \( |\mathbf{v}| = \sqrt{a^2 + b^2} \). The direction is equal to the angle formed with the \(x\)-axis, or with the \(y\)-axis, depending on the application. For a position vector, the direction is found by \( \tan \theta = \left(\frac{b}{a}\right) \Rightarrow \theta = \tan^{-1} \left(\frac{b}{a}\right) \), as illustrated in Figure 8.70.

Figure 8.70

Two vectors \( \mathbf{v} \) and \( \mathbf{u} \) are considered equal if they have the same magnitude and the same direction. Additionally, if both vectors have the same position vector, they are equal.

Example 8.61

Finding the Magnitude and Direction of a Vector

Find the magnitude and direction of the vector with initial point \( P(-8, 1) \) and terminal point \( Q(-2, -5) \). Draw the vector.

Solution

Example 8.62

Showing That Two Vectors Are Equal

Show that vector \( \mathbf{v} \) with initial point at \((5, -3)\) and terminal point at \((-1, 2)\) is equal to vector \( \mathbf{u} \) with initial point at \((-1, -3)\) and terminal point at \((-7, 2)\). Draw the position vector on the same grid as \( \mathbf{v} \) and \( \mathbf{u} \). Next, find the magnitude and direction of each vector.

Solution
Performing Vector Addition and Scalar Multiplication

Now that we understand the properties of vectors, we can perform operations involving them. While it is convenient to think of the vector \( \mathbf{u} = \langle x, y \rangle \) as an arrow or directed line segment from the origin to the point \((x, y)\), vectors can be situated anywhere in the plane. The sum of two vectors \( \mathbf{u} \) and \( \mathbf{v} \), or vector addition, produces a third vector \( \mathbf{u} + \mathbf{v} \), the resultant vector.

To find \( \mathbf{u} + \mathbf{v} \), we first draw the vector \( \mathbf{u} \), and from the terminal end of \( \mathbf{u} \), we draw the vector \( \mathbf{v} \). In other words, we have the initial point of \( \mathbf{v} \) meet the terminal end of \( \mathbf{u} \). This position corresponds to the notion that we move along the first vector and then, from its terminal point, we move along the second vector. The sum \( \mathbf{u} + \mathbf{v} \) is the resultant vector because it results from addition or subtraction of two vectors. The resultant vector travels directly from the beginning of \( \mathbf{u} \) to the end of \( \mathbf{v} \) in a straight path, as shown in Figure 8.71.

![Figure 8.71 - Vector Addition](image)

Vector subtraction is similar to vector addition. To find \( \mathbf{u} - \mathbf{v} \), view it as \( \mathbf{u} + (-\mathbf{v}) \). Adding \(-\mathbf{v}\) is reversing direction of \( \mathbf{v} \) and adding it to the end of \( \mathbf{u} \). The new vector begins at the start of \( \mathbf{u} \) and stops at the end point of \(-\mathbf{v}\). See Figure 8.72 for a visual that compares vector addition and vector subtraction using parallelograms.

![Figure 8.72 - Vector Subtraction](image)

Example 8.63

Adding and Subtracting Vectors

Given \( \mathbf{u} = \langle 3, -2 \rangle \) and \( \mathbf{v} = \langle -1, 4 \rangle \), find two new vectors \( \mathbf{u} + \mathbf{v} \), and \( \mathbf{u} - \mathbf{v} \).

Solution

Multiplying By a Scalar

While adding and subtracting vectors gives us a new vector with a different magnitude and direction, the process of multiplying a vector by a scalar, a constant, changes only the magnitude of the vector or the length of the line. Scalar multiplication has no effect on the direction unless the scalar is negative, in which case the direction of the resulting vector is opposite the direction of the original vector.

Scalar Multiplication

Scalar multiplication involves the product of a vector and a scalar. Each component of the vector is multiplied by the scalar. Thus, to multiply \( \mathbf{v} = \langle a, b \rangle \) by \( k \), we have

\[
kv = \langle ka, kb \rangle
\]

(8.46)

Only the magnitude changes, unless \( k \) is negative, and then the vector reverses direction.
Example 8.64

Performing Scalar Multiplication

Given vector \( \mathbf{v} = \langle 3, 1 \rangle \), find \( 3\mathbf{v} \), \( \frac{1}{2} \mathbf{v} \), and \( -\mathbf{v} \).

Solution

Analysis

Notice that the vector \( 3\mathbf{v} \) is three times the length of \( \mathbf{v} \), \( \frac{1}{2} \mathbf{v} \) is half the length of \( \mathbf{v} \), and \( -\mathbf{v} \) is the same length of \( \mathbf{v} \), but in the opposite direction.

8.40 Find the scalar multiple \( 3 \mathbf{u} \) given \( \mathbf{u} = \langle 5, 4 \rangle \).

Example 8.65

Using Vector Addition and Scalar Multiplication to Find a New Vector

Given \( \mathbf{u} = \langle 3, -2 \rangle \) and \( \mathbf{v} = \langle -1, 4 \rangle \), find a new vector \( \mathbf{w} = 3\mathbf{u} + 2\mathbf{v} \).

Solution

Finding Component Form

In some applications involving vectors, it is helpful for us to be able to break a vector down into its components. Vectors are comprised of two components: the horizontal component is the \( x \) direction, and the vertical component is the \( y \) direction. For example, we can see in the graph in Figure 8.73 that the position vector \( \langle 2, 3 \rangle \) comes from adding the vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \). We have \( \mathbf{v}_1 \) with initial point \( (0, 0) \) and terminal point \( (2, 0) \).

\[
\mathbf{v}_1 = \langle 2 - 0, 0 - 0 \rangle = \langle 2, 0 \rangle \tag{8.47}
\]

We also have \( \mathbf{v}_2 \) with initial point \( (0, 0) \) and terminal point \( (0, 3) \).

\[
\mathbf{v}_2 = \langle 0 - 0, 3 - 0 \rangle = \langle 0, 3 \rangle \tag{8.48}
\]

Therefore, the position vector is

\[
\mathbf{v} = \langle 2 + 0, 3 + 0 \rangle = \langle 2, 3 \rangle \tag{8.49}
\]

Using the Pythagorean Theorem, the magnitude of \( \mathbf{v}_1 \) is 2, and the magnitude of \( \mathbf{v}_2 \) is 3. To find the magnitude of \( \mathbf{v} \), use the formula with the position vector.
The magnitude of \( v \) is \( \sqrt{13} \). To find the direction, we use the tangent function \( \tan \theta = \frac{v_2}{v_1} \).

\[
\tan \theta = \frac{v_2}{v_1} = \frac{3}{2} \\
\theta = \tan^{-1} \left( \frac{3}{2} \right) = 56.3^\circ
\]

Thus, the magnitude of \( v \) is \( \sqrt{13} \) and the direction is 56.3° off the horizontal.

**Example 8.66**

**Finding the Components of the Vector**

Find the components of the vector \( v \) with initial point (3, 2) and terminal point (7, 4).

**Solution**

**Finding the Unit Vector in the Direction of \( v \)**

In addition to finding a vector’s components, it is also useful in solving problems to find a vector in the same direction as the given vector, but of magnitude 1. We call a vector with a magnitude of 1 a **unit vector**. We can then preserve the direction of the original vector while simplifying calculations.

Unit vectors are defined in terms of components. The horizontal unit vector is written as \( \hat{i} = \langle 1, 0 \rangle \) and is directed along the positive horizontal axis. The vertical unit vector is written as \( \hat{j} = \langle 0, 1 \rangle \) and is directed along the positive vertical axis. See **Figure 8.74**.
The Unit Vectors

If \( \mathbf{v} \) is a nonzero vector, then \( \frac{\mathbf{v}}{v} \) is a unit vector in the direction of \( \mathbf{v} \). Any vector divided by its magnitude is a unit vector. Notice that magnitude is always a scalar, and dividing by a scalar is the same as multiplying by the reciprocal of the scalar.

Example 8.67

Finding the Unit Vector in the Direction of \( \mathbf{v} \)

Find a unit vector in the same direction as \( \mathbf{v} = \langle -5, 12 \rangle \).

Solution

Performing Operations with Vectors in Terms of \( i \) and \( j \)

So far, we have investigated the basics of vectors: magnitude and direction, vector addition and subtraction, scalar multiplication, the components of vectors, and the representation of vectors geometrically. Now that we are familiar with the general strategies used in working with vectors, we will represent vectors in rectangular coordinates in terms of \( i \) and \( j \).

Vectors in the Rectangular Plane

Given a vector \( \mathbf{v} \) with initial point \( P = (x_1, y_1) \) and terminal point \( Q = (x_2, y_2) \), \( \mathbf{v} \) is written as

\[
\mathbf{v} = (x_2 - x_1)i + (y_1 - y_2)j
\]

(8.52)

The position vector from \((0, 0)\) to \((a, b)\), where \((x_2 - x_1) = a\) and \((y_2 - y_1) = b\), is written as \( \mathbf{v} = ai + bj \). This vector sum is called a linear combination of the vectors \( i \) and \( j \).

The magnitude of \( \mathbf{v} = ai + bj \) is given as \( |\mathbf{v}| = \sqrt{a^2 + b^2} \). See Figure 8.75.
Example 8.68

**Writing a Vector in Terms of \( i \) and \( j \)**

Given a vector \( \mathbf{v} \) with initial point \( P = (2, -6) \) and terminal point \( Q = (-6, 6) \), write the vector in terms of \( i \) and \( j \).

**Solution**

\[
\mathbf{v} = a\mathbf{i} + b\mathbf{j}
\]

Example 8.69

**Writing a Vector in Terms of \( i \) and \( j \) Using Initial and Terminal Points**

Given initial point \( P_1 = (-1, 3) \) and terminal point \( P_2 = (2, 7) \), write the vector \( \mathbf{v} \) in terms of \( i \) and \( j \).

**Solution**

\[
\mathbf{v} = c\mathbf{i} + d\mathbf{j}
\]

**Performing Operations on Vectors in Terms of \( i \) and \( j \)**

When vectors are written in terms of \( i \) and \( j \), we can carry out addition, subtraction, and scalar multiplication by performing operations on corresponding components.

**Adding and Subtracting Vectors in Rectangular Coordinates**

Given \( \mathbf{v} = a\mathbf{i} + b\mathbf{j} \) and \( \mathbf{u} = c\mathbf{i} + d\mathbf{j} \), then

\[
\begin{align*}
\mathbf{v} + \mathbf{u} &= (a + c)\mathbf{i} + (b + d)\mathbf{j} \\
\mathbf{v} - \mathbf{u} &= (a - c)\mathbf{i} + (b - d)\mathbf{j}
\end{align*}
\]

8.41 Write the vector \( \mathbf{u} \) with initial point \( P = (-1, 6) \) and terminal point \( Q = (7, -5) \) in terms of \( i \) and \( j \).

Example 8.70
Finding the Sum of the Vectors

Find the sum of \( v_1 = 2i - 3j \) and \( v_2 = 4i + 5j \).

Solution

Calculating the Component Form of a Vector: Direction

We have seen how to draw vectors according to their initial and terminal points and how to find the position vector. We have also examined notation for vectors drawn specifically in the Cartesian coordinate plane using \( i \) and \( j \). For any of these vectors, we can calculate the magnitude. Now, we want to combine the key points, and look further at the ideas of magnitude and direction.

Calculating direction follows the same straightforward process we used for polar coordinates. We find the direction of the vector by finding the angle to the horizontal. We do this by using the basic trigonometric identities, but with \( |v| \) replacing \( r \).

Vector Components in Terms of Magnitude and Direction

Given a position vector \( v = (x, y) \) and a direction angle \( \theta \),

\[
\begin{align*}
\cos \theta &= \frac{x}{|v|} \quad \text{and} \quad \sin \theta = \frac{y}{|v|} \\
x &= |v|\cos \theta \\
y &= |v|\sin \theta
\end{align*}
\]

Thus, \( v = xi + yj = |v|\cos \theta i + |v|\sin \theta j \), and magnitude is expressed as \( |v| = \sqrt{x^2 + y^2} \).

Example 8.71

Writing a Vector in Terms of Magnitude and Direction

Write a vector with length 7 at an angle of 135° to the positive \( x \)-axis in terms of magnitude and direction.

Solution

Finding the Dot Product of Two Vectors

As we discussed earlier in the section, scalar multiplication involves multiplying a vector by a scalar, and the result is a vector. As we have seen, multiplying a vector by a number is called scalar multiplication. If we multiply a vector by a vector, there are two possibilities: the dot product and the cross product. We will only examine the dot product here; you may encounter the cross product in more advanced mathematics courses.

The dot product of two vectors involves multiplying two vectors together, and the result is a scalar.
### Dot Product

The **dot product** of two vectors \( \mathbf{v} = \langle a, b \rangle \) and \( \mathbf{u} = \langle c, d \rangle \) is the sum of the product of the horizontal components and the product of the vertical components.

\[
\mathbf{v} \cdot \mathbf{u} = ac + bd \tag{8.55}
\]

To find the angle between the two vectors, use the formula below.

\[
\cos \theta = \frac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{v}\| \|\mathbf{u}\|} \tag{8.56}
\]

---

#### Example 8.72

**Finding the Dot Product of Two Vectors**

Find the dot product of \( \mathbf{v} = \langle 5, 12 \rangle \) and \( \mathbf{u} = \langle -3, 4 \rangle \).

**Solution**

---

#### Example 8.73

**Finding the Dot Product of Two Vectors and the Angle between Them**

Find the dot product of \( \mathbf{v}_1 = 5\mathbf{i} + 2\mathbf{j} \) and \( \mathbf{v}_2 = 3\mathbf{i} + 7\mathbf{j} \). Then, find the angle between the two vectors.

**Solution**

---

#### Example 8.74

**Finding the Angle between Two Vectors**

Find the angle between \( \mathbf{u} = \langle -3, 4 \rangle \) and \( \mathbf{v} = \langle 5, 12 \rangle \).

**Solution**

---

#### Example 8.75

**Finding Ground Speed and Bearing Using Vectors**

We now have the tools to solve the problem we introduced in the opening of the section.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140°. A north wind (from north to south) is blowing at 16.2 miles per hour. What are the ground speed and actual bearing of the plane? See Figure 8.76.
Solution

Access these online resources for additional instruction and practice with vectors.

- Introduction to Vectors (http://openstaxcollege.org/l/introvectors)
- Vector Operations (http://openstaxcollege.org/l/vectoroperation)
- The Unit Vector (http://openstaxcollege.org/l/unitvector)
8.8 EXERCISES

Verbal

495. What are the characteristics of the letters that are commonly used to represent vectors?

496. How is a vector more specific than a line segment?

497. What are \( \mathbf{i} \) and \( \mathbf{j} \), and what do they represent?

498. What is component form?

499. When a unit vector is expressed as \( \langle a, b \rangle \), which letter is the coefficient of the \( \mathbf{i} \) and which the \( \mathbf{j} \)?

Algebraic

500. Given a vector with initial point \((5, 2)\) and terminal point \((-1, -3)\), find an equivalent vector whose initial point is \((0, 0)\). Write the vector in component form \( \langle a, b \rangle \).

501. Given a vector with initial point \((-4, 2)\) and terminal point \((3, -3)\), find an equivalent vector whose initial point is \((0, 0)\). Write the vector in component form \( \langle a, b \rangle \).

502. Given a vector with initial point \((7, -1)\) and terminal point \((-1, -7)\), find an equivalent vector whose initial point is \((0, 0)\). Write the vector in component form \( \langle a, b \rangle \).

For the following exercises, determine whether the two vectors \( \mathbf{u} \) and \( \mathbf{v} \) are equal, where \( \mathbf{u} \) has an initial point \( P_1 \) and a terminal point \( P_2 \) and \( \mathbf{v} \) has an initial point \( P_3 \) and a terminal point \( P_4 \).

503. \( P_1 = (5, 1), P_2 = (3, -2), P_3 = (-1, 3), \) and \( P_4 = (9, -4) \)

504. \( P_1 = (2, -3), P_2 = (5, 1), P_3 = (6, -1), \) and \( P_4 = (9, 3) \)

505. \( P_1 = (-1, -1), P_2 = (-4, 5), P_3 = (-10, 6), \) and \( P_4 = (-13, 12) \)

506. \( P_1 = (3, 7), P_2 = (2, 1), P_3 = (1, 2), \) and \( P_4 = (-1, -4) \)

507. \( P_1 = (8, 3), P_2 = (6, 5), P_3 = (11, 8), \) and \( P_4 = (9, 10) \)

508. Given initial point \( P_1 = (-3, 1) \) and terminal point \( P_2 = (5, 2) \), write the vector \( \mathbf{v} \) in terms of \( \mathbf{i} \) and \( \mathbf{j} \).

509. Given initial point \( P_1 = (6, 0) \) and terminal point \( P_2 = (-1, -3) \), write the vector \( \mathbf{v} \) in terms of \( \mathbf{i} \) and \( \mathbf{j} \).

For the following exercises, use the vectors \( \mathbf{u} = \mathbf{i} + 5\mathbf{j}, \mathbf{v} = -2\mathbf{i} - 3\mathbf{j}, \) and \( \mathbf{w} = 4\mathbf{i} - \mathbf{j} \).

510. Find \( \mathbf{u} + (\mathbf{v} - \mathbf{w}) \)

511. Find \( 4\mathbf{v} + 2\mathbf{u} \)

For the following exercises, use the given vectors to compute \( \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}, \) and \( 2\mathbf{u} - 3\mathbf{v} \).

512. \( \mathbf{u} = \langle 2, -3 \rangle, \mathbf{v} = \langle 1, 5 \rangle \)

513. \( \mathbf{u} = \langle -3, 4 \rangle, \mathbf{v} = \langle -2, 1 \rangle \)

514. Let \( \mathbf{v} = -4\mathbf{i} + 3\mathbf{j} \). Find a vector that is half the length and points in the same direction as \( \mathbf{v} \).

515. Let \( \mathbf{v} = 5\mathbf{i} + 2\mathbf{j} \). Find a vector that is twice the length and points in the opposite direction as \( \mathbf{v} \).

For the following exercises, find a unit vector in the same direction as the given vector.
516. \( \mathbf{a} = 3\mathbf{i} + 4\mathbf{j} \)
517. \( \mathbf{b} = -2\mathbf{i} + 5\mathbf{j} \)
518. \( \mathbf{c} = 10\mathbf{i} - \mathbf{j} \)
519. \( \mathbf{d} = -\frac{1}{3}\mathbf{i} + \frac{5}{2}\mathbf{j} \)
520. \( \mathbf{u} = 100\mathbf{i} + 200\mathbf{j} \)
521. \( \mathbf{u} = -14\mathbf{i} + 2\mathbf{j} \)

For the following exercises, find the magnitude and direction of the vector, \( 0 \leq \theta < 2\pi \).

522. \( \langle 0, 4 \rangle \)
523. \( \langle 6, 5 \rangle \)
524. \( \langle 2, -5 \rangle \)
525. \( \langle -4, -6 \rangle \)

526. Given \( \mathbf{u} = 3\mathbf{i} - 4\mathbf{j} \) and \( \mathbf{v} = -2\mathbf{i} + 3\mathbf{j} \), calculate \( \mathbf{u} \cdot \mathbf{v} \).
527. Given \( \mathbf{u} = -\mathbf{i} - \mathbf{j} \) and \( \mathbf{v} = \mathbf{i} + 5\mathbf{j} \), calculate \( \mathbf{u} \cdot \mathbf{v} \).
528. Given \( \mathbf{u} = \langle -2, 4 \rangle \) and \( \mathbf{v} = \langle -3, 1 \rangle \), calculate \( \mathbf{u} \cdot \mathbf{v} \).
529. Given \( \mathbf{u} = \langle -1, 6 \rangle \) and \( \mathbf{v} = \langle 6, -1 \rangle \), calculate \( \mathbf{u} \cdot \mathbf{v} \).

**Graphical**

For the following exercises, given \( \mathbf{v} \), draw \( \mathbf{v}, 3\mathbf{v} \) and \( \frac{1}{2}\mathbf{v} \).

530. \( \langle 2, -1 \rangle \)
531. \( \langle -1, 4 \rangle \)
532. \( \langle -3, -2 \rangle \)

For the following exercises, use the vectors shown to sketch \( \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}, \) and \( 2\mathbf{u} \).

533.

\[ \text{Diagram of } \mathbf{u} \text{ and } \mathbf{v} \]

534.

\[ \text{Diagram of } \mathbf{u}, \mathbf{v}, \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}, \text{ and } 2\mathbf{u} \]
For the following exercises, use the vectors shown to sketch $2\mathbf{u} + \mathbf{v}$.

535.

For the following exercises, use the vectors shown to sketch $\mathbf{u} - 3\mathbf{v}$.

536.

537.

538.
539. For the following exercises, write the vector shown in component form.

540.

541.

542. Given initial point \( P_1 = (2, 1) \) and terminal point \( P_2 = (-1, 2) \), write the vector \( \mathbf{v} \) in terms of \( \mathbf{i} \) and \( \mathbf{j} \), then draw the vector on the graph.

543. Given initial point \( P_1 = (4, -1) \) and terminal point \( P_2 = (-3, 2) \), write the vector \( \mathbf{v} \) in terms of \( \mathbf{i} \) and \( \mathbf{j} \). Draw the points and the vector on the graph.

544. Given initial point \( P_1 = (3, 3) \) and terminal point \( P_2 = (-3, 3) \), write the vector \( \mathbf{v} \) in terms of \( \mathbf{i} \) and \( \mathbf{j} \). Draw the points and the vector on the graph.

**Extensions**

For the following exercises, use the given magnitude and direction in standard position, write the vector in component form.

545. \( |\mathbf{v}| = 6, \theta = 45^\circ \)
546. \(|v| = 8, \theta = 220^\circ\)
547. \(|v| = 2, \theta = 300^\circ\)
548. \(|v| = 5, \theta = 135^\circ\)

549. A 60-pound box is resting on a ramp that is inclined 12°. Rounding to the nearest tenth,
   a. Find the magnitude of the normal (perpendicular) component of the force.
   b. Find the magnitude of the component of the force that is parallel to the ramp.

550. A 25-pound box is resting on a ramp that is inclined 8°. Rounding to the nearest tenth,
   a. Find the magnitude of the normal (perpendicular) component of the force.
   b. Find the magnitude of the component of the force that is parallel to the ramp.

551. Find the magnitude of the horizontal and vertical components of a vector with magnitude 8 pounds pointed in a direction of 27° above the horizontal. Round to the nearest hundredth.

552. Find the magnitude of the horizontal and vertical components of the vector with magnitude 4 pounds pointed in a direction of 127° above the horizontal. Round to the nearest hundredth.

553. Find the magnitude of the horizontal and vertical components of a vector with magnitude 5 pounds pointed in a direction of 55° above the horizontal. Round to the nearest hundredth.

554. Find the magnitude of the horizontal and vertical components of the vector with magnitude 1 pound pointed in a direction of 8° above the horizontal. Round to the nearest hundredth.

**Real-World Applications**

555. A woman leaves home and walks 3 miles west, then 2 miles southwest. How far from home is she, and in what direction must she walk to head directly home?

556. A boat leaves the marina and sails 6 miles north, then 2 miles northeast. How far from the marina is the boat, and in what direction must it sail to head directly back to the marina?

557. A man starts walking from home and walks 4 miles east, 2 miles southeast, 5 miles south, 4 miles southwest, and 2 miles east. How far has he walked? If he walked straight home, how far would he have to walk?

558. A woman starts walking from home and walks 4 miles east, 7 miles southeast, 6 miles south, 5 miles southwest, and 3 miles east. How far has she walked? If she walked straight home, how far would she have to walk?

559. A man starts walking from home and walks 3 miles at 20° north of west, then 5 miles at 10° west of south, then 4 miles at 15° north of east. If he walked straight home, how far would he have to walk, and in what direction?

560. A woman starts walking from home and walks 6 miles at 40° north of east, then 2 miles at 15° east of south, then 5 miles at 30° south of west. If she walked straight home, how far would she have to walk, and in what direction?

561. An airplane is heading north at an airspeed of 600 km/hr, but there is a wind blowing from the southwest at 80 km/hr. How many degrees off course will the plane end up flying, and what is the plane’s speed relative to the ground?

562. An airplane is heading north at an airspeed of 500 km/hr, but there is a wind blowing from the northwest at 50 km/hr. How many degrees off course will the plane end up flying, and what is the plane’s speed relative to the ground?

563. An airplane needs to head due north, but there is a wind blowing from the southwest at 60 km/hr. The plane flies with an airspeed of 550 km/hr. To end up flying due north, how many degrees west of north will the pilot need to fly the plane?

564. An airplane needs to head due north, but there is a wind blowing from the northwest at 80 km/hr. The plane flies with an airspeed of 500 km/hr. To end up flying due north, how many degrees west of north will the pilot need to fly the plane?

565. As part of a video game, the point (5, 7) is rotated counterclockwise about the origin through an angle of 35°. Find the new coordinates of this point.

566. As part of a video game, the point (7, 3) is rotated counterclockwise about the origin through an angle of 40°. Find the new coordinates of this point.
567. Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 10 mph relative to the car, and the car is traveling 25 mph down the road. If one child doesn't catch the ball, and it flies out the window, in what direction does the ball fly (ignoring wind resistance)?

568. Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 8 mph relative to the car, and the car is traveling 45 mph down the road. If one child doesn't catch the ball, and it flies out the window, in what direction does the ball fly (ignoring wind resistance)?

569. A 50-pound object rests on a ramp that is inclined 19°. Find the magnitude of the components of the force parallel to and perpendicular to (normal) the ramp to the nearest tenth of a pound.

570. Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it upward, and 5 pounds acting on it 45° from the horizontal. What single force is the resultant force acting on the body?

571. Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it —135° from the horizontal, and 5 pounds acting on it directed 150° from the horizontal. What single force is the resultant force acting on the body?

572. The condition of equilibrium is when the sum of the forces acting on a body is the zero vector. Suppose a body has a force of 2 pounds acting on it to the right, 5 pounds acting on it upward, and 3 pounds acting on it 45° from the horizontal. What single force is needed to produce a state of equilibrium on the body?

573. Suppose a body has a force of 3 pounds acting on it to the left, 4 pounds acting on it upward, and 2 pounds acting on it 30° from the horizontal. What single force is needed to produce a state of equilibrium on the body? Draw the vector.
CHAPTER 8 REVIEW

KEY TERMS

Archimedes’ spiral: a polar curve given by \( r = \theta \). When multiplied by a constant, the equation appears as \( r = a\theta \). As \( r = \theta \), the curve continues to widen in a spiral path over the domain.

altitude: a perpendicular line from one vertex of a triangle to the opposite side, or in the case of an obtuse triangle, to the line containing the opposite side, forming two right triangles

ambiguous case: a scenario in which more than one triangle is a valid solution for a given oblique SSA triangle

argument: the angle associated with a complex number; the angle between the line from the origin to the point and the positive real axis

cardioid: a member of the limaçon family of curves, named for its resemblance to a heart; its equation is given as

\[ r = a \pm b\cos \theta \text{ and } r = a \pm b\sin \theta, \text{ where } \frac{a}{b} = 1 \]

cricle limaçon: a type of one-loop limaçon represented by \( r = a \pm b\cos \theta \) and \( r = a \pm b\sin \theta \) such that \( \frac{a}{b} \geq 2 \)

De Moivre’s Theorem: formula used to find the \( n \)th power or \( n \)th roots of a complex number; states that, for a positive integer \( n \), \( z^n \) is found by raising the modulus to the \( n \)th power and multiplying the angles by \( n \)

dimpled limaçon: a type of one-loop limaçon represented by \( r = a \pm b\cos \theta \) and \( r = a \pm b\sin \theta \) such that \( 1 < \frac{a}{b} < 2 \)

dot product: given two vectors, the sum of the product of the horizontal components and the product of the vertical components

Generalized Pythagorean Theorem: an extension of the Law of Cosines; relates the sides of an oblique triangle and is used for SAS and SSS triangles

initial point: the origin of a vector

inner-loop limaçon: a polar curve similar to the cardioid, but with an inner loop; passes through the pole twice; represented by \( r = a \pm b\cos \theta \) and \( r = a \pm b\sin \theta \) where \( a < b \)

Law of Cosines: states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle

Law of Sines: states that the ratio of the measurement of one angle of a triangle to the length of its opposite side is equal to the remaining two ratios of angle measure to opposite side; any pair of proportions may be used to solve for a missing angle or side

lemniscate: a polar curve resembling a figure 8 and given by the equation \( r^2 = a^2 \cos 2\theta \) and \( r^2 = a^2 \sin 2\theta \), \( a \neq 0 \)

magnitude: the length of a vector; may represent a quantity such as speed, and is calculated using the Pythagorean Theorem

modulus: the absolute value of a complex number, or the distance from the origin to the point \((x, y)\); also called the amplitude

oblique triangle: any triangle that is not a right triangle

one-loop limaçon: a polar curve represented by \( r = a \pm b\cos \theta \) and \( r = a \pm b\sin \theta \) such that \( a > 0, b > 0 \), and \( \frac{a}{b} > 1 \); may be dimpled or convex; does not pass through the pole

parameter: a variable, often representing time, upon which \( x \) and \( y \) are both dependent

polar axis: on the polar grid, the equivalent of the positive \( x \)-axis on the rectangular grid
polar coordinates: on the polar grid, the coordinates of a point labeled \((r, \theta)\), where \(\theta\) indicates the angle of rotation from the polar axis and \(r\) represents the radius, or the distance of the point from the pole in the direction of \(\theta\)

polar equation: an equation describing a curve on the polar grid.

polar form of a complex number: a complex number expressed in terms of an angle \(\theta\) and its distance from the origin \(r\); can be found by using conversion formulas \(x = r\cos \theta\), \(y = r\sin \theta\), and \(r = \sqrt{x^2 + y^2}\)

pole: the origin of the polar grid

resultant: a vector that results from addition or subtraction of two vectors, or from scalar multiplication

rose curve: a polar equation resembling a flower, given by the equations \(r = a\cos n\theta\) and \(r = a\sin n\theta\); when \(n\) is even there are \(2n\) petals, and the curve is highly symmetrical; when \(n\) is odd there are \(n\) petals.

scalar multiplication: the product of a constant and each component of a vector

scalar: a quantity associated with magnitude but not direction; a constant

standard position: the placement of a vector with the initial point at \((0, 0)\) and the terminal point \((a, b)\), represented by the change in the \(x\)-coordinates and the change in the \(y\)-coordinates of the original vector

terminal point: the end point of a vector, usually represented by an arrow indicating its direction

unit vector: a vector that begins at the origin and has magnitude of 1; the horizontal unit vector runs along the \(x\)-axis and is defined as \(v_1 = \langle 1, 0 \rangle\); the vertical unit vector runs along the \(y\)-axis and is defined as \(v_2 = \langle 0, 1 \rangle\).

vector addition: the sum of two vectors, found by adding corresponding components

vector: a quantity associated with both magnitude and direction, represented as a directed line segment with a starting point (initial point) and an end point (terminal point)

**KEY EQUATIONS**

| Law of Sines | \[
\begin{align*}
sin \alpha &= \frac{b}{a} = \frac{\sin \beta}{\sin \alpha} \\
\frac{a}{\sin \alpha} &= \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}
\end{align*}
| Law of Cosines | \[
\begin{align*}
a^2 &= b^2 + c^2 - 2bc \cos \alpha \\
b^2 &= a^2 + c^2 - 2ac \cos \beta \\
c^2 &= a^2 + b^2 - 2ab \cos \gamma
\end{align*}
| Heron’s formula | \[
\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}
\]
where \(s = \frac{(a + b + c)}{2}\)

| Conversion formulas | \[
\begin{align*}
\cos \theta &= \frac{x}{r} \rightarrow x = r \cos \theta \\
\sin \theta &= \frac{y}{r} \rightarrow y = r \sin \theta \\
r^2 &= x^2 + y^2 \\
\tan \theta &= \frac{y}{x}
\end{align*}

Table 8.10

Table 8.11

Table 8.12
KEY CONCEPTS

8.1 Non-right Triangles: Law of Sines

- The Law of Sines can be used to solve oblique triangles, which are non-right triangles.
- According to the Law of Sines, the ratio of the measurement of one of the angles to the length of its opposite side equals the other two ratios of angle measure to opposite side.
- There are three possible cases: ASA, AAS, SSA. Depending on the information given, we can choose the appropriate equation to find the requested solution. See Example 8.1.
- The ambiguous case arises when an oblique triangle can have different outcomes.
- There are three possible cases that arise from SSA arrangement—a single solution, two possible solutions, and no solution. See Example 8.2 and Example 8.3.
- The Law of Sines can be used to solve triangles with given criteria. See Example 8.4.
- The general area formula for triangles translates to oblique triangles by first finding the appropriate height value. See Example 8.5.
- There are many trigonometric applications. They can often be solved by first drawing a diagram of the given information and then using the appropriate equation. See Example 8.6.

8.2 Non-right Triangles: Law of Cosines

- The Law of Cosines defines the relationship among angle measurements and lengths of sides in oblique triangles.
- The Generalized Pythagorean Theorem is the Law of Cosines for two cases of oblique triangles: SAS and SSS. Dropping an imaginary perpendicular splits the oblique triangle into two right triangles or forms one right triangle, which allows sides to be related and measurements to be calculated. See Example 8.7 and Example 8.8.
- The Law of Cosines is useful for many types of applied problems. The first step in solving such problems is generally to draw a sketch of the problem presented. If the information given fits one of the three models (the three equations), then apply the Law of Cosines to find a solution. See Example 8.9 and Example 8.10.
- Heron’s formula allows the calculation of area in oblique triangles. All three sides must be known to apply Heron’s formula. See Example 8.11 and See Example 8.12.

8.3 Polar Coordinates

- The polar grid is represented as a series of concentric circles radiating out from the pole, or origin.
- To plot a point in the form \((r, \theta)\), \(\theta > 0\), move in a counterclockwise direction from the polar axis by an angle of \(\theta\), and then extend a directed line segment from the pole the length of \(r\) in the direction of \(\theta\). If \(\theta\) is negative, move in a clockwise direction, and extend a directed line segment the length of \(r\) in the direction of \(\theta\). See Example 8.13.
- If \(r\) is negative, extend the directed line segment in the opposite direction of \(\theta\). See Example 8.14.
- To convert from polar coordinates to rectangular coordinates, use the formulas \(x = r\cos \theta\) and \(y = r\sin \theta\). See Example 8.15 and Example 8.16.
- To convert from rectangular coordinates to polar coordinates, use one or more of the formulas: \(\cos \theta = \frac{x}{r}\), \(\sin \theta = \frac{y}{r}\), \(\tan \theta = \frac{y}{x}\), and \(r = \sqrt{x^2 + y^2}\). See Example 8.17.
- Transforming equations between polar and rectangular forms means making the appropriate substitutions based on the available formulas, together with algebraic manipulations. See Example 8.18, Example 8.19, and Example 8.20.
- Using the appropriate substitutions makes it possible to rewrite a polar equation as a rectangular equation, and then graph it in the rectangular plane. See Example 8.21, Example 8.22, and Example 8.23.

8.4 Polar Coordinates: Graphs

- It is easier to graph polar equations if we can test the equations for symmetry with respect to the line \(\theta = \frac{\pi}{2}\), the polar axis, or the pole.
- There are three symmetry tests that indicate whether the graph of a polar equation will exhibit symmetry. If an equation fails a symmetry test, the graph may or may not exhibit symmetry. See Example 8.24.
• Polar equations may be graphed by making a table of values for $\theta$ and $r$.

• The maximum value of a polar equation is found by substituting the value $\theta$ that leads to the maximum value of the trigonometric expression.

• The zeros of a polar equation are found by setting $r = 0$ and solving for $\theta$. See Example 8.25.

• Some formulas that produce the graph of a circle in polar coordinates are given by $r = a \cos \theta$ and $r = a \sin \theta$. See Example 8.26.

• The formulas that produce the graphs of a cardioid are given by $r = a \pm b \cos \theta$ and $r = a \pm b \sin \theta$, for $a > 0$, $b > 0$, and $\frac{a}{b} = 1$. See Example 8.27.

• The formulas that produce the graphs of a one-loop limaçon are given by $r = a \pm b \cos \theta$ and $r = a \pm b \sin \theta$ for $1 < \frac{a}{b} < 2$. See Example 8.28.

• The formulas that produce the graphs of an inner-loop limaçon are given by $r = a \pm b \cos \theta$ and $r = a \pm b \sin \theta$ for $a > 0$, $b > 0$, and $a < b$. See Example 8.29.

• The formulas that produce the graphs of a lemniscates are given by $r^2 = a^2 \cos 2\theta$ and $r^2 = a^2 \sin 2\theta$, where $a \neq 0$. See Example 8.30.

• The formulas that produce the graphs of rose curves are given by $r = a \cos n\theta$ and $r = a \sin n\theta$, where $a \neq 0$; if $n$ is even, there are $2n$ petals, and if $n$ is odd, there are $n$ petals. See Example 8.31 and Example 8.32.

• The formula that produces the graph of an Archimedes’ spiral is given by $r = \theta$, $\theta \geq 0$. See Example 8.33.

#### 8.5 Polar Form of Complex Numbers

• Complex numbers in the form $a + bi$ are plotted in the complex plane similar to the way rectangular coordinates are plotted in the rectangular plane. Label the $x$-axis as the real axis and the $y$-axis as the imaginary axis. See Example 8.34.

• The absolute value of a complex number is the same as its magnitude. It is the distance from the origin to the point: $|z| = \sqrt{a^2 + b^2}$. See Example 8.35 and Example 8.36.

• To write complex numbers in polar form, we use the formulas $x = r \cos \theta$, $y = r \sin \theta$, and $r = \sqrt{x^2 + y^2}$. Then, $z = r(\cos \theta + i \sin \theta)$. See Example 8.37 and Example 8.38.

• To convert from polar form to rectangular form, first evaluate the trigonometric functions. Then, multiply through by $r$. See Example 8.39 and Example 8.40.

• To find the product of two complex numbers, multiply the two moduli and add the two angles. Evaluate the trigonometric functions, and multiply using the distributive property. See Example 8.41.

• To find the quotient of two complex numbers in polar form, find the quotient of the two moduli and the difference of the two angles. See Example 8.42.

• To find the power of a complex number $z^n$, raise $r$ to the power $n$, and multiply $\theta$ by $n$. See Example 8.43.

• Finding the roots of a complex number is the same as raising a complex number to a power, but using a rational exponent. See Example 8.44.

#### 8.6 Parametric Equations

• Parameterizing a curve involves translating a rectangular equation in two variables, $x$ and $y$, into two equations in three variables, $x$, $y$, and $t$. Often, more information is obtained from a set of parametric equations. See Example 8.45, Example 8.46, and Example 8.47.

• Sometimes equations are simpler to graph when written in rectangular form. By eliminating $t$, an equation in $x$ and $y$ is the result.

• To eliminate $t$, solve one of the equations for $t$, and substitute the expression into the second equation. See Example 8.48, Example 8.49, Example 8.50, and Example 8.51.
• Finding the rectangular equation for a curve defined parametrically is basically the same as eliminating the parameter. Solve for $t$ in one of the equations, and substitute the expression into the second equation. See Example 8.52.

• There are an infinite number of ways to choose a set of parametric equations for a curve defined as a rectangular equation.

• Find an expression for $x$ such that the domain of the set of parametric equations remains the same as the original rectangular equation. See Example 8.53.

8.7 Parametric Equations: Graphs

• When there is a third variable, a third parameter on which $x$ and $y$ depend, parametric equations can be used.

• To graph parametric equations by plotting points, make a table with three columns labeled $t$, $x(t)$, and $y(t)$. Choose values for $t$ in increasing order. Plot the last two columns for $x$ and $y$. See Example 8.54 and Example 8.55.

• When graphing a parametric curve by plotting points, note the associated $t$-values and show arrows on the graph indicating the orientation of the curve. See Example 8.56 and Example 8.57.

• Parametric equations allow the direction or the orientation of the curve to be shown on the graph. Equations that are not functions can be graphed and used in many applications involving motion. See Example 8.58.

• Projectile motion depends on two parametric equations: $x = (v_0 \cos \theta)t$ and $y = -16t^2 + (v_0 \sin \theta)t + h$. Initial velocity is symbolized as $v_0$, $\theta$ represents the initial angle of the object when thrown, and $h$ represents the height at which the object is propelled.

8.8 Vectors

• The position vector has its initial point at the origin. See Example 8.59.

• If the position vector is the same for two vectors, they are equal. See Example 8.60.

• Vectors are defined by their magnitude and direction. See Example 8.61.

• If two vectors have the same magnitude and direction, they are equal. See Example 8.62.

• Vector addition and subtraction result in a new vector found by adding or subtracting corresponding elements. See Example 8.63.

• Scalar multiplication is multiplying a vector by a constant. Only the magnitude changes; the direction stays the same. See Example 8.64 and Example 8.65.

• Vectors are comprised of two components: the horizontal component along the positive $x$-axis, and the vertical component along the positive $y$-axis. See Example 8.66.

• The unit vector in the same direction of any nonzero vector is found by dividing the vector by its magnitude.

• The magnitude of a vector in the rectangular coordinate system is $|v| = \sqrt{a^2 + b^2}$. See Example 8.67.

• In the rectangular coordinate system, unit vectors may be represented in terms of $i$ and $j$ where $i$ represents the horizontal component and $j$ represents the vertical component. Then, $v = ai + bj$ is a scalar multiple of $v$ by real numbers $a$ and $b$. See Example 8.68 and Example 8.69.

• Adding and subtracting vectors in terms of $i$ and $j$ consists of adding or subtracting corresponding coefficients of $i$ and corresponding coefficients of $j$. See Example 8.70.

• A vector $v = ai + bj$ is written in terms of magnitude and direction as $v = |v|\cos \theta i + |v|\sin \theta j$. See Example 8.71.

• The dot product of two vectors is the product of the $i$ terms plus the product of the $j$ terms. See Example 8.72.

• We can use the dot product to find the angle between two vectors. Example 8.73 and Example 8.74.

• Dot products are useful for many types of physics applications. See Example 8.75.

CHAPTER 8 REVIEW EXERCISES
Section 8.1

For the following exercises, assume \( \alpha \) is opposite side \( a \), \( \beta \) is opposite side \( b \), and \( \gamma \) is opposite side \( c \). Solve each triangle, if possible. Round each answer to the nearest tenth.

616. \( \beta = 50^\circ \), \( a = 105 \), \( b = 45 \)

617. \( \alpha = 43.1^\circ \), \( a = 184.2 \), \( b = 242.8 \)

![Image](image1.png)

618. Solve the triangle.

619. Find the area of the triangle.

![Image](image2.png)

620. A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 2.1 km apart, to be 25\(^\circ\) and 49\(^\circ\), as shown in Figure 8.77. Find the distance of the plane from point \( A \) and the elevation of the plane.

![Figure 8.77](image3.png)

Section 8.2

621. Solve the triangle, rounding to the nearest tenth, assuming \( \alpha \) is opposite side \( a \), \( \beta \) is opposite side \( b \), and \( \gamma \) is opposite side \( c \) : \( a = 4 \), \( b = 6 \), \( c = 8 \).

622. Solve the triangle in Figure 8.78, rounding to the nearest tenth.

![Image](image4.png)

623. Find the area of a triangle with sides of length 8.3, 6.6, and 9.1.
To find the distance between two cities, a satellite calculates the distances and angle shown in Figure 8.79 (not to scale). Find the distance between the cities. Round answers to the nearest tenth.

![Figure 8.79](image)

Section 8.3

625. Plot the point with polar coordinates \( \left(3, \frac{\pi}{6}\right) \).

626. Plot the point with polar coordinates \( \left(5, -\frac{2\pi}{3}\right) \).

627. Convert \( \left(6, -\frac{3\pi}{4}\right) \) to rectangular coordinates.

628. Convert \( \left(-2, \frac{3\pi}{2}\right) \) to rectangular coordinates.

629. Convert \( (7, -2) \) to polar coordinates.

630. Convert \( (-9, -4) \) to polar coordinates.

For the following exercises, convert the given Cartesian equation to a polar equation.

631. \( x = -2 \)

632. \( x^2 + y^2 = 64 \)

633. \( x^2 + y^2 = -2y \)

For the following exercises, convert the given polar equation to a Cartesian equation.

634. \( r = 7\cos \theta \)

635. \( r = \frac{-2}{4\cos \theta + \sin \theta} \)

For the following exercises, convert to rectangular form and graph.

636. \( \theta = \frac{3\pi}{4} \)

637. \( r = 5\sec \theta \)
Section 8.4
For the following exercises, test each equation for symmetry.

638. \( r = 4 + 4\sin \theta \)

639. \( r = 7 \)

640. Sketch a graph of the polar equation \( r = 1 - 5\sin \theta \). Label the axis intercepts.

641. Sketch a graph of the polar equation \( r = 5\sin(7\theta) \).

642. Sketch a graph of the polar equation \( r = 3 - 3\cos \theta \)

Section 8.5
For the following exercises, find the absolute value of each complex number.

643. \( -2 + 6i \)

644. \( 4 - 3i \)

Write the complex number in polar form.

645. \( 5 + 9i \)

646. \( \frac{1}{2} - \frac{\sqrt{3}}{2}i \)

For the following exercises, convert the complex number from polar to rectangular form.

647. \( z = 5\text{cis}\left(\frac{5\pi}{6}\right) \)

648. \( z = 3\text{cis}(40^\circ) \)

For the following exercises, find the product \( z_1 z_2 \) in polar form.

649. \( z_1 = 2\text{cis}(89^\circ) \)
\( z_2 = 5\text{cis}(23^\circ) \)

650. \( z_1 = 10\text{cis}\left(\frac{\pi}{6}\right) \)
\( z_2 = 6\text{cis}\left(\frac{\pi}{3}\right) \)

For the following exercises, find the quotient \( \frac{z_1}{z_2} \) in polar form.

651. \( z_1 = 12\text{cis}(55^\circ) \)
\( z_2 = 3\text{cis}(18^\circ) \)

652. \( z_1 = 27\text{cis}\left(\frac{5\pi}{3}\right) \)
\( z_2 = 9\text{cis}\left(\frac{\pi}{3}\right) \)
For the following exercises, find the powers of each complex number in polar form.

653. Find \( z^4 \) when \( z = 2 \cis (70^\circ) \)

654. Find \( z^2 \) when \( z = 5 \cis \left( \frac{3\pi}{4} \right) \)

For the following exercises, evaluate each root.

655. Evaluate the cube root of \( z \) when \( z = 64 \cis (210^\circ) \).

656. Evaluate the square root of \( z \) when \( z = 25 \cis \left( \frac{3\pi}{2} \right) \).

For the following exercises, plot the complex number in the complex plane.

657. \( 6 - 2i \)

658. \( -1 + 3i \)

Section 8.6

For the following exercises, eliminate the parameter \( t \) to rewrite the parametric equation as a Cartesian equation.

659. \[
\begin{align*}
x(t) &= 3t - 1 \\
y(t) &= \sqrt{t}
\end{align*}
\]

660. \[
\begin{align*}
x(t) &= -\cos t \\
y(t) &= 2\sin^2 t
\end{align*}
\]

661. Parameterize (write a parametric equation for) each Cartesian equation by using \( x(t) = a \cos t \) and \( y(t) = b \sin t \) for \( \frac{x^2}{25} + \frac{y^2}{16} = 1 \).

662. Parameterize the line from \((-2, 3)\) to \((4, 7)\) so that the line is at \((-2, 3)\) at \( t = 0 \) and \((4, 7)\) at \( t = 1 \).

Section 8.7

For the following exercises, make a table of values for each set of parametric equations, graph the equations, and include an orientation; then write the Cartesian equation.

663. \[
\begin{align*}
x(t) &= 3t^2 \\
y(t) &= 2t - 1
\end{align*}
\]

664. \[
\begin{align*}
x(t) &= e^t \\
y(t) &= -2e^{5t}
\end{align*}
\]

665. \[
\begin{align*}
x(t) &= 3\cos t \\
y(t) &= 2\sin t
\end{align*}
\]

666. A ball is launched with an initial velocity of 80 feet per second at an angle of 40° to the horizontal. The ball is released at a height of 4 feet above the ground.
   a. Find the parametric equations to model the path of the ball.
   b. Where is the ball after 3 seconds?
   c. How long is the ball in the air?
Section 8.8
For the following exercises, determine whether the two vectors, \( u \) and \( v \), are equal, where \( u \) has an initial point \( P_1 \) and a terminal point \( P_2 \), and \( v \) has an initial point \( P_3 \) and a terminal point \( P_4 \).

667. \( P_1 = (-1, 4) \), \( P_2 = (3, 1) \), \( P_3 = (5, 5) \) and \( P_4 = (9, 2) \)

668. \( P_1 = (6, 11) \), \( P_2 = (-2, 8) \), \( P_3 = (0, -1) \) and \( P_4 = (-8, 2) \)

For the following exercises, use the vectors \( u = 2i - j \), \( v = 4i - 3j \), and \( w = -2i + 5j \) to evaluate the expression.

669. \( u - v \)

670. \( 2v - u + w \)

For the following exercises, find a unit vector in the same direction as the given vector.

671. \( a = 8i - 6j \)

672. \( b = -3i - j \)

For the following exercises, find the magnitude and direction of the vector.

673. \( \langle 6, -2 \rangle \)

674. \( \langle -3, -3 \rangle \)

For the following exercises, calculate \( u \cdot v \).

675. \( u = -2i + j \) and \( v = 3i + 7j \)

676. \( u = i + 4j \) and \( v = 4i + 3j \)

677. Given \( v = \langle -3, 4 \rangle \) draw \( v \), \( 2v \), and \( \frac{1}{2}v \).

678. Given the vectors shown in Figure 8.80, sketch \( u + v \), \( u - v \) and \( 3v \).

Figure 8.80

679. Given initial point \( P_1 = (3, 2) \) and terminal point \( P_2 = (-5, -1) \), write the vector \( v \) in terms of \( i \) and \( j \). Draw the points and the vector on the graph.

CHAPTER 8 PRACTICE TEST
638. Assume \( \alpha \) is opposite side \( a \), \( \beta \) is opposite side \( b \), and \( \gamma \) is opposite side \( c \). Solve the triangle, if possible, and round each answer to the nearest tenth, given \( \beta = 68^\circ \), \( b = 21 \), \( c = 16 \).

639. Find the area of the triangle in Figure 8.81. Round each answer to the nearest tenth.

![Figure 8.81](image)

640. A pilot flies in a straight path for 2 hours. He then makes a course correction, heading 15° to the right of his original course, and flies 1 hour in the new direction. If he maintains a constant speed of 575 miles per hour, how far is he from his starting position?

641. Convert \((2, 2)\) to polar coordinates, and then plot the point.

642. Convert \( \left(2, \frac{\pi}{3}\right) \) to rectangular coordinates.

643. Convert the polar equation to a Cartesian equation: \( x^2 + y^2 = 5y \).

644. Convert to rectangular form and graph: \( r = -3\csc \theta \).

645. Test the equation for symmetry: \( r = -4\sin(2\theta) \).

646. Graph \( r = 3 + 3\cos \theta \).

647. Graph \( r = 3 - 5\sin \theta \).

648. Find the absolute value of the complex number \( 5 - 9i \).

649. Write the complex number in polar form: \( 4 + i \).

650. Convert the complex number from polar to rectangular form: \( z = 5\cis\left(\frac{2\pi}{3}\right) \).

Given \( z_1 = 8\cis(36^\circ) \) and \( z_2 = 2\cis(15^\circ) \), evaluate each expression.

651. \( z_1 z_2 \)

652. \( \frac{z_1}{z_2} \)

653. \( (z_2)^3 \)

654. \( \sqrt[5]{z_1} \)

655. Plot the complex number \(-5 - i\) in the complex plane.
656. Eliminate the parameter $t$ to rewrite the following parametric equations as a Cartesian equation:

\[ \begin{align*}
  x(t) &= t + 1 \\
  y(t) &= 2t^2.
\end{align*} \]

657. Parameterize (write a parametric equation for) the following Cartesian equation by using $x(t) = \cos t$ and $y(t) = \sin t$:

\[ \frac{x^2}{36} + \frac{y^2}{100} = 1. \]

658. Graph the set of parametric equations and find the Cartesian equation:

\[ \begin{align*}
  x(t) &= -2 \sin t \\
  y(t) &= 5 \cos t.
\end{align*} \]

659. A ball is launched with an initial velocity of 95 feet per second at an angle of 52° to the horizontal. The ball is released at a height of 3.5 feet above the ground.

a. Find the parametric equations to model the path of the ball.

b. Where is the ball after 2 seconds?

c. How long is the ball in the air?

For the following exercises, use the vectors $\mathbf{u} = \mathbf{i} - 3\mathbf{j}$ and $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$.

660. Find $2\mathbf{u} - 3\mathbf{v}$.

661. Calculate $\mathbf{u} \cdot \mathbf{v}$.

662. Find a unit vector in the same direction as $\mathbf{v}$.

663. Given vector $\mathbf{v}$ has an initial point $P_1 = (2, 2)$ and terminal point $P_2 = (-1, 0)$. Write the vector $\mathbf{v}$ in terms of $\mathbf{i}$ and $\mathbf{j}$. On the graph, draw $\mathbf{v}$, and $-\mathbf{v}$. 

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9 | SYSTEMS OF EQUATIONS AND INEQUALITIES

Figure 9.1  Enigma machines like this one, once owned by Italian dictator Benito Mussolini, were used by government and military officials for enciphering and deciphering top-secret communications during World War II. (credit: Dave Addey, Flickr)

Chapter Outline

9.1 Systems of Linear Equations: Two Variables
9.2 Systems of Linear Equations: Three Variables
9.3 Systems of Nonlinear Equations and Inequalities: Two Variables
9.4 Partial Fractions
9.5 Matrices and Matrix Operations
9.6 Solving Systems with Gaussian Elimination
9.7 Solving Systems with Inverses
9.8 Solving Systems with Cramer’s Rule

Introduction

By 1943, it was obvious to the Nazi regime that defeat was imminent unless it could build a weapon with unlimited destructive power, one that had never been seen before in the history of the world. In September, Adolf Hitler ordered German scientists to begin building an atomic bomb. Rumors and whispers began to spread from across the ocean. Refugees and diplomats told of the experiments happening in Norway. However, Franklin D. Roosevelt wasn’t sold, and even doubted
British Prime Minister Winston Churchill’s warning. Roosevelt wanted undeniable proof. Fortunately, he soon received the proof he wanted when a group of mathematicians cracked the “Enigma” code, proving beyond a doubt that Hitler was building an atomic bomb. The next day, Roosevelt gave the order that the United States begin work on the same.

The Enigma is perhaps the most famous cryptographic device ever known. It stands as an example of the pivotal role cryptography has played in society. Now, technology has moved cryptanalysis to the digital world.

Many ciphers are designed using invertible matrices as the method of message transference, as finding the inverse of a matrix is generally part of the process of decoding. In addition to knowing the matrix and its inverse, the receiver must also know the key that, when used with the matrix inverse, will allow the message to be read.

In this chapter, we will investigate matrices and their inverses, and various ways to use matrices to solve systems of equations. First, however, we will study systems of equations on their own: linear and nonlinear, and then partial fractions. We will not be breaking any secret codes here, but we will lay the foundation for future courses.

## 9.1 | Systems of Linear Equations: Two Variables

### Learning Objectives

In this section, you will:

- **9.1.1** Solve systems of equations by graphing.
- **9.1.2** Solve systems of equations by substitution.
- **9.1.3** Solve systems of equations by addition.
- **9.1.4** Identify inconsistent systems of equations containing two variables.
- **9.1.5** Express the solution of a system of dependent equations containing two variables.

![Figure 9.2](credit: Thomas Sørenes)

A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible? In this section, we will consider linear equations with two variables to answer these and similar questions.

### Introduction to Systems of Equations

In order to investigate situations such as that of the skateboard manufacturer, we need to recognize that we are dealing with more than one variable and likely more than one equation. A **system of linear equations** consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously. To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables. For example, consider the following system of linear equations in two variables.

\[
\begin{align*}
2x + y &= 15 \\
3x - y &= 5
\end{align*}
\]

The **solution** to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, the ordered pair \((4, 7)\) is the solution to the system of linear equations. We can verify the solution by
substituting the values into each equation to see if the ordered pair satisfies both equations. Shortly we will investigate methods of finding such a solution if it exists.

\[
\begin{align*}
2(4) + (7) &= 15 \quad \text{True} \\
3(4) - (7) &= 5 \quad \text{True}
\end{align*}
\]

In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions. A **consistent system** of equations has at least one solution. A consistent system is considered to be an **independent system** if it has a single solution, such as the example we just explored. The two lines have different slopes and intersect at one point in the plane. A consistent system is considered to be a **dependent system** if the equations have the same slope and the same y-intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is an **inconsistent system**, which is one in which the equations represent two parallel lines. The lines have the same slope and different y-intercepts. There are no points common to both lines; hence, there is no solution to the system.

### Types of Linear Systems

There are three types of systems of linear equations in two variables, and three types of solutions.

- An **independent system** has exactly one solution pair \((x, y)\). The point where the two lines intersect is the only solution.
- An **inconsistent system** has no solution. Notice that the two lines are parallel and will never intersect.
- A **dependent system** has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.

**Figure 9.3** compares graphical representations of each type of system.

![Graphical representations of each type of system](image)

**Figure 9.3**

**How To:** Given a system of linear equations and an ordered pair, determine whether the ordered pair is a solution.

1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

**Example 9.1**
Determining Whether an Ordered Pair Is a Solution to a System of Equations

Determine whether the ordered pair \((5, 1)\) is a solution to the given system of equations.

\[
\begin{align*}
    x + 3y &= 8 \\
    2x - 9 &= y
\end{align*}
\]

**Solution**

**Analysis**

We can see the solution clearly by plotting the graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines. See Figure 9.3.

---

9.1 Determine whether the ordered pair \((8, 5)\) is a solution to the following system.

\[
\begin{align*}
    5x - 4y &= 20 \\
    2x + 1 &= 3y
\end{align*}
\]

Solving Systems of Equations by Graphing

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

**Example 9.2**

**Solving a System of Equations in Two Variables by Graphing**

Solve the following system of equations by graphing. Identify the type of system.

\[
\begin{align*}
    2x + y &= -8 \\
    x - y &= -1
\end{align*}
\]

**Solution**
9.2 Solve the following system of equations by graphing.

\[ \begin{align*} 2x - 5y &= -25 \\ -4x + 5y &= 35 \end{align*} \] (9.4)

**Can graphing be used if the system is inconsistent or dependent?**

Yes, in both cases we can still graph the system to determine the type of system and solution. If the two lines are parallel, the system has no solution and is inconsistent. If the two lines are identical, the system has infinite solutions and is a dependent system.

**Solving Systems of Equations by Substitution**

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a system of linear equations that are more precise than graphing. One such method is solving a system of equations by the **substitution method**, in which we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable. Recall that we can solve for only one variable at a time, which is the reason the substitution method is both valuable and practical.

**Given a system of two equations in two variables, solve using the substitution method.**

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

**Example 9.3**

**Solving a System of Equations in Two Variables by Substitution**

Solve the following system of equations by substitution.

\[ \begin{align*} -x + y &= -5 \\ 2x - 5y &= 1 \end{align*} \]

**Solution**

9.3 Solve the following system of equations by substitution.

\[ \begin{align*} x &= y + 3 \\ 4 &= 3x - 2y \end{align*} \] (9.5)

**Can the substitution method be used to solve any linear system in two variables?**

Yes, but the method works best if one of the equations contains a coefficient of 1 or -1 so that we do not have to deal with fractions.

**Solving Systems of Equations in Two Variables by the Addition Method**

A third method of solving systems of linear equations is the **addition method**. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.
Given a system of equations, solve using the addition method.

1. Write both equations with $x$- and $y$-variables on the left side of the equal sign and constants on the right.

2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.

3. Solve the resulting equation for the remaining variable.

4. Substitute that value into one of the original equations and solve for the second variable.

5. Check the solution by substituting the values into the other equation.

Example 9.4

Solving a System by the Addition Method

Solve the given system of equations by addition.

\[
\begin{align*}
x + 2y &= -1 \\
-x + y &= 3
\end{align*}
\]

Solution

Analysis

We gain an important perspective on systems of equations by looking at the graphical representation. See Figure 9.3 to find that the equations intersect at the solution. We do not need to ask whether there may be a second solution because observing the graph confirms that the system has exactly one solution.

Example 9.5

Using the Addition Method When Multiplication of One Equation Is Required

Solve the given system of equations by the addition method.

\[
\begin{align*}
3x + 5y &= -11 \\
x - 2y &= 11
\end{align*}
\]

Solution
9.4 Solve the system of equations by addition.

\[
\begin{align*}
2x - 7y &= 2 \\
3x + y &= -20
\end{align*}
\]

(9.6)

Example 9.6

**Using the Addition Method When Multiplication of Both Equations Is Required**

Solve the given system of equations in two variables by addition.

\[
\begin{align*}
2x + 3y &= -16 \\
5x - 10y &= 30
\end{align*}
\]

**Solution**

\[
\begin{align*}
2x + 3y &= -16 \\
5x - 10y &= 30
\end{align*}
\]

Example 9.7

**Using the Addition Method in Systems of Equations Containing Fractions**

Solve the given system of equations in two variables by addition.

\[
\begin{align*}
\frac{x}{3} + \frac{y}{6} &= 3 \\
\frac{x}{2} - \frac{y}{4} &= 1
\end{align*}
\]

**Solution**

\[
\begin{align*}
\frac{x}{3} + \frac{y}{6} &= 3 \\
\frac{x}{2} - \frac{y}{4} &= 1
\end{align*}
\]

9.5 Solve the system of equations by addition.

\[
\begin{align*}
2x + 3y &= 8 \\
3x + 5y &= 10
\end{align*}
\]

(9.7)

Identifying Inconsistent Systems of Equations Containing Two Variables

Now that we have several methods for solving systems of equations, we can use the methods to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different \(y\)-intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as \(12 = 0\).

Example 9.8

**Solving an Inconsistent System of Equations**

Solve the following system of equations.

\[
\begin{align*}
x &= 9 - 2y \\
x + 2y &= 13
\end{align*}
\]
9.6 Solve the following system of equations in two variables.

\[
\begin{align*}
2y - 2x &= 2 \\
2y - 2x &= 6
\end{align*}
\]

Expressing the Solution of a System of Dependent Equations Containing Two Variables

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using substitution or addition, the resulting equation will be an identity, such as \(0 = 0\).

Example 9.9

Finding a Solution to a Dependent System of Linear Equations

Find a solution to the system of equations using the addition method.

\[
\begin{align*}
x + 3y &= 2 \\
3x + 9y &= 6
\end{align*}
\]

Solution

Analysis

If we rewrote both equations in the slope-intercept form, we might know what the solution would look like before adding. Let’s look at what happens when we convert the system to slope-intercept form.
Solve the following system of equations in two variables.

\[
\begin{align*}
y - 2x &= 5 \\
-3y + 6x &= -15
\end{align*}
\]

Using Systems of Equations to Investigate Profits

Using what we have learned about systems of equations, we can return to the skateboard manufacturing problem at the beginning of the section. The skateboard manufacturer’s revenue function is the function used to calculate the amount of money that comes into the business. It can be represented by the equation \( R = xp \), where \( x \) = quantity and \( p \) = price.

The revenue function is shown in orange in Figure 9.4.

The cost function is the function used to calculate the costs of doing business. It includes fixed costs, such as rent and salaries, and variable costs, such as utilities. The cost function is shown in blue in Figure 9.4. The \( x \)-axis represents quantity in hundreds of units. The \( y \)-axis represents either cost or revenue in hundreds of dollars.
The point at which the two lines intersect is called the **break-even point**. We can see from the graph that if 700 units are produced, the cost is $3,300 and the revenue is also $3,300. In other words, the company breaks even if they produce and sell 700 units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to the left represents quantities for which the company suffers a loss. The **profit function** is the revenue function minus the cost function, written as \( P(x) = R(x) - C(x) \). Clearly, knowing the quantity for which the cost equals the revenue is of great importance to businesses.

**Example 9.10**

**Finding the Break-Even Point and the Profit Function Using Substitution**

Given the cost function \( C(x) = 0.85x + 35,000 \) and the revenue function \( R(x) = 1.55x \), find the break-even point and the profit function.

**Solution**

**Analysis**

The cost to produce 50,000 units is $77,500, and the revenue from the sales of 50,000 units is also $77,500. To make a profit, the business must produce and sell more than 50,000 units. See Figure 9.4.
We see from the graph in Figure 9.4 that the profit function has a negative value until \( x = 50,000 \) when the graph crosses the x-axis. Then, the graph emerges into positive y-values and continues on this path as the profit function is a straight line. This illustrates that the break-even point for businesses occurs when the profit function is 0. The area to the left of the break-even point represents operating at a loss.

![Graph of profit function]

Example 9.11

Writing and Solving a System of Equations in Two Variables

The cost of a ticket to the circus is $25.00 for children and $50.00 for adults. On a certain day, attendance at the circus is 2,000 and the total gate revenue is $70,000. How many children and how many adults bought tickets?

Solution

The cost of a meal ticket at the circus is $4.00 for children and $12.00 for adults. If 1,650 meal tickets were bought for a total of $14,200, how many children and how many adults bought meal tickets?

Access these online resources for additional instruction and practice with systems of linear equations.

- Solving Systems of Equations Using Substitution (http://openstaxcollege.org/l/syssubst)
- Solving Systems of Equations Using Elimination (http://openstaxcollege.org/l/syselim)
- Applications of Systems of Equations (http://openstaxcollege.org/l/sysapp)
1. Can a system of linear equations have exactly two solutions? Explain why or why not.

2. If you are performing a break-even analysis for a business and their cost and revenue equations are dependent, explain what this means for the company’s profit margins.

3. If you are solving a break-even analysis and get a negative break-even point, explain what this signifies for the company?

4. If you are solving a break-even analysis and there is no break-even point, explain what this means for the company. How should they ensure there is a break-even point?

5. Given a system of equations, explain at least two different methods of solving that system.

**Algebraic**

For the following exercises, determine whether the given ordered pair is a solution to the system of equations.

6. \(5x - y = 4\) and \(x + 6y = 2\) and \((4, 0)\)

7. \(-3x - 5y = 13\) and \(-x + 4y = 10\) and \((-6, 1)\)

8. \(3x + 7y = 1\) and \(2x + 4y = 0\) and \((2, 3)\)

9. \(-2x + 5y = 7\) and \(2x + 9y = 7\) and \((-1, 1)\)

10. \(x + 8y = 43\) and \(3x - 2y = -1\) and \((3, 5)\)

For the following exercises, solve each system by substitution.

11. \(x + 3y = 5\)
    \(2x + 3y = 4\)

12. \(3x - 2y = 18\)
    \(5x + 10y = -10\)

13. \(4x + 2y = -10\)
    \(3x + 9y = 0\)

14. \(2x + 4y = -3.8\)
    \(9x - 5y = 1.3\)

15. \(-2x + 3y = 1.2\)
    \(-3x - 6y = 1.8\)

16. \(x - 0.2y = 1\)
    \(-10x + 2y = 5\)

17. \(3x + 5y = 9\)
    \(30x + 50y = -90\)

18.
\[-3x + y = 2 \]
\[12x - 4y = -8 \]

19. \[\frac{1}{2}x + \frac{1}{3}y = 16 \]
\[\frac{1}{6}x + \frac{1}{4}y = 9 \]

20. \[-\frac{1}{4}x + \frac{3}{2}y = 11 \]
\[-\frac{1}{8}x + \frac{1}{3}y = 3 \]

For the following exercises, solve each system by addition.

21. \[-2x + 5y = -42 \]
\[7x + 2y = 30 \]

22. \[6x - 5y = -34 \]
\[2x + 6y = 4 \]

23. \[5x - y = -2.6 \]
\[-4x - 6y = 1.4 \]

24. \[7x - 2y = 3 \]
\[4x + 5y = 3.25 \]

25. \[-x + 2y = -1 \]
\[5x - 10y = 6 \]

26. \[7x + 6y = 2 \]
\[-28x - 24y = -8 \]

27. \[\frac{5}{6}x + \frac{1}{4}y = 0 \]
\[\frac{1}{8}x - \frac{1}{2}y = -\frac{43}{120} \]

28. \[\frac{1}{3}x + \frac{1}{9}y = \frac{2}{9} \]
\[-\frac{1}{2}x + \frac{4}{5}y = -\frac{1}{3} \]

29. \[-0.2x + 0.4y = 0.6 \]
\[x - 2y = -3 \]

30. \[-0.1x + 0.2y = 0.6 \]
\[5x - 10y = 1 \]

For the following exercises, solve each system by any method.

31. \[5x + 9y = 16 \]
\[x + 2y = 4 \]

32. \[6x - 8y = -0.6 \]
\[3x + 2y = 0.9 \]
Graphical

For the following exercises, graph the system of equations and state whether the system is consistent, inconsistent, or dependent and whether the system has one solution, no solution, or infinite solutions.

41. \(3x - y = 0.6\)
   \(x - 2y = 1.3\)

42. \(-x + 2y = 4\)
   \(2x - 4y = 1\)

43. \(x + 2y = 7\)
   \(2x + 6y = 12\)

44. \(3x - 5y = 7\)
   \(x - 2y = 3\)

45. \(3x - 2y = 5\)
   \(-9x + 6y = -15\)

Technology

For the following exercises, use the intersect function on a graphing device to solve each system. Round all answers to the nearest hundredth.

46. \(0.1x + 0.2y = 0.3\)
   \(-0.3x + 0.5y = 1\)
47. \(-0.01x + 0.12y = 0.62\)
\(0.15x + 0.20y = 0.52\)

48. \(0.5x + 0.3y = 4\)
\(0.25x - 0.9y = 0.46\)

49. \(0.15x + 0.27y = 0.39\)
\(-0.34x + 0.56y = 1.8\)

50. \(-0.71x + 0.92y = 0.13\)
\(0.83x + 0.05y = 2.1\)

**Extensions**

For the following exercises, solve each system in terms of \(A, B, C, D, E,\) and \(F\) where \(A - F\) are nonzero numbers. Note that \(A \neq B\) and \(AE \neq BD\).

51. \(x + y = A\)
\(x - y = B\)

52. \(x + Ay = 1\)
\(x + By = 1\)

53. \(Ax + y = 0\)
\(Bx + y = 1\)

54. \(Ax + By = C\)
\(x + y = 1\)

55. \(Ax + By = C\)
\(Dx + Ey = F\)

**Real-World Applications**

For the following exercises, solve for the desired quantity.

56. A stuffed animal business has a total cost of production \(C = 12x + 30\) and a revenue function \(R = 20x\). Find the break-even point.

57. A fast-food restaurant has a cost of production \(C(x) = 11x + 120\) and a revenue function \(R(x) = 5x\). When does the company start to turn a profit?

58. A cell phone factory has a cost of production \(C(x) = 150x + 10,000\) and a revenue function \(R(x) = 200x\). What is the break-even point?

59. A musician charges \(C(x) = 64x + 20,000\), where \(x\) is the total number of attendees at the concert. The venue charges $80 per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?

60. A guitar factory has a cost of production \(C(x) = 75x + 50,000\). If the company needs to break even after 150 units sold, at what price should they sell each guitar? Round up to the nearest dollar, and write the revenue function.

For the following exercises, use a system of linear equations with two variables and two equations to solve.

61. Find two numbers whose sum is 28 and difference is 13.

62. A number is 9 more than another number. Twice the sum of the two numbers is 10. Find the two numbers.
The startup cost for a restaurant is $120,000, and each meal costs $10 for the restaurant to make. If each meal is then sold for $15, after how many meals does the restaurant break even?

64. A moving company charges a flat rate of $150, and an additional $5 for each box. If a taxi service would charge $20 for each box, how many boxes would you need for it to be cheaper to use the moving company, and what would be the total cost?

65. A total of 1,595 first- and second-year college students gathered at a pep rally. The number of freshmen exceeded the number of sophomores by 15. How many freshmen and sophomores were in attendance?

66. 276 students enrolled in a freshman-level chemistry class. By the end of the semester, 5 times the number of students passed as failed. Find the number of students who passed, and the number of students who failed.

67. There were 130 faculty at a conference. If there were 18 more women than men attending, how many of each gender attended the conference?

68. A jeep and BMW enter a highway running east-west at the same exit heading in opposite directions. The jeep entered the highway 30 minutes before the BMW did, and traveled 7 mph slower than the BMW. After 2 hours from the time the BMW entered the highway, the cars were 306.5 miles apart. Find the speed of each car, assuming they were driven on cruise control.

69. If a scientist mixed 10% saline solution with 60% saline solution to get 25 gallons of 40% saline solution, how many gallons of 10% and 60% solutions were mixed?

70. An investor earned triple the profits of what she earned last year. If she made $500,000.48 total for both years, how much did she earn in profits each year?

71. An investor who dabbles in real estate invested 1.1 million dollars into two land investments. On the first investment, Swan Peak, her return was a 110% increase on the money she invested. On the second investment, Riverside Community, she earned 50% over what she invested. If she earned $1 million in profits, how much did she invest in each of the land deals?

72. If an investor invests a total of $25,000 into two bonds, one that pays 3% simple interest, and the other that pays $2\% interest, and the investor earns $737.50 annual interest, how much was invested in each account?

73. If an investor invests $23,000 into two bonds, one that pays 4% in simple interest, and the other paying 2% simple interest, and the investor earns $710.00 annual interest, how much was invested in each account?

74. CDs cost $5.96 more than DVDs at All Bets Are Off Electronics. How much would 6 CDs and 2 DVDs cost if 5 CDs and 2 DVDs cost $127.73?

75. A store clerk sold 60 pairs of sneakers. The high-tops sold for $98.99 and the low-tops sold for $129.99. If the receipts for the two types of sales totaled $6,404.40, how many of each type of sneaker were sold?

76. A concert manager counted 350 ticket receipts the day after a concert. The price for a student ticket was $12.50, and the price for an adult ticket was $16.00. The register confirms that $5,075 was taken in. How many student tickets and adult tickets were sold?

77. Admission into an amusement park for 4 children and 2 adults is $116.90. For 6 children and 3 adults, the admission is $175.35. Assuming a different price for children and adults, what is the price of the child’s ticket and the price of the adult ticket?
9.2 | Systems of Linear Equations: Three Variables

**Learning Objectives**

In this section, you will:

9.2.1 Solve systems of three equations in three variables.
9.2.2 Identify inconsistent systems of equations containing three variables.
9.2.3 Express the solution of a system of dependent equations containing three variables.

John received an inheritance of $12,000 that he divided into three parts and invested in three ways: in a money-market fund paying 3% annual interest; in municipal bonds paying 4% annual interest; and in mutual funds paying 7% annual interest. John invested $4,000 more in municipal funds than in municipal bonds. He earned $670 in interest the first year. How much did John invest in each type of fund?

Understanding the correct approach to setting up problems such as this one makes finding a solution a matter of following a pattern. We will solve this and similar problems involving three equations and three variables in this section. Doing so uses similar techniques as those used to solve systems of two equations in two variables. However, finding solutions to systems of three equations requires a bit more organization and a touch of visual gymnastics.

**Solving Systems of Three Equations in Three Variables**

In order to solve systems of equations in three variables, known as three-by-three systems, the primary tool we will be using is called Gaussian elimination, named after the prolific German mathematician Karl Friedrich Gauss. While there is no definitive order in which operations are to be performed, there are specific guidelines as to what type of moves can be made. We may number the equations to keep track of the steps we apply. The goal is to eliminate one variable at a time to achieve upper triangular form, the ideal form for a three-by-three system because it allows for straightforward back-substitution to find a solution \((x, y, z)\), which we call an ordered triple. A system in upper triangular form looks like the following:

\[
\begin{align*}
Ax + By + Cz &= D \\
Ey + Fz &= G \\
Hz &= K
\end{align*}
\]  

(9.10)

The third equation can be solved for \(z\), and then we back-substitute to find \(y\) and \(x\). To write the system in upper triangular form, we can perform the following operations:

1. Interchange the order of any two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a nonzero multiple of one equation to another equation.

The solution set to a three-by-three system is an ordered triple \((x, y, z)\). Graphically, the ordered triple defines the point that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a
rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

**Number of Possible Solutions**

*Figure 9.6* and *Figure 9.7* illustrate possible solution scenarios for three-by-three systems.

- Systems that have a single solution are those which, after elimination, result in a **solution set** consisting of an ordered triple \((x, y, z)\). Graphically, the ordered triple defines a point that is the intersection of three planes in space.
- Systems that have an infinite number of solutions are those which, after elimination, result in an expression that is always true, such as \(0 = 0\). Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space.
- Systems that have no solution are those that, after elimination, result in a statement that is a contradiction, such as \(3 = 0\). Graphically, a system with no solution is represented by three planes with no point in common.

---

**Example 9.12**

**Determining Whether an Ordered Triple Is a Solution to a System**

Determine whether the ordered triple \((3, -2, 1)\) is a solution to the system.

\[
\begin{align*}
    x + y + z &= 2 \\
    6x - 4y + 5z &= 31 \\
    5x + 2y + 2z &= 13
\end{align*}
\]

**Solution**


Given a linear system of three equations, solve for three unknowns.

1. Pick any pair of equations and solve for one variable.
2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

**Example 9.13**

**Solving a System of Three Equations in Three Variables by Elimination**

Find a solution to the following system:

\[
\begin{align*}
2x + y - 2z &= 9 \\
-x + 3y - z &= -6 \\
2x - 5y + 5z &= 17
\end{align*}
\]

**Solution**

**Example 9.14**

**Solving a Real-World Problem Using a System of Three Equations in Three Variables**

In the problem posed at the beginning of the section, John invested his inheritance of $12,000 in three different funds: part in a money-market fund paying 3% interest annually; part in municipal bonds paying 4% annually; and the rest in mutual funds paying 7% annually. John invested $4,000 more in mutual funds than he invested in municipal bonds. The total interest earned in one year was $670. How much did he invest in each type of fund?

**Solution**

**Try 9.9**

Solve the system of equations in three variables.

\[
\begin{align*}
2x + y - 2z &= -1 \\
3x - 3y - z &= 5 \\
x - 2y + 3z &= 6
\end{align*}
\]

**Identifying Inconsistent Systems of Equations Containing Three Variables**

Just as with systems of equations in two variables, we may come across an inconsistent system of equations in three variables, which means that it does not have a solution that satisfies all three equations. The equations could represent three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location. The process of elimination will result in a false statement, such as \(3 = 7\) or some other contradiction.

**Example 9.15**

**Solving an Inconsistent System of Three Equations in Three Variables**
Solve the following system.

\[
\begin{align*}
  x - 3y + z &= 4 \hspace{1cm} (1) \\
  -x + 2y - 5z &= 3 \hspace{1cm} (2) \\
  5x - 13y + 13z &= 8 \hspace{1cm} (3)
\end{align*}
\]

**Solution**

**Analysis**

In this system, each plane intersects the other two, but not at the same location. Therefore, the system is inconsistent.

---

9.10 Solve the system of three equations in three variables.

\[
\begin{align*}
  x + y + z &= 2 \\
  y - 3z &= 1 \\
  2x + y + 5z &= 0
\end{align*}
\]

**Expressing the Solution of a System of Dependent Equations Containing Three Variables**

We know from working with systems of equations in two variables that a dependent system of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be the same, so that a solution to one equation will be the solution to the other two equations. All three equations could be different but they intersect on a line, which has infinite solutions. Or two of the equations could be the same and intersect the third on a line.

**Example 9.16**

**Finding the Solution to a Dependent System of Equations**

Find the solution to the given system of three equations in three variables.

\[
\begin{align*}
  2x + y - 3z &= 0 \hspace{1cm} (1) \\
  4x + 2y - 6z &= 0 \hspace{1cm} (2) \\
  x - y + z &= 0 \hspace{1cm} (3)
\end{align*}
\]

**Solution**

**Analysis**

As shown in Figure 9.7, two of the planes are the same and they intersect the third plane on a line. The solution set is infinite, as all points along the intersection line will satisfy all three equations.
Does the generic solution to a dependent system always have to be written in terms of $x$?

No, you can write the generic solution in terms of any of the variables, but it is common to write it in terms of $x$ and if needed $x$ and $y$.

9.11 Solve the following system.

$$
\begin{align*}
    x + y + z &= 7 \\
    3x - 2y - z &= 4 \\
    x + 6y + 5z &= 24
\end{align*}
$$

Access these online resources for additional instruction and practice with systems of equations in three variables.

- Ex 1: System of Three Equations with Three Unknowns Using Elimination (http://openstaxcollege.org/l/systhree)
- Ex. 2: System of Three Equations with Three Unknowns Using Elimination (http://openstaxcollege.org/l/systhelim)
9.2 EXERCISES

Verbal

78. Can a linear system of three equations have exactly two solutions? Explain why or why not.

79. If a given ordered triple solves the system of equations, is that solution unique? If so, explain why. If not, give an example where it is not unique.

80. If a given ordered triple does not solve the system of equations, is there no solution? If so, explain why. If not, give an example.

81. Using the method of addition, is there only one way to solve the system?

82. Can you explain whether there can be only one method to solve a linear system of equations? If yes, give an example of such a system of equations. If not, explain why not.

Algebraic

For the following exercises, determine whether the ordered triple given is the solution to the system of equations.

83. \[2x-6y+6z=-12\]
   \[x+4y+5z=-1\quad\text{and}\quad(0,1,-1)\]
   \[-x+2y+3z=-1\]

84. \[6x-y+3z=6\]
   \[3x+5y+2z=0\quad\text{and}\quad(3,-3,-5)\]
   \[x+y=0\]

85. \[6x-7y+z=2\]
   \[-x-y+3z=4\quad\text{and}\quad(4,2,-6)\]
   \[2x+y-z=1\]

86. \[x-y=0\]
   \[x-z=5\quad\text{and}\quad(4,4,-1)\]
   \[x-y+z=-1\]

87. \[-x-y+2z=3\]
   \[5x+8y-3z=4\quad\text{and}\quad(4,1,-7)\]
   \[-x+3y-5z=-5\]

For the following exercises, solve each system by substitution.

88. \[3x-4y+2z=-15\]
   \[2x+4y+z=16\]
   \[2x+3y+5z=20\]

89. \[5x-2y+3z=20\]
   \[2x-4y-3z=-9\]
   \[x+6y-8z=21\]

90. \[5x+2y+4z=9\]
   \[-3x+2y+z=10\]
   \[4x-3y+5z=-3\]

91. \[4x-3y+5z=31\]
   \[-x+2y+4z=20\]
   \[x+5y-2z=-29\]
92. \[ 5x - 2y + 3z = 4 \]
    \[ -4x + 6y - 7z = -1 \]
    \[ 3x + 2y - z = 4 \]

93. \[ 4x + 6y + 9z = 0 \]
    \[ -5x + 2y - 6z = 3 \]
    \[ 7x - 4y + 3z = -3 \]

For the following exercises, solve each system by Gaussian elimination.

94. \[ 2x - y + 3z = 17 \]
    \[ -5x + 4y - 2z = -46 \]
    \[ 2y + 5z = -7 \]

95. \[ 5x - 6y + 3z = 50 \]
    \[ -x + 4y = 10 \]
    \[ 2x - z = 10 \]

96. \[ 2x + 3y - 6z = 1 \]
    \[ -4x - 6y + 12z = -2 \]
    \[ x + 2y + 5z = 10 \]

97. \[ 4x + 6y - 2z = 8 \]
    \[ 6x + 9y - 3z = 12 \]
    \[ -2x - 3y + z = -4 \]

98. \[ 2x + 3y - 4z = 5 \]
    \[ -3x + 2y + z = 11 \]
    \[ -x + 5y + 3z = 4 \]

99. \[ 10x + 2y - 14z = 8 \]
    \[ -x - 2y - 4z = -1 \]
    \[ -12x - 6y + 6z = -12 \]

100. \[ x + y + z = 14 \]
    \[ 2y + 3z = -14 \]
    \[ -16y - 24z = -112 \]

101. \[ 5x - 3y + 4z = -1 \]
    \[ -4x + 2y - 3z = 0 \]
    \[ -x + 5y + 7z = -11 \]

102. \[ x + y + z = 0 \]
    \[ 2x - y + 3z = 0 \]
    \[ x - z = 0 \]

103. \[ 3x + 2y - 5z = 6 \]
    \[ 5x - 4y + 3z = -12 \]
    \[ 4x + 5y - 2z = 15 \]

104. \[ x + y + z = 0 \]
    \[ 2x - y + 3z = 0 \]
    \[ x - z = 1 \]

105.
\[3x - \frac{1}{2}y - z = -\frac{1}{2}\]
\[4x + z = 3\]
\[-x + \frac{3}{2}y = \frac{5}{2}\]

106. \[6x - 5y + 6z = 38\]
\[\frac{1}{3}x - \frac{1}{2}y + \frac{3}{5}z = 1\]
\[-4x - \frac{3}{2}y - z = -74\]

107. \[\frac{1}{2}x - \frac{1}{5}y + \frac{2}{5}z = -\frac{13}{10}\]
\[\frac{1}{4}x - \frac{2}{5}y - \frac{1}{5}z = -\frac{7}{20}\]
\[-\frac{1}{2}x - \frac{3}{4}y - \frac{1}{2}z = -\frac{5}{4}\]

108. \[-\frac{1}{3}x - \frac{1}{2}y - \frac{1}{4}z = \frac{3}{4}\]
\[-\frac{1}{2}x - \frac{1}{4}y - \frac{1}{2}z = 2\]
\[-\frac{1}{4}x - \frac{3}{4}y - \frac{1}{2}z = -\frac{1}{2}\]

109. \[\frac{1}{2}x - \frac{1}{4}y + \frac{3}{4}z = 0\]
\[\frac{1}{4}x - \frac{1}{10}y + \frac{2}{5}z = -2\]
\[\frac{1}{8}x + \frac{1}{5}y - \frac{1}{8}z = 2\]

110. \[\frac{4}{5}x - \frac{7}{8}y + \frac{1}{2}z = 1\]
\[-\frac{4}{5}x - \frac{3}{4}y + \frac{1}{3}z = -8\]
\[-\frac{2}{5}x - \frac{7}{8}y + \frac{1}{2}z = -5\]

111. \[-\frac{1}{3}x - \frac{1}{8}y + \frac{1}{6}z = -\frac{4}{3}\]
\[-\frac{2}{3}x - \frac{7}{8}y + \frac{1}{3}z = -\frac{23}{3}\]
\[-\frac{1}{3}x - \frac{5}{8}y + \frac{5}{6}z = 0\]

112. \[-\frac{1}{4}x - \frac{5}{4}y + \frac{5}{2}z = -5\]
\[-\frac{1}{2}x - \frac{5}{3}y + \frac{5}{4}z = \frac{55}{12}\]
\[-\frac{1}{3}x - \frac{1}{3}y + \frac{1}{3}z = \frac{5}{3}\]

113. \[\frac{1}{40}x + \frac{1}{60}y + \frac{1}{80}z = \frac{1}{100}\]
\[-\frac{1}{2}x - \frac{3}{4}y - \frac{1}{4}z = -\frac{1}{5}\]
\[\frac{3}{8}x + \frac{3}{12}y + \frac{3}{16}z = \frac{3}{20}\]

114.
0.1x−0.2y + 0.3z = 2
0.5x−0.1y + 0.4z = 8
0.7x−0.2y + 0.3z = 8

115. 0.2x + 0.1y−0.3z = 0.2
     0.8x + 0.4y−1.2z = 0.1
     1.6x + 0.8y−2.4z = 0.2

116. 1.1x + 0.7y−3.1z = −1.79
     2.1x + 0.5y−1.6z = −0.13
     0.5x + 0.4y−0.5z = −0.07

117. 0.5x−0.5y + 0.5z = 10
     0.2x−0.2y + 0.2z = 4
     0.1x−0.1y + 0.1z = 2

118. 0.1x + 0.2y + 0.3z = 0.37
     0.1x−0.2y−0.3z = −0.27
     0.5x−0.1y−0.3z = −0.03

119. 0.5x−0.5y−0.3z = 0.13
     0.4x−0.1y−0.3z = 0.11
     0.2x−0.8y−0.9z = −0.32

120. 0.5x + 0.2y−0.3z = 1
     0.4x−0.6y + 0.7z = 0.8
     0.3x−0.1y−0.9z = 0.6

121. 0.3x + 0.3y + 0.5z = 0.6
     0.4x + 0.4y + 0.4z = 1.8
     0.4x + 0.2y + 0.1z = 1.6

122. 0.8x + 0.8y + 0.8z = 2.4
     0.3x−0.5y + 0.2z = 0
     0.1x + 0.2y + 0.3z = 0.6

Extensions

For the following exercises, solve the system for x, y, and z.

123. \[ \begin{align*}
    x + y + z &= 3 \\
    \frac{x-1}{2} + \frac{y-3}{2} + \frac{z+1}{2} &= 0 \\
    \frac{x-2}{3} + \frac{y+4}{3} + \frac{z-3}{3} &= 2
\end{align*} \]

124. \[ \begin{align*}
    5x−3y − \frac{z+1}{2} &= \frac{1}{2} \\
    6x + \frac{y-9}{2} + 2z &= −3 \\
    \frac{x+8}{2} −4y + z &= 4
\end{align*} \]

125.
Three even numbers sum up to 108. The smaller is half the larger and the middle number is \(\frac{3}{4}\) the larger. What are the three numbers?

Three numbers sum up to 147. The smallest number is half the middle number, which is half the largest number. What are the three numbers?

At a family reunion, there were only blood relatives, consisting of children, parents, and grandparents, in attendance. There were 400 people total. There were twice as many parents as grandparents, and 50 more children than parents. How many children, parents, and grandparents were in attendance?

An animal shelter has a total of 350 animals comprised of cats, dogs, and rabbits. If the number of rabbits is 5 less than one-half the number of cats, and there are 20 more cats than dogs, how many of each animal are at the shelter?

Your roommate, Sarah, offered to buy groceries for you and your other roommate. The total bill was $82. She forgot to save the individual receipts but remembered that your groceries were $0.05 cheaper than half of her groceries, and that your other roommate’s groceries were $2.10 more than your groceries. How much was each of your share of the groceries?

Your roommate, John, offered to buy household supplies for you and your other roommate. You live near the border of three states, each of which has a different sales tax. The total amount of money spent was $100.75. Your supplies were bought with 5% tax, John’s with 8% tax, and your third roommate’s with 9% sales tax. The total amount of money spent without taxes is $93.50. If your supplies before tax were $1 more than half of what your third roommate’s supplies were before tax, how much did each of you spend? Give your answer both with and without taxes.

Three coworkers work for the same employer. Their jobs are warehouse manager, office manager, and truck driver. The sum of the annual salaries of the warehouse manager and office manager is $82,000. The office manager makes $4,000 more than the truck driver annually. The annual salaries of the warehouse manager and the truck driver total $78,000. What is the annual salary of each of the co-workers?

At a carnival, $2,914.25 in receipts were taken at the end of the day. The cost of a child’s ticket was $20.50, an adult ticket was $29.75, and a senior citizen ticket was $15.25. There were twice as many senior citizens as adults in attendance, and 20 more children than senior citizens. How many children, adults, and senior citizen tickets were sold?

A local band sells out for their concert. They sell all 1,175 tickets for a total purse of $28,112.50. The tickets were priced at $20 for student tickets, $22.50 for children, and $29 for adult tickets. If the band sold twice as many adult as children tickets, how many of each type was sold?

In a bag, a child has 325 coins worth $19.50. There were three types of coins: pennies, nickels, and dimes. If the bag contained the same number of nickels as dimes, how many of each type of coin was in the bag?

Last year, at Haven’s Pond Car Dealership, for a particular model of BMW, Jeep, and Toyota, one could purchase all three cars for a total of $140,000. This year, due to inflation, the same cars would cost $151,830. The cost of the BMW
increased by 8%, the Jeep by 5%, and the Toyota by 12%. If the price of last year’s Jeep was $7,000 less than the price of last year’s BMW, what was the price of each of the three cars last year?

139. A recent college graduate took advantage of his business education and invested in three investments immediately after graduating. He invested $80,500 into three accounts, one that paid 4% simple interest, one that paid $1\frac{1}{8}$ simple interest, and one that paid $2\frac{1}{2}$% simple interest. He earned $2,670 interest at the end of one year. If the amount of the money invested in the second account was four times the amount invested in the third account, how much was invested in each account?

140. You inherit one million dollars. You invest it all in three accounts for one year. The first account pays 3% compounded annually, the second account pays 4% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $34,000 in interest. If you invest four times the money into the account that pays 3% compared to 2%, how much did you invest in each account?

141. You inherit one hundred thousand dollars. You invest it all in three accounts for one year. The first account pays 4% compounded annually, the second account pays 3% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $3,650 in interest. If you invest five times the money into the account that pays 4% compared to 3%, how much did you invest in each account?

142. The top three countries in oil consumption in a certain year are as follows: the United States, Japan, and China. In millions of barrels per day, the three top countries consumed 39.8% of the world’s consumed oil. The United States consumed 0.7% more than four times China’s consumption. The United States consumed 5% more than triple Japan’s consumption. What percent of the world oil consumption did the United States, Japan, and China consume?[1]

143. The top three countries in oil production in the same year are Saudi Arabia, the United States, and Russia. In millions of barrels per day, the top three countries produced 31.4% of the world’s produced oil. Saudi Arabia and the United States combined for 22.1% of the world’s production, and Saudi Arabia produced 2% more oil than Russia. What percent of the world oil production did Saudi Arabia, the United States, and Russia produce?[2]

144. The top three sources of oil imports for the United States in the same year were Saudi Arabia, Mexico, and Canada. The three top countries accounted for 47% of oil imports. The United States imported 1.8% more from Saudi Arabia than they did from Mexico, and 1.7% more from Saudi Arabia than they did from Canada. What percent of the United States oil imports were from these three countries?[3]

145. The top three oil producers in the United States in a certain year are the Gulf of Mexico, Texas, and Alaska. The three regions were responsible for 64% of the United States oil production. The Gulf of Mexico and Texas combined for 47% of oil production. Texas produced 3% more than Alaska. What percent of United States oil production came from these regions?[4]

146. At one time, in the United States, 398 species of animals were on the endangered species list. The top groups were mammals, birds, and fish, which comprised 55% of the endangered species. Birds accounted for 0.7% more than fish, and fish accounted for 1.5% more than mammals. What percent of the endangered species came from mammals, birds, and fish?

147. Meat consumption in the United States can be broken into three categories: red meat, poultry, and fish. If fish makes up 4% less than one-quarter of poultry consumption, and red meat consumption is 18.2% higher than poultry consumption, what are the percentages of meat consumption?[5]

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9.3 | Systems of Nonlinear Equations and Inequalities: Two Variables

Learning Objectives

In this section, you will:

9.3.1 Solve a system of nonlinear equations using substitution.
9.3.2 Solve a system of nonlinear equations using elimination.
9.3.3 Graph a nonlinear inequality.
9.3.4 Graph a system of nonlinear inequalities.

Halley’s Comet (Figure 9.8) orbits the sun about once every 75 years. Its path can be considered to be a very elongated ellipse. Other comets follow similar paths in space. These orbital paths can be studied using systems of equations. These systems, however, are different from the ones we considered in the previous section because the equations are not linear.

In this section, we will consider the intersection of a parabola and a line, a circle and a line, and a circle and an ellipse. The methods for solving systems of nonlinear equations are similar to those for linear equations.

Solving a System of Nonlinear Equations Using Substitution

A system of nonlinear equations is a system of two or more equations in two or more variables containing at least one equation that is not linear. Recall that a linear equation can take the form $Ax + By + C = 0$. Any equation that cannot be written in this form is nonlinear. The substitution method we used for linear systems is the same method we will use for nonlinear systems. We solve one equation for one variable and then substitute the result into the second equation to solve for another variable, and so on. There is, however, a variation in the possible outcomes.

Intersection of a Parabola and a Line

There are three possible types of solutions for a system of nonlinear equations involving a parabola and a line.

Possible Types of Solutions for Points of Intersection of a Parabola and a Line

Figure 9.9 illustrates possible solution sets for a system of equations involving a parabola and a line.

- No solution. The line will never intersect the parabola.
- One solution. The line is tangent to the parabola and intersects the parabola at exactly one point.
- Two solutions. The line crosses on the inside of the parabola and intersects the parabola at two points.
Given a system of equations containing a line and a parabola, find the solution.

1. Solve the linear equation for one of the variables.
2. Substitute the expression obtained in step one into the parabola equation.
3. Solve for the remaining variable.
4. Check your solutions in both equations.

**Example 9.17**

**Solving a System of Nonlinear Equations Representing a Parabola and a Line**

Solve the system of equations.

\[
\begin{aligned}
x - y &= -1 \\
y &= x^2 + 1
\end{aligned}
\]
Could we have substituted values for $y$ into the second equation to solve for $x$ in Example 9.17?

Yes, but because $x$ is squared in the second equation this could give us extraneous solutions for $x$.

For $y = 1$

\[
\begin{align*}
y &= x^2 + 1 \\
y &= x^2 + 1 \\
x^2 &= 0 \\
x &= \pm \sqrt{0} = 0
\end{align*}
\]  

This gives us the same value as in the solution.

For $y = 2$

\[
\begin{align*}
y &= x^2 + 1 \\
2 &= x^2 + 1 \\
x^2 &= 1 \\
x &= \pm \sqrt{1} = \pm 1
\end{align*}
\]  

Notice that $-1$ is an extraneous solution.

**Try it**

Solve the given system of equations by substitution.

\[
\begin{align*}
3x - y &= -2 \\
2x^2 - y &= 0
\end{align*}
\]  

Intersection of a Circle and a Line

Just as with a parabola and a line, there are three possible outcomes when solving a system of equations representing a circle and a line.

**Possible Types of Solutions for the Points of Intersection of a Circle and a Line**

Figure 9.10 illustrates possible solution sets for a system of equations involving a circle and a line.

- No solution. The line does not intersect the circle.
- One solution. The line is tangent to the circle and intersects the circle at exactly one point.
- Two solutions. The line crosses the circle and intersects it at two points.

Figure 9.10
Given a system of equations containing a line and a circle, find the solution.

1. Solve the linear equation for one of the variables.
2. Substitute the expression obtained in step one into the equation for the circle.
3. Solve for the remaining variable.
4. Check your solutions in both equations.

Example 9.18

Finding the Intersection of a Circle and a Line by Substitution

Find the intersection of the given circle and the given line by substitution.

\[ x^2 + y^2 = 5 \]
\[ y = 3x - 5 \]

Solution

Solving a System of Nonlinear Equations Using Elimination

We have seen that substitution is often the preferred method when a system of equations includes a linear equation and a nonlinear equation. However, when both equations in the system have like variables of the second degree, solving them using elimination by addition is often easier than substitution. Generally, elimination is a far simpler method when the system involves only two equations in two variables (a two-by-two system), rather than a three-by-three system, as there are fewer steps. As an example, we will investigate the possible types of solutions when solving a system of equations representing a circle and an ellipse.

Possible Types of Solutions for the Points of Intersection of a Circle and an Ellipse

**Figure 9.11** illustrates possible solution sets for a system of equations involving a circle and an ellipse.

- No solution. The circle and ellipse do not intersect. One shape is inside the other or the circle and the ellipse are a distance away from the other.
- One solution. The circle and ellipse are tangent to each other, and intersect at exactly one point.
- Two solutions. The circle and the ellipse intersect at two points.
- Three solutions. The circle and the ellipse intersect at three points.
- Four solutions. The circle and the ellipse intersect at four points.
Example 9.19

Solving a System of Nonlinear Equations Representing a Circle and an Ellipse

Solve the system of nonlinear equations.

\[ x^2 + y^2 = 26 \]  \hspace{1cm} (1)
\[ 3x^2 + 25y^2 = 100 \]  \hspace{1cm} (2)

Solution

Find the solution set for the given system of nonlinear equations.

\[ 4x^2 + y^2 = 13 \]  \hspace{1cm} (9.18)
\[ x^2 + y^2 = 10 \]

Graphing a Nonlinear Inequality

All of the equations in the systems that we have encountered so far have involved equalities, but we may also encounter systems that involve inequalities. We have already learned to graph linear inequalities by graphing the corresponding equation, and then shading the region represented by the inequality symbol. Now, we will follow similar steps to graph a nonlinear inequality so that we can learn to solve systems of nonlinear inequalities. A nonlinear inequality is an inequality containing a nonlinear expression. Graphing a nonlinear inequality is much like graphing a linear inequality.

Recall that when the inequality is greater than, \( y > a \), or less than, \( y < a \), the graph is drawn with a dashed line. When the inequality is greater than or equal to, \( y \geq a \), or less than or equal to, \( y \leq a \), the graph is drawn with a solid line. The graphs will create regions in the plane, and we will test each region for a solution. If one point in the region works, the whole region works. That is the region we shade. See Figure 9.12.
Given an inequality bounded by a parabola, sketch a graph.

1. Graph the parabola as if it were an equation. This is the boundary for the region that is the solution set.
2. If the boundary is included in the region (the operator is \( \leq \) or \( \geq \)), the parabola is graphed as a solid line.
3. If the boundary is not included in the region (the operator is \(<\) or \(>\)), the parabola is graphed as a dashed line.
4. Test a point in one of the regions to determine whether it satisfies the inequality statement. If the statement is true, the solution set is the region including the point. If the statement is false, the solution set is the region on the other side of the boundary line.
5. Shade the region representing the solution set.

Example 9.20

Graphing an Inequality for a Parabola

Graph the inequality \( y > x^2 + 1 \).

Solution

Graphing a System of Nonlinear Inequalities

Now that we have learned to graph nonlinear inequalities, we can learn how to graph systems of nonlinear inequalities. A system of nonlinear inequalities is a system of two or more inequalities in two or more variables containing at least one inequality that is not linear. Graphing a system of nonlinear inequalities is similar to graphing a system of linear inequalities. The difference is that our graph may result in more shaded regions that represent a solution than we find in a system of linear inequalities. The solution to a nonlinear system of inequalities is the region of the graph where the shaded regions of the graph of each inequality overlap, or where the regions intersect, called the feasible region.
Given a system of nonlinear inequalities, sketch a graph.

1. Find the intersection points by solving the corresponding system of nonlinear equations.
2. Graph the nonlinear equations.
3. Find the shaded regions of each inequality.
4. Identify the feasible region as the intersection of the shaded regions of each inequality or the set of points common to each inequality.

Example 9.21

Graphing a System of Inequalities

Graph the given system of inequalities.

\[
\begin{align*}
x^2 - y & \leq 0 \\
2x^2 + y & \leq 12
\end{align*}
\]

Solution

Try It 9.15

Graph the given system of inequalities.

\[
\begin{align*}
y & \geq x^2 - 1 \\
x - y & \geq -1
\end{align*}
\]
9.3 Exercises

Verbal

148. Explain whether a system of two nonlinear equations can have exactly two solutions. What about exactly three? If not, explain why not. If so, give an example of such a system, in graph form, and explain why your choice gives two or three answers.

149. When graphing an inequality, explain why we only need to test one point to determine whether an entire region is the solution?

150. When you graph a system of inequalities, will there always be a feasible region? If so, explain why. If not, give an example of a graph of inequalities that does not have a feasible region. Why does it not have a feasible region?

151. If you graph a revenue and cost function, explain how to determine in what regions there is profit.

152. If you perform your break-even analysis and there is more than one solution, explain how you would determine which x-values are profit and which are not.

Algebraic

For the following exercises, solve the system of nonlinear equations using substitution.

153. \( x + y = 4 \)
   \( x^2 + y^2 = 9 \)

154. \( y = x - 3 \)
   \( x^2 + y^2 = 9 \)

155. \( y = x \)
   \( x^2 + y^2 = 9 \)

156. \( y = -x \)
   \( x^2 + y^2 = 9 \)

157. \( x = 2 \)
   \( x^2 - y^2 = 9 \)

For the following exercises, solve the system of nonlinear equations using elimination.

158. \( 4x^2 - 9y^2 = 36 \)
   \( 4x^2 + 9y^2 = 36 \)

159. \( x^2 + y^2 = 25 \)
   \( x^2 - y^2 = 1 \)

160. \( 2x^2 + 4y^2 = 4 \)
   \( 2x^2 - 4y^2 = 25x - 10 \)

161. \( y^2 - x^2 = 9 \)
   \( 3x^2 + 2y^2 = 8 \)

162. \( x^2 + y^2 + \frac{1}{16} = 2500 \)
   \( y = 2x^2 \)

For the following exercises, use any method to solve the system of nonlinear equations.
163. \(-2x^2 + y = -5\)
    \[6x - y = 9\]

164. \(-x^2 + y = 2\)
    \[-x + y = 2\]

165. \(x^2 + y^2 = 1\)
    \[y = 20x^2 - 1\]

166. \(x^2 + y^2 = 1\)
    \[y = -x^2\]

167. \(2x^3 - x^2 = y\)
    \[y = \frac{1}{2} - x\]

168. \(9x^2 + 25y^2 = 225\)
    \[(x-6)^2 + y^2 = 1\]

169. \(x^4 - x^2 = y\)
    \[x^2 + y = 0\]

170. \(2x^3 - x^2 = y\)
    \[x^2 + y = 0\]

For the following exercises, use any method to solve the nonlinear system.

171. \(x^2 + y^2 = 9\)
    \[y = 3 - x^2\]

172. \(x^2 - y^2 = 9\)
    \[x = 3\]

173. \(x^2 - y^2 = 9\)
    \[y = 3\]

174. \(x^2 - y^2 = 9\)
    \[x - y = 0\]

175. \(-x^2 + y = 2\)
    \[-4x + y = -1\]

176. \(-x^2 + y = 2\)
    \[2y = -x\]

177. \(x^2 + y^2 = 25\)
    \[x^2 - y^2 = 36\]

178. \(x^2 + y^2 = 1\)
    \[y^2 = x^2\]
179. \[ 16x^2 - 9y^2 + 144 = 0 \]
    \[ y^2 + x^2 = 16 \]

180. \[ 3x^2 - y^2 = 12 \]
    \[ (x-1)^2 + y^2 = 1 \]

181. \[ 3x^2 - y^2 = 12 \]
    \[ (x-1)^2 + y^2 = 4 \]

182. \[ 3x^2 - y^2 = 12 \]
    \[ x^2 + y^2 = 16 \]

183. \[ x^2 - y^2 - 6x - 4y - 11 = 0 \]
    \[ -x^2 + y^2 = 5 \]

184. \[ x^2 + y^2 - 6y = 7 \]
    \[ x^2 + y = 1 \]

185. \[ x^2 + y^2 = 6 \]
    \[ xy = 1 \]

**Graphical**

For the following exercises, graph the inequality.

186. \[ x^2 + y < 9 \]

187. \[ x^2 + y^2 < 4 \]

For the following exercises, graph the system of inequalities. Label all points of intersection.

188. \[ x^2 + y < 1 \]
    \[ y > 2x \]

189. \[ x^2 + y < -5 \]
    \[ y > 5x + 10 \]

190. \[ x^2 + y^2 < 25 \]
    \[ 3x^2 - y^2 > 12 \]

191. \[ x^2 - y^2 > -4 \]
    \[ x^2 + y^2 < 12 \]

192. \[ x^2 + 3y^2 > 16 \]
    \[ 3x^2 - y^2 < 1 \]

**Extensions**

For the following exercises, graph the inequality.

193. \[ y \geq e^x \]
    \[ y \leq \ln(x) + 5 \]
194.  
\[
\begin{align*}
\log(x) & \leq -1 \\
y & \leq e^x
\end{align*}
\]

For the following exercises, find the solutions to the nonlinear equations with two variables.

195. 
\[
\frac{4}{x^2} + \frac{1}{y^2} = 24
\]
\[
\frac{5}{x^2} - \frac{2}{y^2} + 4 = 0
\]

196. 
\[
\frac{6}{x^2} - \frac{1}{y^2} = 8
\]
\[
\frac{1}{x^2} - \frac{6}{y^2} = \frac{1}{8}
\]

197. 
\[
x^2 - xy + y^2 - 2 = 0
\]
\[
x + 3y = 4
\]

198. 
\[
x^2 - xy - 2y^2 - 6 = 0
\]
\[
x^2 + y^2 = 1
\]

199. 
\[
x^2 + 4xy - 2y^2 - 6 = 0
\]
\[
x = y + 2
\]

**Technology**

For the following exercises, solve the system of inequalities. Use a calculator to graph the system to confirm the answer.

200. 
\[
x y < 1
\]
\[
y > \sqrt{x}
\]

201. 
\[
x^2 + y < 3
\]
\[
y > 2x
\]

**Real-World Applications**

For the following exercises, construct a system of nonlinear equations to describe the given behavior, then solve for the requested solutions.

202. Two numbers add up to 300. One number is twice the square of the other number. What are the numbers?

203. The squares of two numbers add to 360. The second number is half the value of the first number squared. What are the numbers?

204. A laptop company has discovered their cost and revenue functions for each day: 
\[
C(x) = 3x^2 - 10x + 200
\]
\[
R(x) = -2x^2 + 100x + 50
\]  
If they want to make a profit, what is the range of laptops per day that they should produce? Round to the nearest number which would generate profit.

205. A cell phone company has the following cost and revenue functions: 
\[
C(x) = 8x^2 - 600x + 21,500
\]
\[
R(x) = -3x^2 + 480x
\]  
What is the range of cell phones they should produce each day so there is profit? Round to the nearest number that generates profit.
9.4 | Partial Fractions

### Learning Objectives

In this section, you will:

- **9.4.1** Decompose \( P(x) Q(x) \), where \( Q(x) \) has only nonrepeated linear factors.
- **9.4.2** Decompose \( P(x) Q(x) \), where \( Q(x) \) has repeated linear factors.
- **9.4.3** Decompose \( P(x) Q(x) \), where \( Q(x) \) has a nonrepeated irreducible quadratic factor.
- **9.4.4** Decompose \( P(x) Q(x) \), where \( Q(x) \) has a repeated irreducible quadratic factor.

Earlier in this chapter, we studied systems of two equations in two variables, systems of three equations in three variables, and nonlinear systems. Here we introduce another way that systems of equations can be utilized—the decomposition of rational expressions.

Fractions can be complicated; adding a variable in the denominator makes them even more so. The methods studied in this section will help simplify the concept of a rational expression.

### Decomposing \( \frac{P(x)}{Q(x)} \) Where \( Q(x) \) Has Only Nonrepeated Linear Factors

Recall the algebra regarding adding and subtracting rational expressions. These operations depend on finding a common denominator so that we can write the sum or difference as a single, simplified rational expression. In this section, we will look at partial fraction decomposition, which is the undoing of the procedure to add or subtract rational expressions. In other words, it is a return from the single simplified rational expression to the original expressions, called the **partial fractions**.

For example, suppose we add the following fractions:

\[
\frac{2}{x-3} + \frac{-1}{x+2} \quad (9.20)
\]

We would first need to find a common denominator, \((x + 2)(x - 3)\).

Next, we would write each expression with this common denominator and find the sum of the terms.

\[
\frac{2}{x-3} \left( \frac{x + 2}{x + 2} \right) + \frac{-1}{x + 2} \left( \frac{x - 3}{x - 3} \right) = \frac{2x + 4 - x + 3}{(x + 2)(x - 3)} = \frac{x + 7}{x^2 - x - 6} \quad (9.21)
\]

Partial fraction decomposition is the reverse of this procedure. We would start with the solution and rewrite (decompose) it as the sum of two fractions.

\[
\frac{x + 7}{x^2 - x - 6} = \frac{2}{x-3} + \frac{-1}{x+2} \quad (9.22)
\]

We will investigate rational expressions with linear factors and quadratic factors in the denominator where the degree of the numerator is less than the degree of the denominator. Regardless of the type of expression we are decomposing, the first and most important thing to do is factor the denominator.

When the denominator of the simplified expression contains distinct linear factors, it is likely that each of the original rational expressions, which were added or subtracted, had one of the linear factors as the denominator. In other words, using the example above, the factors of \(x^2 - x - 6\) are \((x-3)(x+2)\), the denominators of the decomposed rational expression. So we will rewrite the simplified form as the sum of individual fractions and use a variable for each numerator. Then, we will solve for each numerator using one of several methods available for partial fraction decomposition.
Partial Fraction Decomposition of $\frac{P(x)}{Q(x)}$ : $Q(x)$ Has Nonrepeated Linear Factors

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$ when $Q(x)$ has nonrepeated linear factors and the degree of $P(x)$ is less than the degree of $Q(x)$ is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1 x + b_1)} + \frac{A_2}{(a_2 x + b_2)} + \frac{A_3}{(a_3 x + b_3)} + \cdots + \frac{A_n}{(a_n x + b_n)}.$$  

(9.23)

Given a rational expression with distinct linear factors in the denominator, decompose it.

1. Use a variable for the original numerators, usually $A$, $B$, or $C$, depending on the number of factors, placing each variable over a single factor. For the purpose of this definition, we use $A_n$ for each numerator

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1 x + b_1)} + \frac{A_2}{(a_2 x + b_2)} + \cdots + \frac{A_n}{(a_n x + b_n)}.$$  

(9.24)

2. Multiply both sides of the equation by the common denominator to eliminate fractions.

3. Expand the right side of the equation by the common denominator to eliminate fractions.

4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 9.22

Decomposing a Rational Function with Distinct Linear Factors

Decompose the given rational expression with distinct linear factors.

$$\frac{3}{x(x+2)(x-1)}$$

Solution

Find the partial fraction decomposition of the following expression.

$$\frac{x}{(x-3)(x-2)}$$  

(9.25)

Decomposing $\frac{P(x)}{Q(x)}$ Where $Q(x)$ Has Repeated Linear Factors

Some fractions we may come across are special cases that we can decompose into partial fractions with repeated linear factors. We must remember that we account for repeated factors by writing each factor in increasing powers.

Partial Fraction Decomposition of $\frac{P(x)}{Q(x)}$ : $Q(x)$ Has Repeated Linear Factors

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$ when $Q(x)$ has a repeated linear factor occurring $n$ times and the degree of $P(x)$ is less than the degree of $Q(x)$, is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(ax + b)} + \frac{A_2}{(ax + b)^2} + \frac{A_3}{(ax + b)^3} + \cdots + \frac{A_n}{(ax + b)^n}.$$  

(9.26)
Write the denominator powers in increasing order.

### Given a rational expression with repeated linear factors, decompose it.

1. Use a variable like \( A \), \( B \), or \( C \) for the numerators and account for increasing powers of the denominators.

\[
\frac{P(x)}{Q(x)} = \frac{A_1}{(ax + b)^1} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_n}{(ax + b)^n} \quad (9.27)
\]

2. Multiply both sides of the equation by the common denominator to eliminate fractions.

3. Expand the right side of the equation and collect like terms.

4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

### Example 9.23

#### Decomposing with Repeated Linear Factors

Decompose the given rational expression with repeated linear factors.

\[
\frac{-x^2 + 2x + 4}{x^3 - 4x^2 + 4x}
\]

#### Solution

The decomposition may contain more rational expressions if there are linear factors. Each linear factor will have a different constant numerator: \( A \), \( B \), \( C \), and so on.

### Try It

Find the partial fraction decomposition of the expression with repeated linear factors.

\[
\frac{6x - 11}{(x-1)^2}
\]
Given a rational expression where the factors of the denominator are distinct, irreducible quadratic factors, decompose it.

1. Use variables such as \( A, B, \) or \( C \) for the constant numerators over linear factors, and linear expressions such as \( A_1x + B_1, A_2x + B_2, \) etc., for the numerators of each quadratic factor in the denominator.

\[
\frac{P(x)}{Q(x)} = \frac{A}{ax + b} + \frac{A_1x + B_1}{(a_1x^2 + b_1x + c_1)} + \frac{A_2x + B_2}{(a_2x^2 + b_2x + c_2)} + \cdots + \frac{A_nx + B_n}{(a_nx^2 + b_nx + c_n)} \quad (9.30)
\]

2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 9.24

Decomposing \( \frac{P(x)}{Q(x)} \) When \( Q(x) \) Contains a Nonrepeated Irreducible Quadratic Factor

Find a partial fraction decomposition of the given expression.

\[
\frac{8x^2 + 12x - 20}{(x + 3)(x^2 + x + 2)}
\]

Solution

Could we have just set up a system of equations to solve Example 9.24?

Yes, we could have solved it by setting up a system of equations without solving for \( A \) first. The expansion on the right would be:

\[
8x^2 + 12x - 20 = Ax^2 + Ax + 2A + Bx^2 + 3B + Cx + 3C \quad (9.31)
\]

So the system of equations would be:

\[
\begin{align*}
A + B &= 8 \\
A + 3B + C &= 12 \\
2A + 3C &= -20
\end{align*}
\]

9.18 Find the partial fraction decomposition of the expression with a nonrepeating irreducible quadratic factor.

\[
\frac{5x^2 - 6x + 7}{(x-1)(x^2 + 1)} \quad (9.33)
\]

Decomposing \( \frac{P(x)}{Q(x)} \) When \( Q(x) \) Has a Repeated Irreducible Quadratic Factor

Now that we can decompose a simplified rational expression with an irreducible quadratic factor, we will learn how to do partial fraction decomposition when the simplified rational expression has repeated irreducible quadratic factors. The decomposition will consist of partial fractions with linear numerators over each irreducible quadratic factor represented in increasing powers.
The partial fraction decomposition of \( \frac{P(x)}{Q(x)} \) when \( Q(x) \) has a repeated irreducible quadratic factor and the degree of \( P(x) \) is less than the degree of \( Q(x) \), is

\[
\frac{P(x)}{(ax^2 + bx + c)^n} = \frac{A_1 x + B_1}{(ax^2 + bx + c)} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \frac{A_3 x + B_3}{(ax^2 + bx + c)^3} + \cdots + \frac{A_n x + B_n}{(ax^2 + bx + c)^n} \tag{9.34}
\]

Write the denominators in increasing powers.

**Given a rational expression that has a repeated irreducible factor, decompose it.**

1. Use variables like \( A, B, \) or \( C \) for the constant numerators over linear factors, and linear expressions such as \( A_1 x + B_1, A_2 x + B_2, \) etc., for the numerators of each quadratic factor in the denominator written in increasing powers, such as

\[
\frac{P(x)}{Q(x)} = \frac{A}{ax + b} + \frac{A_1 x + B_1}{(ax^2 + bx + c)} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_n + B_n}{(ax^2 + bx + c)^n} \tag{9.35}
\]

2. Multiply both sides of the equation by the common denominator to eliminate fractions.

3. Expand the right side of the equation and collect like terms.

4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

**Example 9.25**

**Decomposing a Rational Function with a Repeated Irreducible Quadratic Factor in the Denominator**

Decompose the given expression that has a repeated irreducible factor in the denominator.

\[
x^4 + x^3 + x^2 - x + 1 \quad \frac{x^3 - 4x^2 + 9x - 5}{(x^2 - 2x + 3)^2}
\]

**Solution**

9.19 Find the partial fraction decomposition of the expression with a repeated irreducible quadratic factor.

\[
\frac{x^3 - 4x^2 + 9x - 5}{(x^2 - 2x + 3)^2} \tag{9.36}
\]
9.4 EXERCISES

Verbal

206. Can any quotient of polynomials be decomposed into at least two partial fractions? If so, explain why, and if not, give an example of such a fraction.

207. Can you explain why a partial fraction decomposition is unique? (Hint: Think about it as a system of equations.)

208. Can you explain how to verify a partial fraction decomposition graphically?

209. You are unsure if you correctly decomposed the partial fraction correctly. Explain how you could double-check your answer.

210. Once you have a system of equations generated by the partial fraction decomposition, can you explain another method to solve it? For example if you had \( \frac{7x + 13}{3x^2 + 8x + 15} = \frac{A}{x + 1} + \frac{B}{3x + 5} \), we eventually simplify to \( 7x + 13 = A(3x + 5) + B(x + 1) \). Explain how you could intelligently choose an \( x \)-value that will eliminate either \( A \) or \( B \) and solve for \( A \) and \( B \).

Algebraic

For the following exercises, find the decomposition of the partial fraction for the nonrepeating linear factors.

211. \( \frac{5x + 16}{x^2 + 10x + 24} \)

212. \( \frac{3x-79}{x^2-5x-24} \)

213. \( \frac{-x-24}{x^2-2x-24} \)

214. \( \frac{10x + 47}{x^2 + 7x + 10} \)

215. \( \frac{x}{6x^2 + 25x + 25} \)

216. \( \frac{32x-11}{20x^2-13x + 2} \)

217. \( \frac{x + 1}{x^2 + 7x + 10} \)

218. \( \frac{5x}{x^2-9} \)

219. \( \frac{10x}{x^2-25} \)

220. \( \frac{6x}{x^2-4} \)

221. \( \frac{2x-3}{x^2-6x + 5} \)

222. \( \frac{4x-1}{x^2 - x-6} \)

223. \( \frac{4x + 3}{x^2 + 8x + 15} \)
For the following exercises, find the decomposition of the partial fraction for the repeating linear factors.

224. \(\frac{3x-1}{x^2-5x+6}\)

225. \(\frac{-5x-19}{(x+4)^2}\)

226. \(\frac{x}{(x-2)^2}\)

227. \(\frac{7x+14}{(x+3)^2}\)

228. \(\frac{-24x-27}{(4x+5)^2}\)

229. \(\frac{-24x-27}{(6x-7)^2}\)

230. \(\frac{5-x}{(x-7)^2}\)

231. \(\frac{5x+14}{2x^2+12x+18}\)

232. \(\frac{5x^2+20x+8}{2x(x+1)^2}\)

233. \(\frac{4x^2+55x+25}{5x(3x+5)^2}\)

234. \(\frac{54x^3+127x^2+80x+16}{2x^2(3x+2)^2}\)

235. \(\frac{x^3-5x^2+12x+144}{x^2(x^2+12x+36)}\)

For the following exercises, find the decomposition of the partial fraction for the irreducible nonrepeating quadratic factor.

236. \(\frac{4x^2+6x+11}{(x+2)(x^2+x+3)}\)

237. \(\frac{4x^2+9x+23}{(x-1)(x^2+6x+11)}\)

238. \(\frac{-2x^2+10x+4}{(x-1)(x^2+3x+8)}\)

239. \(\frac{x^2+3x+1}{(x+1)(x^2+5x-2)}\)

240. \(\frac{4x^2+17x-1}{(x+3)(x^2+6x+1)}\)
\[
\frac{4x^2}{(x + 5)(x^2 + 7x - 5)}
\]

242. \[
\frac{4x^2 + 5x + 3}{x^3 - 1}
\]

243. \[
\frac{-5x^2 + 18x - 4}{x^3 + 8}
\]

244. \[
\frac{3x^2 - 7x + 33}{x^3 + 27}
\]

245. \[
\frac{x^2 + 2x + 40}{x^3 - 125}
\]

246. \[
\frac{4x^2 + 4x + 12}{8x^3 - 27}
\]

247. \[
\frac{-50x^2 + 5x - 3}{125x^3 - 1}
\]

248. \[
\frac{-2x^3 - 30x^2 + 36x + 216}{x^4 + 216x}
\]

For the following exercises, find the decomposition of the partial fraction for the irreducible repeating quadratic factor.

249. \[
\frac{3x^3 + 2x^2 + 14x + 15}{(x^2 + 4)^2}
\]

250. \[
\frac{x^3 + 6x^2 + 5x + 9}{(x^2 + 1)^2}
\]

251. \[
\frac{x^3 - x^2 + x - 1}{(x^2 - 3)^2}
\]

252. \[
\frac{x^2 + 5x + 5}{(x + 2)^2}
\]

253. \[
\frac{x^3 + 2x^2 + 4x}{(x^2 + 2x + 9)^2}
\]

254. \[
\frac{x^2 + 25}{(x^2 + 3x + 25)^2}
\]

255. \[
\frac{2x^3 + 11x + 7x + 70}{(2x^2 + x + 14)^2}
\]

256. \[
\frac{5x + 2}{x(x^2 + 4)^2}
\]

257. 

This content is available for free at http://legacy.cnx.org/content/col11667/1.2
\[
\frac{x^4 + x^3 + 8x^2 + 6x + 36}{x(x^2 + 6)^2}
\]

258. \[
\frac{2x - 9}{(x^2 - x)^2}
\]

259. \[
\frac{5x^3 - 2x + 1}{(x^2 + 2x)^2}
\]

**Extensions**

For the following exercises, find the partial fraction expansion.

260. \[
\frac{x^2 + 4}{(x + 1)^3}
\]

261. \[
\frac{x^3 - 4x^2 + 5x + 4}{(x - 2)^3}
\]

For the following exercises, perform the operation and then find the partial fraction decomposition.

262. \[
\frac{7}{x + 8} + \frac{5}{x - 2} - \frac{x - 1}{x^2 - 6x - 16}
\]

263. \[
\frac{1}{x - 4} - \frac{3}{x + 6} - \frac{2x + 7}{x^2 + 2x - 24}
\]

264. \[
\frac{2x}{x^2 - 16} - \frac{1 - 2x}{x^2 + 6x + 8} - \frac{x - 5}{x^2 - 4x}
\]
9.5 | Matrices and Matrix Operations

Learning Objectives

In this section, you will:

- 9.5.1 Find the sum and difference of two matrices.
- 9.5.2 Find scalar multiples of a matrix.
- 9.5.3 Find the product of two matrices.

Two club soccer teams, the Wildcats and the Mud Cats, are hoping to obtain new equipment for an upcoming season. Table 9.1 shows the needs of both teams.

<table>
<thead>
<tr>
<th></th>
<th>Wildcats</th>
<th>Mud Cats</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goals</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>Balls</td>
<td>30</td>
<td>24</td>
</tr>
<tr>
<td>Jerseys</td>
<td>14</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 9.1

A goal costs $300; a ball costs $10; and a jersey costs $30. How can we find the total cost for the equipment needed for each team? In this section, we discover a method in which the data in the soccer equipment table can be displayed and used for calculating other information. Then, we will be able to calculate the cost of the equipment.

Finding the Sum and Difference of Two Matrices

To solve a problem like the one described for the soccer teams, we can use a matrix, which is a rectangular array of numbers. A row in a matrix is a set of numbers that are aligned horizontally. A column in a matrix is a set of numbers that are aligned...
Matrices are enclosed in \([\ ]\) or \(\(\)\), and are usually named with capital letters. For example, three matrices named \(A\), \(B\), and \(C\) are shown below.

\[
A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 7 \\ 0 & -5 & 6 \\ 7 & 8 & 2 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 3 \\ 0 & 2 \\ 3 & 1 \end{bmatrix}
\]  

### Describing Matrices

A matrix is often referred to by its size or dimensions: \(m \times n\) indicating \(m\) rows and \(n\) columns. Matrix entries are defined first by row and then by column. For example, to locate the entry in matrix \(A\) identified as \(a_{ij}\), we look for the entry in row \(i\), column \(j\). In matrix \(A\), shown below, the entry in row 2, column 3 is \(a_{23}\).

\[
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
\]

A square matrix is a matrix with dimensions \(n \times n\), meaning that it has the same number of rows as columns. The \(3\times3\) matrix above is an example of a square matrix.

A row matrix is a matrix consisting of one row with dimensions \(1 \times n\).

\[
[a_{11} \ a_{12} \ a_{13}]
\]

A column matrix is a matrix consisting of one column with dimensions \(m \times 1\).

\[
\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}
\]

A matrix may be used to represent a system of equations. In these cases, the numbers represent the coefficients of the variables in the system. Matrices often make solving systems of equations easier because they are not encumbered with variables. We will investigate this idea further in the next section, but first we will look at basic matrix operations.

---

### Example 9.26

**Finding the Dimensions of the Given Matrix and Locating Entries**

Given matrix \(A\):

a. What are the dimensions of matrix \(A\)?

b. What are the entries at \(a_{31}\) and \(a_{22}\)?

\[
A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 4 & 7 \\ 3 & 1 & -2 \end{bmatrix}
\]
Adding and Subtracting Matrices

We use matrices to list data or to represent systems. Because the entries are numbers, we can perform operations on matrices. We add or subtract matrices by adding or subtracting corresponding entries.

In order to do this, the entries must correspond. Therefore, *addition and subtraction of matrices is only possible when the matrices have the same dimensions*. We can add or subtract a $3 \times 3$ matrix and another $3 \times 3$ matrix, but we cannot add or subtract a $2 \times 3$ matrix and a $3 \times 3$ matrix because some entries in one matrix will not have a corresponding entry in the other matrix.

### Adding and Subtracting Matrices

Given matrices $A$ and $B$ of like dimensions, addition and subtraction of $A$ and $B$ will produce matrix $C$ or matrix $D$ of the same dimension.

\[
A + B = C \quad \text{such that} \quad a_{ij} + b_{ij} = c_{ij} \tag{9.41}
\]

\[
A - B = D \quad \text{such that} \quad a_{ij} - b_{ij} = d_{ij} \tag{9.42}
\]

Matrix addition is commutative.

\[
A + B = B + A \tag{9.43}
\]

It is also associative.

\[
(A + B) + C = A + (B + C) \tag{9.44}
\]

#### Example 9.27

**Finding the Sum of Matrices**

Find the sum of $A$ and $B$, given

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}
\]

**Solution**

#### Example 9.28

**Adding Matrix $A$ and Matrix $B$**

Find the sum of $A$ and $B$.

\[
A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 9 \\ 0 & 7 \end{bmatrix}
\]

**Solution**

#### Example 9.29
Finding the Difference of Two Matrices

Find the difference of $A$ and $B$.

$$A = \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix}$$

Example 9.30

Finding the Sum and Difference of Two 3 x 3 Matrices

Given $A$ and $B$:

a. Find the sum.

b. Find the difference.

$$A = \begin{bmatrix} 2 & -10 & -2 \\ 14 & 12 & 10 \\ 4 & -2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & 10 & -2 \\ 0 & -12 & -4 \\ -5 & 2 & -2 \end{bmatrix}$$

Finding Scalar Multiples of a Matrix

Besides adding and subtracting whole matrices, there are many situations in which we need to multiply a matrix by a constant called a scalar. Recall that a scalar is a real number quantity that has magnitude, but not direction. For example, time, temperature, and distance are scalar quantities. The process of scalar multiplication involves multiplying each entry in a matrix by a scalar. A scalar multiple is any entry of a matrix that results from scalar multiplication.

Consider a real-world scenario in which a university needs to add to its inventory of computers, computer tables, and chairs in two of the campus labs due to increased enrollment. They estimate that 15% more equipment is needed in both labs. The school’s current inventory is displayed in Table 9.2.

<table>
<thead>
<tr>
<th></th>
<th>Lab A</th>
<th>Lab B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computers</td>
<td>15</td>
<td>27</td>
</tr>
<tr>
<td>Computer Tables</td>
<td>16</td>
<td>34</td>
</tr>
<tr>
<td>Chairs</td>
<td>16</td>
<td>34</td>
</tr>
</tbody>
</table>

Table 9.2
Converting the data to a matrix, we have

\[ C_{2013} = \begin{bmatrix} 15 & 27 \\ 16 & 34 \\ 16 & 34 \end{bmatrix} \] \hspace{0.5in} (9.46)

To calculate how much computer equipment will be needed, we multiply all entries in matrix \( C \) by 0.15.

\[ (0.15)C_{2013} = \begin{bmatrix} (0.15)15 & (0.15)27 \\ (0.15)16 & (0.15)34 \\ (0.15)16 & (0.15)34 \end{bmatrix} = \begin{bmatrix} 2.25 & 4.05 \\ 2.4 & 5.1 \\ 2.4 & 5.1 \end{bmatrix} \] \hspace{0.5in} (9.47)

We must round up to the next integer, so the amount of new equipment needed is

\[ \begin{bmatrix} 3 \\ 5 \\ 3 \\ 6 \\ 3 \\ 6 \end{bmatrix} \] \hspace{0.5in} (9.48)

Adding the two matrices as shown below, we see the new inventory amounts.

\[ \begin{bmatrix} 15 & 27 \\ 16 & 34 \\ 16 & 34 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 3 & 6 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 18 & 32 \\ 19 & 40 \\ 19 & 40 \end{bmatrix} \] \hspace{0.5in} (9.49)

This means

\[ C_{2014} = \begin{bmatrix} 18 & 32 \\ 19 & 40 \\ 19 & 40 \end{bmatrix} \] \hspace{0.5in} (9.50)

Thus, Lab A will have 18 computers, 19 computer tables, and 19 chairs; Lab B will have 32 computers, 40 computer tables, and 40 chairs.

**Scalar Multiplication**

Scalar multiplication involves finding the product of a constant by each entry in the matrix. Given

\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \] \hspace{0.5in} (9.51)

the scalar multiple \( cA \) is

\[ cA = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix} \] \hspace{0.5in} (9.52)

Scalar multiplication is distributive. For the matrices \( A, B, \) and \( C \) with scalars \( a \) and \( b \),

\[ a(A + B) = aA + aB \]
\[ (a + b)A = aA + bA \] \hspace{0.5in} (9.53)

**Example 9.31**

**Multiplying the Matrix by a Scalar**

Multiply matrix \( A \) by the scalar 3.

\[ A = \begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix} \]
9.21 Given matrix \( B \), find \(-2B\) where

\[
B = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}
\]  

(9.54)

Example 9.32

Finding the Sum of Scalar Multiples

Find the sum \( 3A + 2B \).

\[
A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 2 \\ 4 & 3 & -6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & 1 & -4 \end{bmatrix}
\]

Solution

Finding the Product of Two Matrices

In addition to multiplying a matrix by a scalar, we can multiply two matrices. Finding the product of two matrices is only possible when the inner dimensions are the same, meaning that the number of columns of the first matrix is equal to the number of rows of the second matrix. If \( A \) is an \( m \times r \) matrix and \( B \) is an \( r \times n \) matrix, then the product matrix \( AB \) is an \( m \times n \) matrix. For example, the product \( AB \) is possible because the number of columns in \( A \) is the same as the number of rows in \( B \). If the inner dimensions do not match, the product is not defined.

\[
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}
\]

(9.55)

Multiply and add as follows to obtain the first entry of the product matrix \( AB \).

1. To obtain the entry in row 1, column 1 of \( AB \), multiply the first row in \( A \) by the first column in \( B \), and add.

\[
\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \cdot \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31}
\]

(9.56)

2. To obtain the entry in row 1, column 2 of \( AB \), multiply the first row in \( A \) by the second column in \( B \), and add.

\[
\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \cdot \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32}
\]

(9.57)

3. To obtain the entry in row 1, column 3 of \( AB \), multiply the first row in \( A \) by the third column in \( B \), and add.
\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
b_{21} & b_{22} & b_{23} \\
b_{31} & b_{32} & b_{33}
\end{bmatrix}
\begin{bmatrix}
b_{13} \\
b_{23} \\
b_{33}
\end{bmatrix}
= a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33}
\]

We proceed the same way to obtain the second row of \( AB \). In other words, row 2 of \( A \) times column 1 of \( B \); row 2 of \( A \) times column 2 of \( B \); row 2 of \( A \) times column 3 of \( B \). When complete, the product matrix will be

\[
AB = \begin{bmatrix}
a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} & a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} \\
a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} & a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33}
\end{bmatrix}
\]

### Properties of Matrix Multiplication

For the matrices \( A, B, \) and \( C \) the following properties hold.

- Matrix multiplication is associative: \((AB)C = A(BC)\).
- Matrix multiplication is distributive: \(C(A + B) = CA + CB\), \((A + B)C = AC + BC\).

Note that matrix multiplication is not commutative.

### Example 9.33

**Multiplying Two Matrices**

Multiply matrix \( A \) and matrix \( B \).

\[
A = \begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
5 & 6 \\
7 & 8
\end{bmatrix}
\]

**Solution**

### Example 9.34

**Multiplying Two Matrices**

Given \( A \) and \( B \):

a. Find \( AB \).

b. Find \( BA \).

\[
A = \begin{bmatrix}
-1 & 2 & 3 \\
4 & 0 & 5
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
5 & -1 \\
-4 & 0 \\
2 & 3
\end{bmatrix}
\]

**Solution**

**Analysis**

Notice that the products \( AB \) and \( BA \) are not equal.
AB = \begin{bmatrix} -7 & 10 \\ 30 & 11 \end{bmatrix} \neq \begin{bmatrix} -9 & 10 & 10 \\ 4 & -8 & -12 \\ 10 & 4 & 21 \end{bmatrix} = BA

This illustrates the fact that matrix multiplication is not commutative.

**Q&A**

*Is it possible for AB to be defined but not BA?*

Yes, consider a matrix A with dimension 3 × 4 and matrix B with dimension 4 × 2. For the product AB the inner dimensions are 4 and the product is defined, but for the product BA the inner dimensions are 2 and 3 so the product is undefined.

**Example 9.35**

**Using Matrices in Real-World Problems**

Let’s return to the problem presented at the opening of this section. We have Table 9.3, representing the equipment needs of two soccer teams.

<table>
<thead>
<tr>
<th></th>
<th>Wildcats</th>
<th>Mud Cats</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goals</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>Balls</td>
<td>30</td>
<td>24</td>
</tr>
<tr>
<td>Jerseys</td>
<td>14</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 9.3

We are also given the prices of the equipment, as shown in Table 9.4.

<table>
<thead>
<tr>
<th>Equipment</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goal</td>
<td>$300</td>
</tr>
<tr>
<td>Ball</td>
<td>$10</td>
</tr>
<tr>
<td>Jersey</td>
<td>$30</td>
</tr>
</tbody>
</table>

Table 9.4

We will convert the data to matrices. Thus, the equipment need matrix is written as

\[ E = \begin{bmatrix} 6 & 10 \\ 30 & 24 \\ 14 & 20 \end{bmatrix} \]

The cost matrix is written as

\[ C = [300 \ 10 \ 30] \]

We perform matrix multiplication to obtain costs for the equipment.
The total cost for equipment for the Wildcats is $2,520, and the total cost for equipment for the Mud Cats is $3,840.

Given a matrix operation, evaluate using a calculator.

1. Save each matrix as a matrix variable \([A], [B], [C], ...\)
2. Enter the operation into the calculator, calling up each matrix variable as needed.
3. If the operation is defined, the calculator will present the solution matrix; if the operation is undefined, it will display an error message.

Example 9.36

Using a Calculator to Perform Matrix Operations

Find \(AB - C\) given

\[
A = \begin{bmatrix} -15 & 25 & 32 \\ 41 & -7 & -28 \\ 10 & 34 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 45 & 21 & -37 \\ -24 & 52 & 19 \\ 6 & -48 & -31 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} -100 & -89 & -98 \\ 25 & -56 & 74 \\ -67 & 42 & -75 \end{bmatrix}.
\]

Solution

Access these online resources for additional instruction and practice with matrices and matrix operations.

- Dimensions of a Matrix (http://openstaxcollege.org/l/matrixdimen)
- Matrix Addition and Subtraction (http://openstaxcollege.org/l/matrixaddsub)
- Matrix Operations (http://openstaxcollege.org/l/matrixoper)
- Matrix Multiplication (http://openstaxcollege.org/l/matrixmult)
9.5 EXERCISES

Verbal

265. Can we add any two matrices together? If so, explain why; if not, explain why not and give an example of two matrices that cannot be added together.

266. Can we multiply any column matrix by any row matrix? Explain why or why not.

267. Can both the products \( AB \) and \( BA \) be defined? If so, explain how; if not, explain why.

268. Can any two matrices of the same size be multiplied? If so, explain why, and if not, explain why not and give an example of two matrices of the same size that cannot be multiplied together.

269. Does matrix multiplication commute? That is, does \( AB = BA \)? If so, prove why it does. If not, explain why it does not.

Algebraic

For the following exercises, use the matrices below and perform the matrix addition or subtraction. Indicate if the operation is undefined.

\[
\begin{align*}
A &= \begin{bmatrix} 1 & 3 \\ 0 & 7 \end{bmatrix}, & B &= \begin{bmatrix} 2 & 14 \\ 22 & 6 \end{bmatrix}, & C &= \begin{bmatrix} 1 & 5 \\ 8 & 92 \end{bmatrix}, & D &= \begin{bmatrix} 10 & 14 \\ 7 & 2 \end{bmatrix}, & E &= \begin{bmatrix} 6 & 12 \\ 14 & 5 \end{bmatrix}, & F &= \begin{bmatrix} 0 & 9 \\ 78 & 17 \end{bmatrix} \\
\end{align*}
\]

270. \( A + B \)

271. \( C + D \)

272. \( A + C \)

273. \( B - E \)

274. \( C + F \)

275. \( D - B \)

For the following exercises, use the matrices below to perform scalar multiplication.

\[
\begin{align*}
A &= \begin{bmatrix} 4 & 6 \\ 13 & 12 \end{bmatrix}, & B &= \begin{bmatrix} 3 & 9 \\ 21 & 12 \end{bmatrix}, & C &= \begin{bmatrix} 16 & 3 & 7 & 18 \\ 90 & 5 & 3 & 29 \end{bmatrix}, & D &= \begin{bmatrix} 18 & 12 & 13 \\ 8 & 14 & 6 \\ 7 & 4 & 21 \end{bmatrix} \\
\end{align*}
\]

276. \( 5A \)

277. \( 3B \)

278. \( -2B \)

279. \( -4C \)

280. \( \frac{1}{2}C \)

281. \( 100D \)

For the following exercises, use the matrices below to perform matrix multiplication.

\[
\begin{align*}
A &= \begin{bmatrix} -1 & 5 \\ 3 & 2 \end{bmatrix}, & B &= \begin{bmatrix} 3 & 6 & 4 \\ -8 & 0 & 12 \end{bmatrix}, & C &= \begin{bmatrix} 4 & 10 \\ -2 & 6 \\ -2 & 5 \end{bmatrix}, & D &= \begin{bmatrix} 2 & -3 & 12 \\ 9 & 3 & 1 \\ 0 & 8 & -10 \end{bmatrix} \\
\end{align*}
\]

282. \( AB \)
283. $BC$
284. $CA$
285. $BD$
286. $DC$
287. $CB$

For the following exercises, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed.

$A = \begin{bmatrix} 2 & -5 \\ 6 & 7 \end{bmatrix}$  $B = \begin{bmatrix} -9 & 6 \\ -4 & 2 \end{bmatrix}$  $C = \begin{bmatrix} 0 & 9 \\ 7 & 11 \end{bmatrix}$  $D = \begin{bmatrix} -8 & 7 & -5 \\ 4 & 3 & 2 \\ 0 & 9 & 2 \end{bmatrix}$  $E = \begin{bmatrix} 4 & 5 & 3 \\ 7 & -6 & -5 \\ 1 & 0 & 9 \end{bmatrix}$  \hspace{1cm} (9.63)

288. $A + B - C$
289. $4A + 5D$
290. $2C + B$
291. $3D + 4E$
292. $C - 0.5D$
293. $100D - 10E$

For the following exercises, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed. (Hint: $A^2 = A \cdot A$)

$A = \begin{bmatrix} -10 & 20 \\ 5 & 25 \end{bmatrix}$  $B = \begin{bmatrix} -40 & 10 \\ -20 & 30 \end{bmatrix}$  $C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$  \hspace{1cm} (9.64)

294. $AB$
295. $BA$
296. $CA$
297. $BC$
298. $A^2$
299. $B^2$
300. $C^2$
301. $B^2 A^2$
302. $A^2 B^2$
303. $(AB)^2$
304. $(BA)^2$

For the following exercises, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed. (Hint: $A^2 = A \cdot A$)
Technology

For the following exercises, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed. Use a calculator to verify your solution.

\[
A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 3 & 4 \\ -1 & 1 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 0.5 & 0.1 \\ 1 & 0.2 \\ -0.5 & 0.3 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & -1 \\ -6 & 7 & 5 \\ 4 & 2 & 1 \end{bmatrix}
\]

(9.65)

305. \(AB\)
306. \(BA\)
307. \(BD\)
308. \(DC\)
309. \(D^2\)
310. \(A^2\)
311. \(D^3\)
312. \((AB)C\)
313. \(A(BC)\)

Extensions

For the following exercises, use the matrix below to perform the indicated operation on the given matrix.

\[
A = \begin{bmatrix} -2 & 0 & 9 \\ 1 & 8 & -3 \\ 0.5 & 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & 3 & 0 \\ -4 & 1 & 6 \\ 8 & 7 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}
\]

(9.66)

314. \(AB\)
315. \(BA\)
316. \(CA\)
317. \(BC\)
318. \(ABC\)

Using the above questions, find a formula for \(B^n\). Test the formula for \(B^{201}\) and \(B^{202}\), using a calculator.
9.6 | Solving Systems with Gaussian Elimination

Learning Objectives

In this section, you will:

9.6.1 Write the augmented matrix of a system of equations.
9.6.2 Write the system of equations from an augmented matrix.
9.6.3 Perform row operations on a matrix.
9.6.4 Solve a system of linear equations using matrices.

Figure 9.14  German mathematician Carl Friedrich Gauss (1777–1855).

Carl Friedrich Gauss lived during the late 18th century and early 19th century, but he is still considered one of the most prolific mathematicians in history. His contributions to the science of mathematics and physics span fields such as algebra, number theory, analysis, differential geometry, astronomy, and optics, among others. His discoveries regarding matrix theory changed the way mathematicians have worked for the last two centuries.

We first encountered Gaussian elimination in m10420 (http://legacy.cnx.org/content/m10420/latest/) . In this section, we will revisit this technique for solving systems, this time using matrices.

Writing the Augmented Matrix of a System of Equations

A matrix can serve as a device for representing and solving a system of equations. To express a system in matrix form, we extract the coefficients of the variables and the constants, and these become the entries of the matrix. We use a vertical line to separate the coefficient entries from the constants, essentially replacing the equal signs. When a system is written in this form, we call it an augmented matrix.

For example, consider the following $2 \times 2$ system of equations.

\[
\begin{align*}
3x + 4y &= 7 \\
4x - 2y &= 5
\end{align*}
\] (9.68)
We can write this system as an augmented matrix:
\[
\begin{bmatrix}
3 & 4 & 7 \\
4 & -2 & 5
\end{bmatrix}
\]
We can also write a matrix containing just the coefficients. This is called the \textbf{coefficient matrix}.
\[
\begin{bmatrix}
3 & 4 \\
4 & -2
\end{bmatrix}
\]
A three-by-three system of equations such as
\[
\begin{align*}
3x - y - z &= 0 \\
x + y &= 5 \\
2x - 3z &= 2
\end{align*}
\]
has a coefficient matrix
\[
\begin{bmatrix}
3 & -1 & -1 \\
1 & 1 & 0 \\
2 & 0 & -3
\end{bmatrix}
\]
and is represented by the augmented matrix
\[
\begin{bmatrix}
3 & -1 & -1 & | & 0 \\
1 & 1 & 0 & | & 5 \\
2 & 0 & -3 & | & 2
\end{bmatrix}
\]
Notice that the matrix is written so that the variables line up in their own columns: \(x\)-terms go in the first column, \(y\)-terms in the second column, and \(z\)-terms in the third column. It is very important that each equation is written in standard form \(ax + by + cz = d\) so that the variables line up. When there is a missing variable term in an equation, the coefficient is 0.

\textbf{Given a system of equations, write an augmented matrix.}

1. Write the coefficients of the \(x\)-terms as the numbers down the first column.
2. Write the coefficients of the \(y\)-terms as the numbers down the second column.
3. If there are \(z\)-terms, write the coefficients as the numbers down the third column.
4. Draw a vertical line and write the constants to the right of the line.

\textbf{Example 9.37}

\textbf{Writing the Augmented Matrix for a System of Equations}

Write the augmented matrix for the given system of equations.
\[
\begin{align*}
x + 2y - z &= 3 \\
2x - y + 2z &= 6 \\
x - 3y + 3z &= 4
\end{align*}
\]

\textbf{Solution}

\textbf{Try it} 

Write the augmented matrix of the given system of equations.
\[
4x - 3y = 11 \\
3x + 2y = 4
\]

\textbf{Writing a System of Equations from an Augmented Matrix}

We can use augmented matrices to help us solve systems of equations because they simplify operations when the systems are not encumbered by the variables. However, it is important to understand how to move back and forth between formats.
in order to make finding solutions smoother and more intuitive. Here, we will use the information in an augmented matrix to write the system of equations in standard form.

Example 9.38

**Writing a System of Equations from an Augmented Matrix Form**

Find the system of equations from the augmented matrix.

\[
\begin{bmatrix}
1 & -3 & -5 & | & -2 \\
2 & -5 & -4 & | & 5 \\
-3 & 5 & 4 & | & 6
\end{bmatrix}
\]

**Solution**

Write the system of equations from the augmented matrix.

(9.75)

\[
\begin{bmatrix}
1 & -1 & 1 & | & 5 \\
2 & -1 & 3 & | & 1 \\
0 & 1 & 1 & | & -9
\end{bmatrix}
\]

**Performing Row Operations on a Matrix**

Now that we can write systems of equations in augmented matrix form, we will examine the various row operations that can be performed on a matrix, such as addition, multiplication by a constant, and interchanging rows.

Performing row operations on a matrix is the method we use for solving a system of equations. In order to solve the system of equations, we want to convert the matrix to row-echelon form, in which there are ones down the main diagonal from the upper left corner to the lower right corner, and zeros in every position below the main diagonal as shown.

(9.76)

\[
\begin{bmatrix}
1 & a & b \\
0 & 1 & d \\
0 & 0 & 1
\end{bmatrix}
\]

We use row operations corresponding to equation operations to obtain a new matrix that is row-equivalent in a simpler form. Here are the guidelines to obtaining row-echelon form.

1. In any nonzero row, the first nonzero number is a 1. It is called a leading 1.
2. Any all-zero rows are placed at the bottom on the matrix.
3. Any leading 1 is below and to the right of a previous leading 1.
4. Any column containing a leading 1 has zeros in all other positions in the column.

To solve a system of equations we can perform the following row operations to convert the coefficient matrix to row-echelon form and do back-substitution to find the solution.

1. Interchange rows. (Notation: \( R_i \leftrightarrow R_j \))
2. Multiply a row by a constant. (Notation: \( cR_i \))
3. Add the product of a row multiplied by a constant to another row. (Notation: \( R_i + cR_j \))

Each of the row operations corresponds to the operations we have already learned to solve systems of equations in three variables. With these operations, there are some key moves that will quickly achieve the goal of writing a matrix in row-echelon form. To obtain a matrix in row-echelon form for finding solutions, we use Gaussian elimination, a method that uses row operations to obtain a 1 as the first entry so that row 1 can be used to convert the remaining rows.
Gaussian Elimination

The Gaussian elimination method refers to a strategy used to obtain the row-echelon form of a matrix. The goal is to write matrix $A$ with the number 1 as the entry down the main diagonal and have all zeros below.

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix} \quad \text{After Gaussian elimination} \quad A = \begin{bmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{bmatrix}
\]  \hspace{1cm} (9.77)

The first step of the Gaussian strategy includes obtaining a 1 as the first entry, so that row 1 may be used to alter the rows below.

**How To:**

1. The first equation should have a leading coefficient of 1. Interchange rows or multiply by a constant, if necessary.
2. Use row operations to obtain zeros down the first column below the first entry of 1.
3. Use row operations to obtain a 1 in row 2, column 2.
4. Use row operations to obtain zeros down column 2, below the entry of 1.
5. Use row operations to obtain a 1 in row 3, column 3.
6. Continue this process for all rows until there is a 1 in every entry down the main diagonal and there are only zeros below.
7. If any rows contain all zeros, place them at the bottom.

**Example 9.39**

**Solving a 2x2 System by Gaussian Elimination**

Solve the given system by Gaussian elimination.

\[
\begin{align*}
2x + 3y &= 6 \\
x - y &= \frac{1}{2}
\end{align*}
\]

**Solution**

**Example 9.40**

**Using Gaussian Elimination to Solve a System of Equations**

Use Gaussian elimination to solve the given $2 \times 2$ system of equations.

\[
\begin{align*}
2x + y &= 1 \\
4x + 2y &= 6
\end{align*}
\]
Example 9.41

Solving a Dependent System

Solve the system of equations.

\[\begin{align*}
3x + 4y &= 12 \\
6x + 8y &= 24
\end{align*}\]

Solution

Example 9.42

Performing Row Operations on a 3x3 Augmented Matrix to Obtain Row-Echelon Form

Perform row operations on the given matrix to obtain row-echelon form.

\[
\begin{bmatrix}
1 & -3 & 4 & 3 \\
2 & -5 & 6 & 6 \\
-3 & 3 & 4 & 6
\end{bmatrix}
\]

Solution

9.25 Write the system of equations in row-echelon form.

\[
\begin{align*}
x - 2y + 3z &= 9 \\
- x + 3y &= -4 \\
2x - 5y + 5z &= 17
\end{align*}
\]

Solving a System of Linear Equations Using Matrices

We have seen how to write a system of equations with an augmented matrix, and then how to use row operations and back-substitution to obtain row-echelon form. Now, we will take row-echelon form a step farther to solve a 3 by 3 system of linear equations. The general idea is to eliminate all but one variable using row operations and then back-substitute to solve for the other variables.

Example 9.43

Solving a System of Linear Equations Using Matrices

Solve the system of linear equations using matrices.
Example 9.44

Solving a Dependent System of Linear Equations Using Matrices

Solve the following system of linear equations using matrices.

\[
\begin{align*}
-x - 2y + z &= -1 \\
2x + 3y &= 2 \\
y - 2z &= 0
\end{align*}
\]

Solution

Example 9.45

Solving Systems of Equations with Matrices Using a Calculator

Solve the system of equations.

\[
\begin{align*}
5x + 3y + 9z &= -1 \\
-x - 4y + 5z &= 1
\end{align*}
\]

Solution
Example 9.46

**Applying 2 × 2 Matrices to Finance**

Carolyn invests a total of $12,000 in two municipal bonds, one paying 10.5% interest and the other paying 12% interest. The annual interest earned on the two investments last year was $1,335. How much was invested at each rate?

**Solution**

Example 9.47

**Applying 3 × 3 Matrices to Finance**

Ava invests a total of $10,000 in three accounts, one paying 5% interest, another paying 8% interest, and the third paying 9% interest. The annual interest earned on the three investments last year was $770. The amount invested at 9% was twice the amount invested at 5%. How much was invested at each rate?

**Solution**

9.27 A small shoe company took out a loan of $1,500,000 to expand their inventory. Part of the money was borrowed at 7%, part was borrowed at 8%, and part was borrowed at 10%. The amount borrowed at 10% was four times the amount borrowed at 7%, and the annual interest on all three loans was $130,500. Use matrices to find the amount borrowed at each rate.

Access these online resources for additional instruction and practice with solving systems of linear equations using Gaussian elimination.

- Solve a System of Two Equations Using an Augmented Matrix (http://openstaxcollege.org/l/system2augmat)
- Solve a System of Three Equations Using an Augmented Matrix (http://openstaxcollege.org/l/system3augmat)
- Augmented Matrices on the Calculator (http://openstaxcollege.org/l/augmatcalc)
9.6 EXERCISES

Verbal

324. Can any system of linear equations be written as an augmented matrix? Explain why or why not. Explain how to write that augmented matrix.

325. Can any matrix be written as a system of linear equations? Explain why or why not. Explain how to write that system of equations.

326. Is there only one correct method of using row operations on a matrix? Try to explain two different row operations possible to solve the augmented matrix \[
\begin{bmatrix}
9 & 3 & 0 \\
1 & -2 & 6
\end{bmatrix}
\]

327. Can a matrix whose entry is 0 on the diagonal be solved? Explain why or why not. What would you do to remedy the situation?

328. Can a matrix that has 0 entries for an entire row have one solution? Explain why or why not.

Algebraic

For the following exercises, write the augmented matrix for the linear system.

329. \[8x-37y = 8 \\
2x + 12y = 3\]

330. \[16y = 4 \\
9x - y = 2\]

331. \[3x + 2y + 10z = 3 \\
-6x + 2y + 5z = 13 \\
4x + z = 18\]

332. \[x + 5y + 8z = 19 \\
12x + 3y = 4 \\
3x + 4y + 9z = -7\]

333. \[6x + 12y + 16z = 4 \\
19x - 5y + 3z = -9 \\
x + 2y = -8\]

For the following exercises, write the linear system from the augmented matrix.

334. \[
\begin{bmatrix}
-2 & 5 & 5 \\
6 & -18 & 26
\end{bmatrix}
\]

335. \[
\begin{bmatrix}
3 & 4 & 10 \\
10 & 17 & 439
\end{bmatrix}
\]

336. \[
\begin{bmatrix}
3 & 2 & 0 & 3 \\
-1 & -9 & 4 & -1 \\
8 & 5 & 7 & 8
\end{bmatrix}
\]

337. \[
\begin{bmatrix}
8 & 29 & 1 & 43 \\
-1 & 7 & 5 & 38 \\
0 & 0 & 3 & 10
\end{bmatrix}
\]

338. \[
\begin{bmatrix}
4 & 5 & -2 & 12 \\
0 & 1 & 58 & 2 \\
8 & 7 & -3 & -5
\end{bmatrix}
\]
For the following exercises, solve the system by Gaussian elimination.

339. \[
\begin{bmatrix}
1 & 0 & | & 3 \\
0 & 0 & | & 0
\end{bmatrix}
\]

340. \[
\begin{bmatrix}
1 & 0 & | & 1 \\
1 & 0 & | & 2
\end{bmatrix}
\]

341. \[
\begin{bmatrix}
1 & 2 & | & 3 \\
4 & 5 & | & 6
\end{bmatrix}
\]

342. \[
\begin{bmatrix}
-1 & 2 & | & -3 \\
4 & -5 & | & 6
\end{bmatrix}
\]

343. \[
\begin{bmatrix}
-2 & 0 & | & 1 \\
0 & 2 & | & -1
\end{bmatrix}
\]

344. \[
\begin{array}{l}
2x - 3y = -9 \\
5x + 4y = 58
\end{array}
\]

345. \[
\begin{array}{l}
6x + 2y = -4 \\
3x + 4y = -17
\end{array}
\]

346. \[
\begin{array}{l}
2x + 3y = 12 \\
4x + y = 14
\end{array}
\]

347. \[
\begin{array}{l}
-4x - 3y = -2 \\
3x - 5y = -13
\end{array}
\]

348. \[
\begin{array}{l}
-5x + 8y = 3 \\
10x + 6y = 5
\end{array}
\]

349. \[
\begin{array}{l}
3x + 4y = 12 \\
-6x - 8y = -24
\end{array}
\]

350. \[
\begin{array}{l}
-60x + 45y = 12 \\
20x - 15y = -4
\end{array}
\]

351. \[
\begin{array}{l}
11x + 10y = 43 \\
15x + 20y = 65
\end{array}
\]

352. \[
\begin{array}{l}
2x - y = 2 \\
3x + 2y = 17
\end{array}
\]

353. \[
\begin{array}{l}
-1.06x - 2.25y = 5.51 \\
-5.03x - 1.08y = 5.40
\end{array}
\]

354. \[
\begin{array}{l}
\frac{3}{4}x - \frac{3}{5}y = 4 \\
\frac{1}{4}x + \frac{2}{5}y = 1
\end{array}
\]

355. \[
\begin{array}{l}
\frac{1}{4}x - \frac{2}{3}y = -1 \\
\frac{1}{2}x + \frac{1}{3}y = 3
\end{array}
\]

356.
\[
\begin{bmatrix}
1 & 0 & 0 & | & 31 \\
0 & 1 & 1 & | & 45 \\
0 & 0 & 1 & | & 87
\end{bmatrix}
\]

357. \[
\begin{bmatrix}
1 & 0 & 1 & | & 50 \\
1 & 1 & 0 & | & 20 \\
0 & 1 & 1 & | & -90
\end{bmatrix}
\]

358. \[
\begin{bmatrix}
1 & 2 & 3 & | & 4 \\
0 & 5 & 6 & | & 7 \\
0 & 0 & 8 & | & 9
\end{bmatrix}
\]

359. \[
\begin{bmatrix}
-0.1 & 0.3 & -0.1 & | & 0.27 \\
-0.4 & 0.2 & 0.1 & | & 0.8 \\
0.6 & 0.1 & 0.7 & | & -0.8
\end{bmatrix}
\]

360. 
\[-2x + 3y - 2z = 3 \\
4x + 2y - z = 9 \\
4x - 8y + 2z = -6\]

361. 
\[x + y - 4z = -4 \\
5x - 3y - 2z = 0 \\
2x + 6y + 7z = 30\]

362. 
\[2x + 3y + 2z = 1 \\
-4x - 6y - 4z = -2 \\
10x + 15y + 10z = 5\]

363. 
\[x + 2y - z = 1 \\
-x - 2y + 2z = -2 \\
3x + 6y - 3z = 5\]

364. 
\[x + 2y - z = 1 \\
-x - 2y + 2z = -2 \\
3x + 6y - 3z = 3\]

365. 
\[x + y = 2 \\
x + z = 1 \\
-y - z = -3\]

366. 
\[x + y + z = 100 \\
x + 2z = 125 \\
-y + 2z = 25\]

367. 
\[\frac{1}{4}x - \frac{2}{3}z = -\frac{1}{2} \\
\frac{1}{5}x + \frac{1}{3}y = \frac{4}{7} \\
\frac{1}{5}y - \frac{1}{3}z = \frac{2}{9}\]

368. 
\[-\frac{1}{2}x + \frac{1}{2}y + \frac{1}{7}z = -\frac{53}{14} \\
\frac{1}{2}x - \frac{1}{2}y + \frac{1}{4}z = 3 \\
\frac{1}{4}x + \frac{1}{5}y + \frac{1}{3}z = \frac{23}{15}\]

369.
\[-\frac{1}{2}x - \frac{1}{3}y + \frac{1}{4}z = -\frac{29}{6} \]
\[\frac{1}{5}x + \frac{1}{6}y - \frac{1}{7}z = \frac{431}{210} \]
\[-\frac{1}{8}x + \frac{1}{9}y + \frac{1}{10}z = -\frac{49}{45} \]

**Extensions**

For the following exercises, use Gaussian elimination to solve the system.

370. \[\frac{x-1}{7} + \frac{y-2}{8} + \frac{z-3}{4} = 0 \]
\[x + y + z = 6 \]
\[\frac{x+2}{3} + 2y + \frac{z-3}{3} = 5 \]

371. \[\frac{x-1}{4} - \frac{y+1}{4} + 3z = -1 \]
\[\frac{x+5}{2} + \frac{y+7}{4} - z = 4 \]
\[x + y - \frac{z-2}{2} = 1 \]

372. \[\frac{x-3}{4} - \frac{y-1}{3} + 2z = -1 \]
\[\frac{x+5}{2} + \frac{y+5}{2} + \frac{z+5}{2} = 8 \]
\[x + y + z = 1 \]

373. \[\frac{x-3}{10} + \frac{y+3}{2} - 2z = 3 \]
\[\frac{x+5}{4} - \frac{y-1}{8} + z = \frac{3}{2} \]
\[\frac{x-1}{4} + \frac{y+4}{2} + 3z = \frac{3}{2} \]

374. \[\frac{x-3}{4} - \frac{y-1}{3} + 2z = -1 \]
\[\frac{x+5}{2} + \frac{y+5}{2} + \frac{z+5}{2} = 7 \]
\[x + y + z = 1 \]

**Real-World Applications**

For the following exercises, set up the augmented matrix that describes the situation, and solve for the desired solution.

375. Every day, a cupcake store sells 5,000 cupcakes in chocolate and vanilla flavors. If the chocolate flavor is 3 times as popular as the vanilla flavor, how many of each cupcake sell per day?

376. At a competing cupcake store, $4,520 worth of cupcakes are sold daily. The chocolate cupcakes cost $2.25 and the red velvet cupcakes cost $1.75. If the total number of cupcakes sold per day is 2,200, how many of each flavor are sold each day?

377. You invested $10,000 into two accounts: one that has simple 3% interest, the other with 2.5% interest. If your total interest payment after one year was $283.50, how much was in each account after the year passed?

378. You invested $2,300 into account 1, and $2,700 into account 2. If the total amount of interest after one year is $254, and account 2 has 1.5 times the interest rate of account 1, what are the interest rates? Assume simple interest rates.

379. Bikes’R’Us manufactures bikes, which sell for $250. It costs the manufacturer $180 per bike, plus a startup fee of $3,500. After how many bikes sold will the manufacturer break even?

380.
A major appliance store is considering purchasing vacuums from a small manufacturer. The store would be able to purchase the vacuums for $86 each, with a delivery fee of $9,200, regardless of how many vacuums are sold. If the store needs to start seeing a profit after 230 units are sold, how much should they charge for the vacuums?

381. The three most popular ice cream flavors are chocolate, strawberry, and vanilla, comprising 83% of the flavors sold at an ice cream shop. If vanilla sells 1% more than twice strawberry, and chocolate sells 11% more than vanilla, how much of the total ice cream consumption are the vanilla, chocolate, and strawberry flavors?

382. At an ice cream shop, three flavors are increasing in demand. Last year, banana, pumpkin, and rocky road ice cream made up 12% of total ice cream sales. This year, the same three ice creams made up 16.9% of ice cream sales. The rocky road sales doubled, the banana sales increased by 50%, and the pumpkin sales increased by 20%. If the rocky road ice cream had one less percent of sales than the banana ice cream, find out the percentage of ice cream sales each individual ice cream made last year.

383. A bag of mixed nuts contains cashews, pistachios, and almonds. There are 1,000 total nuts in the bag, and there are 100 less almonds than pistachios. The cashews weigh 3 g, pistachios weigh 4 g, and almonds weigh 5 g. If the bag weighs 3.7 kg, find out how many of each type of nut is in the bag.

384. A bag of mixed nuts contains cashews, pistachios, and almonds. Originally there were 900 nuts in the bag. 30% of the almonds, 20% of the cashews, and 10% of the pistachios were eaten, and now there are 770 nuts left in the bag. Originally, there were 100 more cashews than almonds. Figure out how many of each type of nut was in the bag to begin with.
9.7 | Solving Systems with Inverses

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<td><strong>9.7.1</strong> Find the inverse of a matrix.</td>
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<td><strong>9.7.2</strong> Solve a system of linear equations using an inverse matrix.</td>
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Nancy plans to invest $10,500 into two different bonds to spread out her risk. The first bond has an annual return of 10%, and the second bond has an annual return of 6%. In order to receive an 8.5% return from the two bonds, how much should Nancy invest in each bond? What is the best method to solve this problem?

There are several ways we can solve this problem. As we have seen in previous sections, systems of equations and matrices are useful in solving real-world problems involving finance. After studying this section, we will have the tools to solve the bond problem using the inverse of a matrix.

**Finding the Inverse of a Matrix**

We know that the multiplicative inverse of a real number \(a\) is \(a^{-1}\), and \(aa^{-1} = a^{-1}a = \left(\frac{1}{a}\right)a = 1\). For example, \(2^{-1} = \frac{1}{2}\) and \(\left(\frac{1}{2}\right)2 = 1\). The multiplicative inverse of a matrix is similar in concept, except that the product of matrix \(A\) and its inverse \(A^{-1}\) equals the identity matrix. The identity matrix is a square matrix containing ones down the main diagonal and zeros everywhere else. We identify identity matrices by \(I_n\) where \(n\) represents the dimension of the matrix. **Equation 9.81** and **Equation 9.82** are the identity matrices for a \(2\times2\) matrix and a \(3\times3\) matrix, respectively.

**Equation 9.81**

\[
I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

**Equation 9.82**

\[
I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

The identity matrix acts as a 1 in matrix algebra. For example, \(AI = IA = A\).

A matrix that has a multiplicative inverse has the properties

\[
AA^{-1} = I \\
A^{-1}A = I
\]

**Equation 9.83**

A matrix that has a multiplicative inverse is called an invertible matrix. Only a square matrix may have a multiplicative inverse, as the reversibility, \(AA^{-1} = A^{-1}A = I\), is a requirement. Not all square matrices have an inverse, but if \(A\) is invertible, then \(A^{-1}\) is unique. We will look at two methods for finding the inverse of a \(2\times2\) matrix and a third method that can be used on both \(2\times2\) and \(3\times3\) matrices.

**The Identity Matrix and Multiplicative Inverse**

The identity matrix, \(I_n\), is a square matrix containing ones down the main diagonal and zeros everywhere else.

**Equation 9.84**

\[
I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

If \(A\) is an \(n \times n\) matrix and \(B\) is an \(n \times n\) matrix such that \(AB = BA = I_n\), then \(B = A^{-1}\), the multiplicative inverse of a matrix \(A\).
Example 9.48

**Showing That the Identity Matrix Acts as a 1**

Given matrix $A$, show that $AI = IA = A$.

$$A = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

Solution

**Example 9.49**

**Showing That Matrix $A$ Is the Multiplicative Inverse of Matrix $B$**

Show that the given matrices are multiplicative inverses of each other.

$$A = \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix}, \quad B = \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix}$$

Solution

**9.28** Show that the following two matrices are inverses of each other.

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \quad (9.85)$$

**Finding the Multiplicative Inverse Using Matrix Multiplication**

We can now determine whether two matrices are inverses, but how would we find the inverse of a given matrix? Since we know that the product of a matrix and its inverse is the identity matrix, we can find the inverse of a matrix by setting up an equation using matrix multiplication.

**Example 9.50**

**Finding the Multiplicative Inverse Using Matrix Multiplication**

Use matrix multiplication to find the inverse of the given matrix.

$$A = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$$

Solution
Finding the Multiplicative Inverse by Augmenting with the Identity

Another way to find the multiplicative inverse is by augmenting with the identity. When matrix $A$ is transformed into $I$, the augmented matrix $I$ transforms into $A^{-1}$.

For example, given

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \quad (9.86)$$

augment $A$ with the identity

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{bmatrix} \quad (9.87)$$

Perform row operations with the goal of turning $A$ into the identity.

1. Switch row 1 and row 2.

$$\begin{bmatrix} 5 & 3 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix} \quad (9.88)$$

2. Multiply row 2 by $-2$ and add to row 1.

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix} \quad (9.89)$$

3. Multiply row 1 by $-2$ and add to row 2.

$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ 0 & -1 & 5 & -2 \end{bmatrix} \quad (9.90)$$

4. Add row 2 to row 1.

$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & -1 & 5 & -2 \end{bmatrix} \quad (9.91)$$

5. Multiply row 2 by $-1$.

$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -5 & 2 \end{bmatrix} \quad (9.92)$$

The matrix we have found is $A^{-1}$.

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \quad (9.93)$$

Finding the Multiplicative Inverse of $2\times2$ Matrices Using a Formula

When we need to find the multiplicative inverse of a $2 \times 2$ matrix, we can use a special formula instead of using matrix multiplication or augmenting with the identity.

If $A$ is a $2 \times 2$ matrix, such as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (9.94)$$

the multiplicative inverse of $A$ is given by the formula

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (9.95)$$

where $ad-bc \neq 0$. If $ad-bc = 0$, then $A$ has no inverse.

Example 9.51

Using the Formula to Find the Multiplicative Inverse of Matrix $A$

Use the formula to find the multiplicative inverse of
\[ A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \]

**Solution**

**Analysis**

We can check that our formula works by using one of the other methods to calculate the inverse. Let’s augment \( A \) with the identity.

\[
\begin{bmatrix}
1 & -2 & | & 1 & 0 \\
2 & -3 & | & 0 & 1
\end{bmatrix}
\]

Perform row operations with the goal of turning \( A \) into the identity.

1. Multiply row 1 by \(-2\) and add to row 2.

\[
\begin{bmatrix}
1 & -2 & | & 1 & 0 \\
0 & 1 & | & -2 & 1
\end{bmatrix}
\]

2. Multiply row 1 by 2 and add to row 1.

\[
\begin{bmatrix}
1 & 0 & | & -3 & 2 \\
0 & 1 & | & -2 & 1
\end{bmatrix}
\]

So, we have verified our original solution.

\[ A^{-1} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \]

**Try It** 9.29 Use the formula to find the inverse of matrix \( A \). Verify your answer by augmenting with the identity matrix.

\[ A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \] (9.96)

**Example 9.52**

**Finding the Inverse of the Matrix, If It Exists**

Find the inverse, if it exists, of the given matrix.

\[ A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} \]

**Solution**

**Finding the Multiplicative Inverse of 3×3 Matrices**

Unfortunately, we do not have a formula similar to the one for a 2×2 matrix to find the inverse of a 3×3 matrix. Instead, we will augment the original matrix with the identity matrix and use row operations to obtain the inverse.

Given a 3×3 matrix

\[ A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \] (9.97)

augment \( A \) with the identity matrix
To begin, we write the augmented matrix with the identity on the right and \( A \) on the left. Performing elementary row operations so that the identity matrix appears on the left, we will obtain the inverse matrix on the right. We will find the inverse of this matrix in the next example.

**Given a 3 \times 3 matrix, find the inverse**

1. Write the original matrix augmented with the identity matrix on the right.
2. Use elementary row operations so that the identity appears on the left.
3. What is obtained on the right is the inverse of the original matrix.
4. Use matrix multiplication to show that \( AA^{-1} = I \) and \( A^{-1}A = I \).

### Example 9.53

**Finding the Inverse of a 3 \times 3 Matrix**

Given the 3 \times 3 matrix \( A \), find the inverse.

\[
A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}
\]

**Solution**

**Analysis**

To prove that \( B = A^{-1} \), let’s multiply the two matrices together to see if the product equals the identity, if \( AA^{-1} = I \) and \( A^{-1}A = I \).

\[
AA^{-1} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} = \begin{bmatrix} -1 \cdot (-1) + 3(-1) + 1(6) & 2(1) + 3(0) + 1(-2) & 2(0) + 3(1) + 1(-3) \\ -1 \cdot (-1) + 3(-1) + 1(6) & 3(1) + 3(0) + 1(-2) & 3(0) + 3(1) + 1(-3) \\ 2(-1) + 4(-1) + 1(6) & 2(1) + 4(0) + 1(-2) & 2(0) + 4(1) + 1(-3) \end{bmatrix}
\]

\[
A^{-1}A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} -1 \cdot (2) + 1(3) + 0(2) & -1(3) + 1(3) + 0(4) & -1(1) + 1(1) + 0(1) \\ -1 \cdot (2) + 0(3) + 1(2) & -1(3) + 0(3) + 1(4) & -1(1) + 0(1) + 1(1) \\ 6(2) + -2(3) + -3(2) & 6(3) + -2(3) + -3(4) & 6(1) + -2(1) + -3(1) \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
Solving a System of Linear Equations Using the Inverse of a Matrix

Solving a system of linear equations using the inverse of a matrix requires the definition of two new matrices: \( X \) is the matrix representing the variables of the system, and \( B \) is the matrix representing the constants. Using matrix multiplication, we may define a system of equations with the same number of equations as variables as

\[
AX = B
\]

To solve a system of linear equations using an inverse matrix, let \( A \) be the coefficient matrix, let \( X \) be the variable matrix, and let \( B \) be the constant matrix. Thus, we want to solve a system \( AX = B \). For example, look at the following system of equations.

\[
\begin{align*}
a_1 x + b_1 y &= c_1 \\
a_2 x + b_2 y &= c_2
\end{align*}
\]

From this system, the coefficient matrix is

\[
A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}
\]

The variable matrix is

\[
X = \begin{bmatrix} x \\ y \end{bmatrix}
\]

And the constant matrix is

\[
B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}
\]

Then \( AX = B \) looks like

\[
\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}
\]

Recall the discussion earlier in this section regarding multiplying a real number by its inverse, \((2^{-1}) 2 = \left(\frac{1}{2}\right)2 = 1\). To solve a single linear equation \( ax = b \) for \( x \), we would simply multiply both sides of the equation by the multiplicative inverse (reciprocal) of \( a \). Thus,

\[
\begin{align*}
ax &= b \\
\left(\frac{1}{a}\right)ax &= \left(\frac{1}{a}\right)b \\
(a^{-1})x &= (a^{-1})b \\
[x(a^{-1})]x &= (a^{-1})b \\
x &= (a^{-1})b
\end{align*}
\]

The only difference between solving a linear equation and a system of equations written in matrix form is that finding the inverse of a matrix is more complicated, and matrix multiplication is a longer process. However, the goal is the same—to isolate the variable.

We will investigate this idea in detail, but it is helpful to begin with a \( 2 \times 2 \) system and then move on to a \( 3 \times 3 \) system.

**Solving a System of Equations Using the Inverse of a Matrix**

Given a system of equations, write the coefficient matrix \( A \), the variable matrix \( X \), and the constant matrix \( B \). Then

\[
AX = B
\]
Multiply both sides by the inverse of $A$ to obtain the solution.

\[
(A^{-1})AX = (A^{-1})B \tag{9.108}
\]

\[
[(A^{-1})A]X = (A^{-1})B \]

\[
IX = (A^{-1})B \]

\[
X = (A^{-1})B
\]

**If the coefficient matrix does not have an inverse, does that mean the system has no solution?**

No, if the coefficient matrix is not invertible, the system could be inconsistent and have no solution, or be dependent and have infinitely many solutions.

---

**Example 9.54**

**Solving a 2 × 2 System Using the Inverse of a Matrix**

Solve the given system of equations using the inverse of a matrix.

\[
\begin{align*}
3x + 8y &= 5 \\
4x + 11y &= 7
\end{align*}
\]

**Solution**

Can we solve for $X$ by finding the product $BA^{-1}$?

No, recall that matrix multiplication is not commutative, so $A^{-1}B \neq BA^{-1}$. Consider our steps for solving the matrix equation.

\[
(A^{-1})AX = (A^{-1})B \tag{9.109}
\]

\[
[(A^{-1})A]X = (A^{-1})B \]

\[
IX = (A^{-1})B \]

\[
X = (A^{-1})B
\]

Notice in the first step we multiplied both sides of the equation by $A^{-1}$, but the $A^{-1}$ was to the left of $A$ on the left side and to the left of $B$ on the right side. Because matrix multiplication is not commutative, order matters.

---

**Example 9.55**

**Solving a 3 × 3 System Using the Inverse of a Matrix**

Solve the following system using the inverse of a matrix.

\[
\begin{align*}
5x + 15y + 56z &= 35 \\
-4x - 11y - 41z &= -26 \\
-x - 3y - 11z &= -7
\end{align*}
\]

**Solution**
Solution

Try It 9.31 Solve the system using the inverse of the coefficient matrix.

\[
\begin{align*}
2x - 17y + 11z &= 0 \\
-x + 11y - 7z &= 8 \\
3y - 2z &= -2
\end{align*}
\] (9.10)

Given a system of equations, solve with matrix inverses using a calculator.

1. Save the coefficient matrix and the constant matrix as matrix variables \([A]\) and \([B]\).
2. Enter the multiplication into the calculator, calling up each matrix variable as needed.
3. If the coefficient matrix is invertible, the calculator will present the solution matrix; if the coefficient matrix is not invertible, the calculator will present an error message.

Example 9.56

Using a Calculator to Solve a System of Equations with Matrix Inverses

Solve the system of equations with matrix inverses using a calculator

\[
\begin{align*}
2x + 3y + z &= 32 \\
3x + 3y + z &= -27 \\
2x + 4y + z &= -2
\end{align*}
\]

Solution

Access these online resources for additional instruction and practice with solving systems with inverses.

- The Identity Matrix (http://openstaxcollege.org/l/identmatrix)
- Determining Inverse Matrices (http://openstaxcollege.org/l/inversematrix)
- Using a Matrix Equation to Solve a System of Equations (http://openstaxcollege.org/l/matrixsystem)
9.7 EXERCISES

Verbal

385. In a previous section, we showed that matrix multiplication is not commutative, that is, \( AB \neq BA \) in most cases. Can you explain why matrix multiplication is commutative for matrix inverses, that is, \( A^{-1} A = AA^{-1} \)?

386. Does every \( 2 \times 2 \) matrix have an inverse? Explain why or why not. Explain what condition is necessary for an inverse to exist.

387. Can you explain whether a \( 2 \times 2 \) matrix with an entire row of zeros can have an inverse?

388. Can a matrix with an entire column of zeros have an inverse? Explain why or why not.

389. Can a matrix with zeros on the diagonal have an inverse? If so, find an example. If not, prove why not. For simplicity, assume a \( 2 \times 2 \) matrix.

Algebraic

In the following exercises, show that matrix \( A \) is the inverse of matrix \( B \).

390. \( A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)

391. \( A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix} \)

392. \( A = \begin{bmatrix} 4 & 5 \\ 7 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1/7 \\ 1/5 & -4/35 \end{bmatrix} \)

393. \( A = \begin{bmatrix} -2 & 1/2 \\ 3 & -1 \end{bmatrix}, B = \begin{bmatrix} -2 & -1 \\ -6 & -4 \end{bmatrix} \)

394. \( A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \)

395. \( A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 2 \\ 1 & 6 & 9 \end{bmatrix}, B = \begin{bmatrix} 1/2 & 6 & 0 & -2 \\ 7 & -3 & -5 \\ 12 & 2 & 4 \end{bmatrix} \)

396. \( A = \begin{bmatrix} 3 & 8 & 2 \\ 1 & 1 & 1 \\ 5 & 6 & 12 \end{bmatrix}, B = \begin{bmatrix} 1/36 & -6 & 84 & -6 \\ 7 & -26 & 1 \\ -1 & -22 & 5 \end{bmatrix} \)

For the following exercises, find the multiplicative inverse of each matrix, if it exists.

397. \( \begin{bmatrix} 3 & -2 \\ 1 & 9 \end{bmatrix} \)

398. \( \begin{bmatrix} -2 & 2 \\ 3 & 1 \end{bmatrix} \)

399. \( \begin{bmatrix} -3 & 7 \\ 9 & 2 \end{bmatrix} \)

400. \( \begin{bmatrix} -4 & -3 \\ -5 & 8 \end{bmatrix} \)
For the following exercises, solve the system using the inverse of a 2 x 2 matrix.

411. \[
\begin{align*}
5x - 6y &= -61 \\
4x + 3y &= -2
\end{align*}
\]

412. \[
\begin{align*}
8x + 4y &= -100 \\
3x - 4y &= 1
\end{align*}
\]

413. \[
\begin{align*}
3x - 2y &= 6 \\
-x + 5y &= -2
\end{align*}
\]

414. \[
\begin{align*}
5x - 4y &= -5 \\
4x + y &= 2.3
\end{align*}
\]

415. \[
\begin{align*}
-3x - 4y &= 9 \\
12x + 4y &= -6
\end{align*}
\]
\[-2x + 3y = \frac{3}{10}\]
\[-x + 5y = \frac{1}{2}\]

417. \[\frac{8}{5}x - \frac{4}{5}y = \frac{2}{5}\]
\[-\frac{8}{5}x + \frac{1}{5}y = \frac{7}{10}\]

418. \[\frac{1}{2}x + \frac{1}{5}y = -\frac{1}{4}\]
\[\frac{1}{2}x - \frac{3}{5}y = -\frac{9}{4}\]

For the following exercises, solve a system using the inverse of a 3x3 matrix.

419. \[3x - 2y + 5z = 21\]
\[5x + 4y = 37\]
\[x - 2y - 5z = 5\]

420. \[4x + 4y + 4z = 40\]
\[2x - 3y + 4z = -12\]
\[-x + 3y + 4z = 9\]

421. \[6x - 5y - z = 31\]
\[-x + 2y + z = -6\]
\[3x + 3y + 2z = 13\]

422. \[6x - 5y + 2z = -4\]
\[2x + 5y - z = 12\]
\[2x + 5y + z = 12\]

423. \[4x - 2y + 3z = -12\]
\[2x + 2y - 9z = 33\]
\[6y - 4z = 1\]

424. \[\frac{1}{10}x - \frac{1}{5}y + 4z = -\frac{41}{2}\]
\[\frac{1}{3}x - 20y + \frac{2}{5}z = -101\]
\[\frac{3}{10}x + 4y - \frac{3}{10}z = 23\]

425. \[\frac{1}{2}x - \frac{1}{5}y + \frac{1}{5}z = \frac{31}{100}\]
\[-\frac{3}{4}x - \frac{1}{4}y + \frac{1}{2}z = \frac{7}{40}\]
\[-\frac{4}{5}x - \frac{1}{5}y + \frac{3}{2}z = \frac{1}{4}\]

426. \[0.1x + 0.2y + 0.3z = -1.4\]
\[0.1x - 0.2y + 0.3z = 0.6\]
\[0.4y + 0.9z = -2\]

**Technology**

For the following exercises, use a calculator to solve the system of equations with matrix inverses.

427. \[2x - y = -3\]
\[-x + 2y = 2.3\]
428. \[-\frac{1}{2}x - \frac{3}{2}y = -\frac{43}{20} \]
\[\frac{5}{2}x + \frac{11}{5}y = \frac{31}{4} \]

429. \[12.3x - 2y - 2.5z = 2 \]
\[36.9x + 7y - 7.5z = -7 \]
\[8y - 5z = -10 \]

430. \[0.5x - 3y + 6z = -0.8 \]
\[0.7x - 2y = -0.06 \]
\[0.5x + 4y + 5z = 0 \]

**Extensions**

For the following exercises, find the inverse of the given matrix.

431. \[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]

432. \[
\begin{bmatrix}
-1 & 0 & 2 & 5 \\
0 & 0 & 0 & 2 \\
0 & 2 & -1 & 0 \\
1 & -3 & 0 & 1
\end{bmatrix}
\]

433. \[
\begin{bmatrix}
1 & -2 & 3 & 0 \\
0 & 1 & 0 & 2 \\
1 & 4 & -2 & 3 \\
-5 & 0 & 1 & 1
\end{bmatrix}
\]

434. \[
\begin{bmatrix}
1 & 2 & 0 & 2 & 3 \\
0 & 2 & 1 & 0 & 0 \\
0 & 0 & 3 & 0 & 1 \\
0 & 2 & 0 & 0 & 1 \\
0 & 0 & 1 & 2 & 0
\end{bmatrix}
\]

435. \[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]

**Real-World Applications**

For the following exercises, write a system of equations that represents the situation. Then, solve the system using the inverse of a matrix.

436. 2,400 tickets were sold for a basketball game. If the prices for floor 1 and floor 2 were different, and the total amount of money brought in is $64,000, how much was the price of each ticket?

437. In the previous exercise, if you were told there were 400 more tickets sold for floor 2 than floor 1, how much was the price of each ticket?

438. A food drive collected two different types of canned goods, green beans and kidney beans. The total number of collected cans was 350 and the total weight of all donated food was 348 lb, 12 oz. If the green bean cans weigh 2 oz less than the kidney bean cans, how many of each can was donated?

439. Students were asked to bring their favorite fruit to class. 95% of the fruits consisted of banana, apple, and oranges. If oranges were twice as popular as bananas, and apples were 5% less popular than bananas, what are the percentages of each individual fruit?
440. A sorority held a bake sale to raise money and sold brownies and chocolate chip cookies. They priced the brownies at $1 and the chocolate chip cookies at $0.75. They raised $700 and sold 850 items. How many brownies and how many cookies were sold?

441. A clothing store needs to order new inventory. It has three different types of hats for sale: straw hats, beanies, and cowboy hats. The straw hat is priced at $13.99, the beanie at $7.99, and the cowboy hat at $14.49. If 100 hats were sold this past quarter, $1,119 was taken in by sales, and the amount of beanies sold was 10 more than cowboy hats, how many of each should the clothing store order to replace those already sold?

442. Anna, Ashley, and Andrea weigh a combined 370 lb. If Andrea weighs 20 lb more than Ashley, and Anna weighs 1.5 times as much as Ashley, how much does each girl weigh?

443. Three roommates shared a package of 12 ice cream bars, but no one remembers who ate how many. If Tom ate twice as many ice cream bars as Joe, and Albert ate three less than Tom, how many ice cream bars did each roommate eat?

444. A farmer constructed a chicken coop out of chicken wire, wood, and plywood. The chicken wire cost $2 per square foot, the wood $10 per square foot, and the plywood $5 per square foot. The farmer spent a total of $51, and the total amount of materials used was 14 ft². He used 3 ft² more chicken wire than plywood. How much of each material in did the farmer use?

445. Jay has lemon, orange, and pomegranate trees in his backyard. An orange weighs 8 oz, a lemon 5 oz, and a pomegranate 11 oz. Jay picked 142 pieces of fruit weighing a total of 70 lb, 10 oz. He picked 15.5 times more oranges than pomegranates. How many of each fruit did Jay pick?
9.8 | Solving Systems with Cramer's Rule

Learning Objectives

In this section, you will:

9.8.1 Evaluate 2 × 2 determinants.
9.8.2 Use Cramer's Rule to solve a system of equations in two variables.
9.8.3 Evaluate 3 × 3 determinants.
9.8.4 Use Cramer's Rule to solve a system of three equations in three variables.
9.8.5 Know the properties of determinants.

We have learned how to solve systems of equations in two variables and three variables, and by multiple methods: substitution, addition, Gaussian elimination, using the inverse of a matrix, and graphing. Some of these methods are easier to apply than others and are more appropriate in certain situations. In this section, we will study two more strategies for solving systems of equations.

Evaluating the Determinant of a 2×2 Matrix

A determinant is a real number that can be very useful in mathematics because it has multiple applications, such as calculating area, volume, and other quantities. Here, we will use determinants to reveal whether a matrix is invertible by using the entries of a square matrix to determine whether there is a solution to the system of equations. Perhaps one of the more interesting applications, however, is their use in cryptography. Secure signals or messages are sometimes sent encoded in a matrix. The data can only be decrypted with an invertible matrix and the determinant. For our purposes, we focus on the determinant as an indication of the invertibility of the matrix. Calculating the determinant of a matrix involves following the specific patterns that are outlined in this section.

Find the Determinant of a 2 × 2 Matrix

The determinant of a 2 × 2 matrix, given

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

(9.111)

is defined as

\[
\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb
\]

Notice the change in notation. There are several ways to indicate the determinant, including \(\det(A)\) and replacing the brackets in a matrix with straight lines, \(|A|\).

Example 9.57

Finding the Determinant of a 2 × 2 Matrix

Find the determinant of the given matrix.

\[
A = \begin{bmatrix} 5 & 2 \\ -6 & 3 \end{bmatrix}
\]

Solution
Using Cramer’s Rule to Solve a System of Two Equations in Two Variables

We will now introduce a final method for solving systems of equations that uses determinants. Known as Cramer’s Rule, this technique dates back to the middle of the 18th century and is named for its innovator, the Swiss mathematician Gabriel Cramer (1704-1752), who introduced it in 1750 in *Introduction à l’Analyse des lignes Courbes algébriques*. Cramer’s Rule is a viable and efficient method for finding solutions to systems with an arbitrary number of unknowns, provided that we have the same number of equations as unknowns.

Cramer’s Rule will give us the unique solution to a system of equations, if it exists. However, if the system has no solution or an infinite number of solutions, this will be indicated by a determinant of zero. To find out if the system is inconsistent or dependent, another method, such as elimination, will have to be used.

To understand Cramer’s Rule, let’s look closely at how we solve systems of linear equations using basic row operations. Consider a system of two equations in two variables.

\[ \begin{align*}
  a_1x + b_1y &= c_1 \quad (1) \\
  a_2x + b_2y &= c_2 \quad (2)
\end{align*} \]

We eliminate one variable using row operations and solve for the other. Say that we wish to solve for \( x \). If equation (2) is multiplied by the opposite of the coefficient of \( y \) in equation (1), equation (1) is multiplied by the coefficient of \( y \) in equation (2), and we add the two equations, the variable \( y \) will be eliminated.

\[ \begin{align*}
  b_2a_1x + b_2b_1y &= b_2c_1 \\
  -b_1a_2x - b_1b_2y &= -b_1c_2
\end{align*} \]

Multiply \( R_1 \) by \( b_2 \) \hspace{1cm} \text{Multiply} \hspace{0.5cm} R_2 \hspace{0.5cm} \text{by} \hspace{0.5cm} -b_1

\[ b_2a_1x - b_1a_2x = b_2c_1 - b_1c_2 \]

Now, solve for \( x \).

\[ x = \frac{b_2c_1 - b_1c_2}{b_2a_1 - b_1a_2} = \frac{\begin{bmatrix} c_1 & b_1 \\ b_2 & b_2 \end{bmatrix}}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}} \]

Similarly, to solve for \( y \), we will eliminate \( x \).

\[ \begin{align*}
  a_2a_1x + a_2b_1y &= a_2c_1 \\
  -a_1a_2x - a_1b_2y &= -a_1c_2
\end{align*} \]

Multiply \( R_1 \) by \( a_2 \) \hspace{1cm} \text{Multiply} \hspace{0.5cm} R_2 \hspace{0.5cm} \text{by} \hspace{0.5cm} -a_1

\[ a_2b_1y - a_1b_2y = a_2c_1 - a_1c_2 \]

Solving for \( y \) gives

\[ y = \frac{a_2c_1 - a_1c_2}{a_2b_1 - a_1b_2} = \frac{\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}} \]

Notice that the denominator for both \( x \) and \( y \) is the determinant of the coefficient matrix.

We can use these formulas to solve for \( x \) and \( y \), but Cramer’s Rule also introduces new notation:

- \( D \): determinant of the coefficient matrix
- \( D_x \): determinant of the numerator in the solution of \( x \)
- \( D_y \): determinant of the numerator in the solution of \( y \)

\[ x = \frac{D_x}{D} \hspace{1cm} (9.117) \]

\[ y = \frac{D_y}{D} \]
The key to Cramer’s Rule is replacing the variable column of interest with the constant column and calculating the determinants. We can then express \( x \) and \( y \) as a quotient of two determinants.

Cramer’s Rule for 2×2 Systems

Cramer’s Rule is a method that uses determinants to solve systems of equations that have the same number of equations as variables.

Consider a system of two linear equations in two variables.

\[
\begin{align*}
 a_1 x + b_1 y &= c_1 \\
 a_2 x + b_2 y &= c_2
\end{align*}
\]

The solution using Cramer’s Rule is given as

\[
\begin{align*}
 x &= \frac{D_x}{D} = \frac{c_1 b_2 - b_1 c_2}{D} \quad D \neq 0; \\
 y &= \frac{D_y}{D} = \frac{a_1 c_2 - a_2 c_1}{D} \quad D \neq 0.
\end{align*}
\]

If we are solving for \( x \), the \( x \) column is replaced with the constant column. If we are solving for \( y \), the \( y \) column is replaced with the constant column.

Example 9.58

Using Cramer’s Rule to Solve a 2 × 2 System

Solve the following 2 × 2 system using Cramer’s Rule.

\[
\begin{align*}
 12x + 3y &= 15 \\
 2x - 3y &= 13
\end{align*}
\]

Solution

9.32 Use Cramer’s Rule to solve the 2 × 2 system of equations.

\[
\begin{align*}
 x + 2y &= -11 \\
 -2x + y &= -13
\end{align*}
\]

Evaluating the Determinant of a 3 × 3 Matrix

Finding the determinant of a 2×2 matrix is straightforward, but finding the determinant of a 3×3 matrix is more complicated. One method is to augment the 3×3 matrix with a repetition of the first two columns, giving a 3×5 matrix. Then we calculate the sum of the products of entries down each of the three diagonals (upper left to lower right), and subtract the products of entries up each of the three diagonals (lower left to upper right). This is more easily understood with a visual and an example.

Find the determinant of the 3×3 matrix.

\[
A = \begin{bmatrix}
 a_1 & b_1 & c_1 \\
 a_2 & b_2 & c_2 \\
 a_3 & b_3 & c_3
\end{bmatrix}
\]

1. Augment \( A \) with the first two columns.
2. From upper left to lower right: Multiply the entries down the first diagonal. Add the result to the product of entries down the second diagonal. Add this result to the product of the entries down the third diagonal.

3. From lower left to upper right: Subtract the product of entries up the first diagonal. From this result subtract the product of entries up the second diagonal. From this result, subtract the product of entries up the third diagonal.

\[
\text{det}(A) = \begin{vmatrix} 
|a_1 & b_1 & c_1| \\
|a_2 & b_2 & c_2| \\
|a_3 & b_3 & c_3|
\end{vmatrix}
\]

The algebra is as follows:

\[ |A| = a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - a_3 b_2 c_1 - b_3 c_2 a_1 - c_3 a_2 b_1 \]  

\[ (9.124) \]

### Example 9.59

**Finding the Determinant of a 3 × 3 Matrix**

Find the determinant of the 3 × 3 matrix given

\[
A = \begin{bmatrix} 
0 & 2 & 1 \\
3 & -1 & 1 \\
4 & 0 & 1
\end{bmatrix}
\]

**Solution**

Find the determinant of the 3 × 3 matrix.

\[
\text{det}(A) = \begin{vmatrix} 
1 & -3 & 7 \\
1 & 1 & 1 \\
1 & -2 & 3
\end{vmatrix}
\]

\[ (9.125) \]

**Can we use the same method to find the determinant of a larger matrix?**

No, this method only works for 2 × 2 and 3 × 3 matrices. For larger matrices it is best to use a graphing utility or computer software.

**Using Cramer’s Rule to Solve a System of Three Equations in Three Variables**

Now that we can find the determinant of a 3 × 3 matrix, we can apply Cramer’s Rule to solve a system of three equations in three variables. Cramer’s Rule is straightforward, following a pattern consistent with Cramer’s Rule for 2 × 2 matrices. As the order of the matrix increases to 3 × 3, however, there are many more calculations required.

When we calculate the determinant to be zero, Cramer’s Rule gives no indication as to whether the system has no solution or an infinite number of solutions. To find out, we have to perform elimination on the system.

Consider a 3 × 3 system of equations.
\[
a_{1}x + \bar{b}_1 y + c_1 z = d_1 \\
a_{2}x + \bar{b}_2 y + c_2 z = d_2 \\
a_{3}x + \bar{b}_3 y + c_3 z = d_3
\]

\[
x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D}, \quad D \neq 0 \quad (9.126)
\]

where

\[
D = \begin{vmatrix}
    a_1 & b_1 & c_1 \\
    a_2 & b_2 & c_2 \\
    a_3 & b_3 & c_3
\end{vmatrix}, \quad D_x = \begin{vmatrix}
    d_1 & b_1 & c_1 \\
    d_2 & b_2 & c_2 \\
    d_3 & b_3 & c_3
\end{vmatrix}, \quad D_y = \begin{vmatrix}
    a_1 & d_1 & c_1 \\
    a_2 & d_2 & c_2 \\
    a_3 & d_3 & c_3
\end{vmatrix}, \quad D_z = \begin{vmatrix}
    a_1 & b_1 & d_1 \\
    a_2 & b_2 & d_2 \\
    a_3 & b_3 & d_3
\end{vmatrix}
\]

If we are writing the determinant \( D_x \), we replace the \( x \) column with the constant column. If we are writing the determinant \( D_y \), we replace the \( y \) column with the constant column. If we are writing the determinant \( D_z \), we replace the \( z \) column with the constant column. Always check the answer.

Example 9.60

Solving a 3 \times 3 System Using Cramer’s Rule

Find the solution to the given 3 \times 3 system using Cramer’s Rule.

\[
\begin{align*}
x + y - z &= 6 \\
3x - 2y + z &= -5 \\
x + 3y - 2z &= 14
\end{align*}
\]

Solution

Example 9.61

Using Cramer’s Rule to Solve an Inconsistent System

Solve the system of equations using Cramer’s Rule.

\[
\begin{align*}
3x - 2y &= 4 \quad (1) \\
6x - 4y &= 0 \quad (2)
\end{align*}
\]

Solution

Example 9.62
Use Cramer’s Rule to Solve a Dependent System

Solve the system with an infinite number of solutions.

\[
\begin{align*}
  x - 2y + 3z &= 0 \quad (1) \\
  3x + y - 2z &= 0 \quad (2) \\
  2x - 4y + 6z &= 0 \quad (3)
\end{align*}
\]

Solution

Understanding Properties of Determinants

There are many properties of determinants. Listed here are some properties that may be helpful in calculating the determinant of a matrix.

<table>
<thead>
<tr>
<th>Properties of Determinants</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. If the matrix is in upper triangular form, the determinant equals the product of entries down the main diagonal.</td>
</tr>
<tr>
<td>2. When two rows are interchanged, the determinant changes sign.</td>
</tr>
<tr>
<td>3. If either two rows or two columns are identical, the determinant equals zero.</td>
</tr>
<tr>
<td>4. If a matrix contains either a row of zeros or a column of zeros, the determinant equals zero.</td>
</tr>
<tr>
<td>5. The determinant of an inverse matrix $A^{-1}$ is the reciprocal of the determinant of the matrix $A$.</td>
</tr>
<tr>
<td>6. If any row or column is multiplied by a constant, the determinant is multiplied by the same factor.</td>
</tr>
</tbody>
</table>

Example 9.63

Illustrating Properties of Determinants

Illustrate each of the properties of determinants.

Solution

Example 9.64

Using Cramer’s Rule and Determinant Properties to Solve a System

Find the solution to the given $3 \times 3$ system.

\[
\begin{align*}
  2x + 4y + 4z &= 2 \quad (1) \\
  3x + 7y + 7z &= -5 \quad (2) \\
  x + 2y + 2z &= 4 \quad (3)
\end{align*}
\]

Solution
Access these online resources for additional instruction and practice with Cramer’s Rule.

- Solve a System of Two Equations Using Cramer’s Rule (http://openstaxcollege.org/l/system2cramer)
- Solve a Systems of Three Equations using Cramer’s Rule (http://openstaxcollege.org/l/system3cramer)
9.8 EXERCISES

Verbal

446. Explain why we can always evaluate the determinant of a square matrix.

447. Examining Cramer’s Rule, explain why there is no unique solution to the system when the determinant of your matrix is 0. For simplicity, use a $2 \times 2$ matrix.

448. Explain what it means in terms of an inverse for a matrix to have a 0 determinant.

449. The determinant of $2 \times 2$ matrix $A$ is 3. If you switch the rows and multiply the first row by 6 and the second row by 2, explain how to find the determinant and provide the answer.

Algebraic

For the following exercises, find the determinant.

450. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

451. $\begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix}$

452. $\begin{vmatrix} 2 & -5 \\ -1 & 6 \end{vmatrix}$

453. $\begin{vmatrix} -8 & 4 \\ -1 & 5 \end{vmatrix}$

454. $\begin{vmatrix} 1 & 0 \\ 3 & -4 \end{vmatrix}$

455. $\begin{vmatrix} 10 & 20 \\ 0 & -10 \end{vmatrix}$

456. $\begin{vmatrix} 10 & 0.2 \\ 5 & 0.1 \end{vmatrix}$

457. $\begin{vmatrix} 6 & -3 \\ 8 & 4 \end{vmatrix}$

458. $\begin{vmatrix} -2 & -3 \\ 3.1 & 4 \end{vmatrix}$

459. $\begin{vmatrix} -1.1 & 0.6 \\ 7.2 & -0.5 \end{vmatrix}$

460. $\begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{vmatrix}$

461. $\begin{vmatrix} -1 & 4 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{vmatrix}$

462. $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$

463.
For the following exercises, solve the system of linear equations using Cramer’s Rule.

464. \[
\begin{vmatrix}
2 & -3 & 1 \\
3 & -4 & 1 \\
5 & 6 & 1 \\
\end{vmatrix}
\]

465. \[
\begin{vmatrix}
2 & 1 & 4 \\
4 & 2 & -8 \\
2 & -8 & -3 \\
\end{vmatrix}
\]

466. \[
\begin{vmatrix}
5 & 1 & -1 \\
2 & 3 & 1 \\
3 & -6 & -3 \\
\end{vmatrix}
\]

467. \[
\begin{vmatrix}
1.1 & 2 & -1 \\
-4 & 0 & 0 \\
4.1 & -0.4 & 2.5 \\
\end{vmatrix}
\]

468. \[
\begin{vmatrix}
2 & -1.6 & 3.1 \\
1.1 & 3 & -8 \\
-9.3 & 0 & 2 \\
\end{vmatrix}
\]

469. \[
\begin{vmatrix}
-1 & 1 & 1 \\
2 & 3 & 4 \\
1 & -1 & 1 \\
1 & 5 & -1 \\
\end{vmatrix}
\]

For the following exercises, solve the system of linear equations using Cramer’s Rule.

470. \[
2x - 3y = -1 \\
4x + 5y = 9
\]

471. \[
5x - 4y = 2 \\
-4x + 7y = 6
\]

472. \[
6x - 3y = 2 \\
-8x + 9y = -1
\]

473. \[
2x + 6y = 12 \\
5x - 2y = 13
\]

474. \[
4x + 3y = 23 \\
2x - y = -1
\]

475. \[
10x - 6y = 2 \\
-5x + 8y = -1
\]

476. \[
4x - 3y = -3 \\
2x + 6y = -4
\]

477. \[
4x - 5y = 7 \\
-3x + 9y = 0
\]

478. \[
4x + 10y = 180 \\
-3x - 5y = -105
\]
For the following exercises, solve the system of linear equations using Cramer’s Rule.

479. \[ \begin{align*}
8x - 2y &= -3 \\
-4x + 6y &= 4
\end{align*} \]

480. \[ \begin{align*}
x + 2y - 4z &= -1 \\
7x + 3y + 5z &= 26 \\
-2x - 6y + 7z &= -6
\end{align*} \]

481. \[ \begin{align*}
-5x + 2y - 4z &= -47 \\
4x - 3y - z &= -94 \\
3x - 3y + 2z &= 94
\end{align*} \]

482. \[ \begin{align*}
4x + 5y - z &= -7 \\
-2x - 9y + 2z &= 8 \\
5y + 7z &= 21
\end{align*} \]

483. \[ \begin{align*}
4x - 3y + 4z &= 10 \\
5x - 2z &= -2 \\
3x + 2y - 5z &= -9
\end{align*} \]

484. \[ \begin{align*}
4x - 2y + 3z &= 6 \\
-6x + y &= -2 \\
2x + 7y + 8z &= 24
\end{align*} \]

485. \[ \begin{align*}
5x + 2y - z &= 1 \\
-7x - 8y + 3z &= 1.5 \\
6x - 12y + z &= 7
\end{align*} \]

486. \[ \begin{align*}
13x - 17y + 16z &= 73 \\
-11x + 15y + 17z &= 61 \\
46x + 10y - 30z &= -18
\end{align*} \]

487. \[ \begin{align*}
-4x - 3y - 8z &= -7 \\
2x - 9y + 5z &= 0.5 \\
5x - 6y - 5z &= -2
\end{align*} \]

488. \[ \begin{align*}
4x - 6y + 8z &= 10 \\
-2x + 3y - 4z &= -5 \\
x + y + z &= 1
\end{align*} \]

489. \[ \begin{align*}
4x - 6y + 8z &= 10 \\
-2x + 3y - 4z &= -5 \\
12x + 18y - 24z &= -30
\end{align*} \]

**Technology**

For the following exercises, use the determinant function on a graphing utility.

490. \[ \begin{vmatrix}
1 & 0 & 8 & 9 \\
0 & 2 & 1 & 0 \\
1 & 0 & 3 & 0 \\
0 & 2 & 4 & 3
\end{vmatrix} \]

491.
For the following exercises, create a system of linear equations to describe the behavior. Then, calculate the determinant. Will there be a unique solution? If so, find the unique solution.

494. Two numbers add up to 56. One number is 20 less than the other.

495. Two numbers add up to 104. If you add two times the first number plus two times the second number, your total is 208.

496. Three numbers add up to 106. The first number is 3 less than the second number. The third number is 4 more than the first number.

497. Three numbers add to 216. The sum of the first two numbers is 112. The third number is 8 less than the first two numbers combined.

For the following exercises, create a system of linear equations to describe the behavior. Then, solve the system for all solutions using Cramer’s Rule.

498. You invest $10,000 into two accounts, which receive 8% interest and 5% interest. At the end of a year, you had $10,710 in your combined accounts. How much was invested in each account?

499. You invest $80,000 into two accounts, $22,000 in one account, and $58,000 in the other account. At the end of one year, assuming simple interest, you have earned $2,470 in interest. The second account receives half a percent less than twice the interest on the first account. What are the interest rates for your accounts?

500. A movie theater needs to know how many adult tickets and children tickets were sold out of the 1,200 total tickets. If children’s tickets are $5.95, adult tickets are $11.15, and the total amount of revenue was $12,756, how many children’s tickets and adult tickets were sold?

501. A concert venue sells single tickets for $40 each and couple’s tickets for $65. If the total revenue was $18,090 and the 321 tickets were sold, how many single tickets and how many couple’s tickets were sold?

502. You decide to paint your kitchen green. You create the color of paint by mixing yellow and blue paints. You cannot remember how many gallons of each color went into your mix, but you know there were 10 gal total. Additionally, you kept your receipt, and know the total amount spent was $29.50. If each gallon of yellow costs $2.59, and each gallon of blue costs $3.19, how many gallons of each color go into your green mix?

503. You sold two types of scarves at a farmers’ market and would like to know which one was more popular. The total number of scarves sold was 56, the yellow scarf cost $10, and the purple scarf cost $11. If you had total revenue of $583, how many yellow scarves and how many purple scarves were sold?

504. Your garden produced two types of tomatoes, one green and one red. The red weigh 10 oz, and the green weigh 4 oz. You have 30 tomatoes, and a total weight of 13 lb, 14 oz. How many of each type of tomato do you have?

505. At a market, the three most popular vegetables make up 53% of vegetable sales. Corn has 4% higher sales than broccoli, which has 5% more sales than onions. What percentage does each vegetable have in the market share?
506. At the same market, the three most popular fruits make up 37% of the total fruit sold. Strawberries sell twice as much as oranges, and kiwis sell one more percentage point than oranges. For each fruit, find the percentage of total fruit sold.

507. Three bands performed at a concert venue. The first band charged $15 per ticket, the second band charged $45 per ticket, and the final band charged $22 per ticket. There were 510 tickets sold, for a total of $12,700. If the first band had 40 more audience members than the second band, how many tickets were sold for each band?

508. A movie theatre sold tickets to three movies. The tickets to the first movie were $5, the tickets to the second movie were $11, and the third movie was $12. 100 tickets were sold to the first movie. The total number of tickets sold was 642, for a total revenue of $6,774. How many tickets for each movie were sold?

509. Men aged 20–29, 30–39, and 40–49 made up 78% of the population at a prison last year. This year, the same age groups made up 82.08% of the population. The 20–29 age group increased by 20%, the 30–39 age group increased by 2%, and the 40–49 age group decreased to \( \frac{3}{4} \) of their previous population. Originally, the 30–39 age group had 2% more prisoners than the 20–29 age group. Determine the prison population percentage for each age group last year.

510. At a women’s prison down the road, the total number of inmates aged 20–49 totaled 5,525. This year, the 20–29 age group increased by 10%, the 30–39 age group decreased by 20%, and the 40–49 age group doubled. There are now 6,040 prisoners. Originally, there were 500 more in the 30–39 age group than the 20–29 age group. Determine the prison population for each age group last year.

For the following exercises, use this scenario: A health-conscious company decides to make a trail mix out of almonds, dried cranberries, and chocolate-covered cashews. The nutritional information for these items is shown in Table 9.5.

<table>
<thead>
<tr>
<th></th>
<th>Almonds (10)</th>
<th>Cranberries (10)</th>
<th>Cashews (10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fat (g)</td>
<td>6</td>
<td>0.02</td>
<td>7</td>
</tr>
<tr>
<td>Protein (g)</td>
<td>2</td>
<td>0</td>
<td>3.5</td>
</tr>
<tr>
<td>Carbohydrates(g)</td>
<td>3</td>
<td>8</td>
<td>5.5</td>
</tr>
</tbody>
</table>

Table 9.5

511. For the special “low-carb” trail mix, there are 1,000 pieces of mix. The total number of carbohydrates is 425 g, and the total amount of fat is 570.2 g. If there are 200 more pieces of cashews than cranberries, how many of each item is in the trail mix?

512. For the “hiking” mix, there are 1,000 pieces in the mix, containing 390.8 g of fat, and 165 g of protein. If there is the same amount of almonds as cashews, how many of each item is in the trail mix?

513. For the “energy-booster” mix, there are 1,000 pieces in the mix, containing 145 g of protein and 625 g of carbohydrates. If the number of almonds and cashews summed together is equivalent to the amount of cranberries, how many of each item is in the trail mix?
CHAPTER 9 REVIEW

KEY TERMS

addition method: an algebraic technique used to solve systems of linear equations in which the equations are added in a way that eliminates one variable, allowing the resulting equation to be solved for the remaining variable; substitution is then used to solve for the first variable

augmented matrix: a coefficient matrix adjoined with the constant column separated by a vertical line within the matrix brackets

break-even point: the point at which a cost function intersects a revenue function; where profit is zero

Cramer’s Rule: a method for solving systems of equations that have the same number of equations as variables using determinants

coefficient matrix: a matrix that contains only the coefficients from a system of equations

column: a set of numbers aligned vertically in a matrix

consistent system: a system for which there is a single solution to all equations in the system and it is an independent system, or if there are an infinite number of solutions and it is a dependent system

cost function: the function used to calculate the costs of doing business; it usually has two parts, fixed costs and variable costs

dependent system: a system of linear equations in which the two equations represent the same line; there are an infinite number of solutions to a dependent system

determinant: a number calculated using the entries of a square matrix that determines such information as whether there is a solution to a system of equations

entry: an element, coefficient, or constant in a matrix

feasible region: the solution to a system of nonlinear inequalities that is the region of the graph where the shaded regions of each inequality intersect

Gaussian elimination: using elementary row operations to obtain a matrix in row-echelon form

identity matrix: a square matrix containing ones down the main diagonal and zeros everywhere else; it acts as a 1 in matrix algebra

inconsistent system: a system of linear equations with no common solution because they represent parallel lines, which have no point or line in common

independent system: a system of linear equations with exactly one solution pair \((x, y)\)

main diagonal: entries from the upper left corner diagonally to the lower right corner of a square matrix

matrix: a rectangular array of numbers

multiplicative inverse of a matrix: a matrix that, when multiplied by the original, equals the identity matrix

nonlinear inequality: an inequality containing a nonlinear expression

partial fraction decomposition: the process of returning a simplified rational expression to its original form, a sum or difference of simpler rational expressions

partial fractions: the individual fractions that make up the sum or difference of a rational expression before combining them into a simplified rational expression

profit function: the profit function is written as \(P(x) = R(x) - C(x)\), revenue minus cost
revenue function: the function that is used to calculate revenue, simply written as \( R = xp \), where \( x \) = quantity and \( p \) = price

row operations: adding one row to another row, multiplying a row by a constant, interchanging rows, and so on, with the goal of achieving row-echelon form

row-echelon form: after performing row operations, the matrix form that contains ones down the main diagonal and zeros at every space below the diagonal

row-equivalent: two matrices \( A \) and \( B \) are row-equivalent if one can be obtained from the other by performing basic row operations

row: a set of numbers aligned horizontally in a matrix

scalar multiple: an entry of a matrix that has been multiplied by a scalar

solution set: the set of all ordered pairs or triples that satisfy all equations in a system of equations

substitution method: an algebraic technique used to solve systems of linear equations in which one of the two equations is solved for one variable and then substituted into the second equation to solve for the second variable

system of linear equations: a set of two or more equations in two or more variables that must be considered simultaneously.

system of nonlinear equations: a system of equations containing at least one equation that is of degree larger than one

system of nonlinear inequalities: a system of two or more inequalities in two or more variables containing at least one inequality that is not linear

**KEY EQUATIONS**

<table>
<thead>
<tr>
<th>Identity matrix for a 2\times2 matrix</th>
<th>( I_2 = \begin{bmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{bmatrix} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identity matrix for a 3\times3 matrix</td>
<td>( I_3 = \begin{bmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{bmatrix} )</td>
</tr>
<tr>
<td>Multiplicative inverse of a 2\times2 matrix</td>
<td>( A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d &amp; -b \ -c &amp; a \end{bmatrix} ) where ( ad - bc \neq 0 )</td>
</tr>
</tbody>
</table>

Table 9.6

**KEY CONCEPTS**

9.1 Systems of Linear Equations: Two Variables

- A system of linear equations consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.
- The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. See Example 9.1.
- Systems of equations are classified as independent with one solution, dependent with an infinite number of solutions, or inconsistent with no solution.
• One method of solving a system of linear equations in two variables is by graphing. In this method, we graph the equations on the same set of axes. See Example 9.2.
• Another method of solving a system of linear equations is by substitution. In this method, we solve for one variable in one equation and substitute the result into the second equation. See Example 9.3.
• A third method of solving a system of linear equations is by addition, in which we can eliminate a variable by adding opposite coefficients of corresponding variables. See Example 9.4.
• It is often necessary to multiply one or both equations by a constant to facilitate elimination of a variable when adding the two equations together. See Example 9.5, Example 9.6, and Example 9.7.
• Either method of solving a system of equations results in a false statement for inconsistent systems because they are made up of parallel lines that never intersect. See Example 9.8.
• The solution to a system of dependent equations will always be true because both equations describe the same line. See Example 9.9.
• Systems of equations can be used to solve real-world problems that involve more than one variable, such as those relating to revenue, cost, and profit. See Example 9.10 and Example 9.11.

9.2 Systems of Linear Equations: Three Variables

• A solution set is an ordered triple \((x, y, z)\) that represents the intersection of three planes in space. See Example 9.12.
• A system of three equations in three variables can be solved by using a series of steps that forces a variable to be eliminated. The steps include interchanging the order of equations, multiplying both sides of an equation by a nonzero constant, and adding a nonzero multiple of one equation to another equation. See Example 9.13.
• Systems of three equations in three variables are useful for solving many different types of real-world problems. See Example 9.14.
• A system of equations in three variables is inconsistent if no solution exists. After performing elimination operations, the result is a contradiction. See Example 9.15.
• Systems of equations in three variables that are inconsistent could result from three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location.
• A system of equations in three variables is dependent if it has an infinite number of solutions. After performing elimination operations, the result is an identity. See Example 9.16.
• Systems of equations in three variables that are dependent could result from three identical planes, three planes intersecting at a line, or two identical planes that intersect the third on a line.

9.3 Systems of Nonlinear Equations and Inequalities: Two Variables

• There are three possible types of solutions to a system of equations representing a line and a parabola: (1) no solution, the line does not intersect the parabola; (2) one solution, the line is tangent to the parabola; and (3) two solutions, the line intersects the parabola in two points. See Example 9.17.
• There are three possible types of solutions to a system of equations representing a circle and a line: (1) no solution, the line does not intersect the circle; (2) one solution, the line is tangent to the parabola; (3) two solutions, the line intersects the circle in two points. See Example 9.18.
• There are five possible types of solutions to the system of nonlinear equations representing an ellipse and a circle: (1) no solution, the circle and the ellipse do not intersect; (2) one solution, the circle and the ellipse are tangent to each other; (3) two solutions, the circle and the ellipse intersect in two points; (4) three solutions, the circle and ellipse intersect in three places; (5) four solutions, the circle and the ellipse intersect in four points. See Example 9.19.
• An inequality is graphed in much the same way as an equation, except for > or <, we draw a dashed line and shade the region containing the solution set. See Example 9.20.
• Inequalities are solved the same way as equalities, but solutions to systems of inequalities must satisfy both inequalities. See Example 9.21.
9.4 Partial Fractions

- Decompose \( \frac{P(x)}{Q(x)} \) by writing the partial fractions as \( \frac{A}{a_1x + b_1} + \frac{B}{a_2x + b_2} \). Solve by clearing the fractions, expanding the right side, collecting like terms, and setting corresponding coefficients equal to each other, then setting up and solving a system of equations. See Example 9.22.

- The decomposition of \( \frac{P(x)}{Q(x)} \) with repeated linear factors must account for the factors of the denominator in increasing powers. See Example 9.23.

- The decomposition of \( \frac{P(x)}{Q(x)} \) with a nonrepeated irreducible quadratic factor needs a linear numerator over the quadratic factor, as in \( \frac{A}{x} + \frac{Bx + C}{ax^2 + bx + c} \). See Example 9.24.

- In the decomposition of \( \frac{P(x)}{Q(x)} \) where \( Q(x) \) has a repeated irreducible quadratic factor, when the irreducible quadratic factors are repeated, powers of the denominator factors must be represented in increasing powers as

\[
\frac{Ax + B}{(ax^2 + bx + c)^1} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \ldots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}.
\]

See Example 9.25.

9.5 Matrices and Matrix Operations

- A matrix is a rectangular array of numbers. Entries are arranged in rows and columns.

- The dimensions of a matrix refer to the number of rows and the number of columns. A 3×2 matrix has three rows and two columns. See Example 9.26.

- We add and subtract matrices of equal dimensions by adding and subtracting corresponding entries of each matrix. See Example 9.27, Example 9.28, Example 9.29, and Example 9.30.

- Scalar multiplication involves multiplying each entry in a matrix by a constant. See Example 9.31.

- Scalar multiplication is often required before addition or subtraction can occur. See Example 9.32.

- Multiplying matrices is possible when inner dimensions are the same—the number of columns in the first matrix must match the number of rows in the second.

- The product of two matrices, \( A \) and \( B \), is obtained by multiplying each entry in row 1 of \( A \) by each entry in column 1 of \( B \); then multiply each entry of row 1 of \( A \) by each entry in columns 2 of \( B \), and so on. See Example 9.33 and Example 9.34.

- Many real-world problems can often be solved using matrices. See Example 9.35.

- We can use a calculator to perform matrix operations after saving each matrix as a matrix variable. See Example 9.36.

9.6 Solving Systems with Gaussian Elimination

- An augmented matrix is one that contains the coefficients and constants of a system of equations. See Example 9.37.

- A matrix augmented with the constant column can be represented as the original system of equations. See Example 9.38.

- Row operations include multiplying a row by a constant, adding one row to another row, and interchanging rows.

- We can use Gaussian elimination to solve a system of equations. See Example 9.39, Example 9.40, and Example 9.41.

- Row operations are performed on matrices to obtain row-echelon form. See Example 9.42.

- To solve a system of equations, write it in augmented matrix form. Perform row operations to obtain row-echelon form. Back-substitute to find the solutions. See Example 9.43 and Example 9.44.

- A calculator can be used to solve systems of equations using matrices. See Example 9.45.
Many real-world problems can be solved using augmented matrices. See Example 9.46 and Example 9.47.

9.7 Solving Systems with Inverses

- An identity matrix has the property \( AI = IA = A \). See Example 9.48.
- An invertible matrix has the property \( AA^{-1} = A^{-1} A = I \). See Example 9.49.
- Use matrix multiplication and the identity to find the inverse of a 2×2 matrix. See Example 9.50.
- The multiplicative inverse can be found using a formula. See Example 9.51.
- Another method of finding the inverse is by augmenting with the identity. See Example 9.52.
- We can augment a 3×3 matrix with the identity on the right and use row operations to turn the original matrix into the identity, and the matrix on the right becomes the inverse. See Example 9.53.
- Write the system of equations as \( AX = B \), and multiply both sides by the inverse of \( A : A^{-1} AX = A^{-1} B \). See Example 9.54 and Example 9.55.
- We can also use a calculator to solve a system of equations with matrix inverses. See Example 9.56.

9.8 Solving Systems with Cramer’s Rule

- The determinant for \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is \( ad - bc \). See Example 9.57.
- Cramer’s Rule replaces a variable column with the constant column. Solutions are \( x = \frac{D_x}{D}, y = \frac{D_y}{D} \). See Example 9.58.
- To find the determinant of a 3×3 matrix, augment with the first two columns. Add the three diagonal entries (upper left to lower right) and subtract the three diagonal entries (lower left to upper right). See Example 9.59.
- To solve a system of three equations in three variables using Cramer’s Rule, replace a variable column with the constant column for each desired solution: \( x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D} \). See Example 9.60.
- Cramer’s Rule is also useful for finding the solution of a system of equations with no solution or infinite solutions. See Example 9.61 and Example 9.62.
- Certain properties of determinants are useful for solving problems. For example:
  - If the matrix is in upper triangular form, the determinant equals the product of entries down the main diagonal.
  - When two rows are interchanged, the determinant changes sign.
  - If either two rows or two columns are identical, the determinant equals zero.
  - If a matrix contains either a row of zeros or a column of zeros, the determinant equals zero.
  - The determinant of an inverse matrix \( A^{-1} \) is the reciprocal of the determinant of the matrix \( A \).
  - If any row or column is multiplied by a constant, the determinant is multiplied by the same factor. See Example 9.63 and Example 9.64.

CHAPTER 9 REVIEW EXERCISES

Section 9.1

For the following exercises, determine whether the ordered pair is a solution to the system of equations.

548. \[ \begin{align*}
3x - y &= 4 \\
x + 4y &= -3
\end{align*} \quad \text{and} \quad (-1, 1) \]

549. \[ \begin{align*}
6x - 2y &= 24 \\
-3x + 3y &= 18
\end{align*} \quad \text{and} \quad (9, 15) \]

For the following exercises, use substitution to solve the system of equations.
550. \[\begin{align*}
10x + 5y &= -5 \\
3x - 2y &= -12
\end{align*}\]

551. \[\begin{align*}
\frac{4}{7}x + \frac{1}{3}y &= \frac{43}{7} \\
\frac{5}{6}x - \frac{1}{3}y &= -\frac{2}{3}
\end{align*}\]

552. \[\begin{align*}
5x + 6y &= 14 \\
4x + 8y &= 8
\end{align*}\]

For the following exercises, use addition to solve the system of equations.

553. \[\begin{align*}
3x + 2y &= -7 \\
2x + 4y &= 6
\end{align*}\]

554. \[\begin{align*}
3x + 4y &= 2 \\
9x + 12y &= 3
\end{align*}\]

555. \[\begin{align*}
8x + 4y &= 2 \\
6x - 5y &= 0.7
\end{align*}\]

For the following exercises, write a system of equations to solve each problem. Solve the system of equations.

556. A factory has a cost of production \(C(x) = 150x + 15,000\) and a revenue function \(R(x) = 200x\). What is the break-even point?

557. A performer charges \(C(x) = 50x + 10,000\), where \(x\) is the total number of attendees at a show. The venue charges $75 per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?

Section 9.2

For the following exercises, solve the system of three equations using substitution or addition.

558. \[\begin{align*}
0.5x - 0.5y &= 10 \\
-0.2y + 0.2x &= 4 \\
0.1x + 0.1z &= 2
\end{align*}\]

559. \[\begin{align*}
5x + 3y - z &= 5 \\
3x - 2y + 4z &= 13 \\
4x + 3y + 5z &= 22
\end{align*}\]

560. \[\begin{align*}
x + y + z &= 1 \\
2x + 2y + 2z &= 1 \\
3x + 3y &= 2
\end{align*}\]

561. \[\begin{align*}
2x - 3y + z &= -1 \\
x + y + z &= -4 \\
4x + 2y - 3z &= 33
\end{align*}\]

562. \[\begin{align*}
3x + 2y - z &= -10 \\
x - y + 2z &= 7 \\
-x + 3y + z &= -2
\end{align*}\]
For the following exercises, write a system of equations to solve each problem. Solve the system of equations.

566. Three odd numbers sum up to 61. The smaller is one-third the larger and the middle number is 16 less than the larger. What are the three numbers?

567. A local theatre sells out for their show. They sell all 500 tickets for a total purse of $8,070.00. The tickets were priced at $15 for students, $12 for children, and $18 for adults. If the band sold three times as many adult tickets as children’s tickets, how many of each type was sold?

Section 9.3

For the following exercises, solve the system of nonlinear equations.

568. 
\[
\begin{align*}
y &= x^2 - 7 \\
y &= 5x - 13
\end{align*}
\]

569. 
\[
\begin{align*}
y &= x^2 - 4 \\
y &= 5x + 10
\end{align*}
\]

570. 
\[
\begin{align*}
x^2 + y^2 &= 16 \\
y &= x - 8
\end{align*}
\]

571. 
\[
\begin{align*}
x^2 + y^2 &= 25 \\
y &= x^2 + 5
\end{align*}
\]

572. 
\[
\begin{align*}
x^2 + y^2 &= 4 \\
y - x^2 &= 3
\end{align*}
\]

For the following exercises, graph the inequality.

573. 
\[y > x^2 - 1\]

574. 
\[\frac{1}{4}x^2 + y^2 < 4\]

For the following exercises, graph the system of inequalities.

575. 
\[
\begin{align*}
x^2 + y^2 + 2x &< 3 \\
y &> -x^2 - 3
\end{align*}
\]
576. \[ x^2 - 2x + y^2 - 4x < 4 \]
\[ y < -x + 4 \]

577. \[ x^2 + y^2 < 1 \]
\[ y^2 < x \]

**Section 9.4**

For the following exercises, decompose into partial fractions.

578. \[ \frac{-2x + 6}{x^2 + 3x + 2} \]

579. \[ \frac{10x + 2}{4x^2 + 4x + 1} \]

580. \[ \frac{7x + 20}{x^2 + 10x + 25} \]

581. \[ \frac{x - 18}{x^2 - 12x + 36} \]

582. \[ \frac{-x^2 + 36x + 70}{x^3 - 125} \]

583. \[ \frac{-5x^2 + 6x - 2}{x^3 + 27} \]

584. \[ \frac{x^3 - 4x^2 + 3x + 11}{(x^2 - 2)^2} \]

585. \[ \frac{4x^4 - 2x^3 + 22x^2 - 6x + 48}{x(x^2 + 4)^2} \]

**Section 9.5**

For the following exercises, perform the requested operations on the given matrices.

\[
A = \begin{bmatrix} 4 & -2 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 7 & -3 \\ 11 & -2 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 7 \\ 11 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & -4 & 9 \\ 10 & 5 & -7 \end{bmatrix}, \quad E = \begin{bmatrix} 7 & -14 & 3 \\ 2 & -1 & 3 \end{bmatrix}
\]

586. \[ -4A \]

587. \[ 10D - 6E \]

588. \[ B + C \]

589. \[ AB \]

590. \[ BA \]

591. \[ BC \]

592. \[ CB \]
For the following exercises, write the system of linear equations from the augmented matrix. Indicate whether there will be a unique solution.

598.  
\[
\begin{bmatrix}
1 & 0 & -3 & | & 7 \\
0 & 1 & 2 & | & -5 \\
0 & 0 & 0 & | & 0
\end{bmatrix}
\]

599.  
\[
\begin{bmatrix}
1 & 0 & 5 & | & -9 \\
0 & 1 & -2 & | & 4 \\
0 & 0 & 0 & | & 3
\end{bmatrix}
\]

For the following exercises, write the augmented matrix from the system of linear equations.

600.  
\[-2x + 2y + z = 7 \\
2x - 8y + 5z = 0 \\
19x - 10y + 22z = 3 \]

601.  
\[-12x + 3y + z = 100 \\
x + 9y - 6y + 2z = 31 \]

602.  
\[-x + 4y = 0 \\
y + 2z = -7 \]

For the following exercises, solve the system of linear equations using Gaussian elimination.

603.  
\[3x - 4y = -7 \\
-6x + 8y = 14 \]

604.  
\[3x - 4y = 1 \\
-6x + 8y = 6 \]

605.  
\[-1.1x - 2.3y = 6.2 \\
-5.2x - 4.1y = 4.3 \]

606.  
\[2x + 3y + 2z = 1 \\
-4x - 6y - 4z = -2 \\
10x + 15y + 10z = 0 \]
\[-x + 2y - 4z = 8\]
\[3y + 8z = -4\]
\[-7x + y + 2z = 1\]

Section 9.7
For the following exercises, find the inverse of the matrix.

608. \[
\begin{bmatrix}
-0.2 & 1.4 \\
1.2 & -0.4
\end{bmatrix}
\]

609. \[
\begin{bmatrix}
\frac{1}{2} & -\frac{1}{2} \\
-\frac{1}{4} & \frac{3}{4}
\end{bmatrix}
\]

610. \[
\begin{bmatrix}
12 & 9 & -6 \\
-1 & 3 & 2 \\
-4 & -3 & 2
\end{bmatrix}
\]

611. \[
\begin{bmatrix}
2 & 1 & 3 \\
1 & 2 & 3 \\
3 & 2 & 1
\end{bmatrix}
\]

For the following exercises, find the solutions by computing the inverse of the matrix.

612. \[
0.3x - 0.1y = -10
\]
\[-0.1x + 0.3y = 14
\]

613. \[
0.4x - 0.2y = -0.6
\]
\[-0.1x + 0.05y = 0.3
\]

614. \[
4x + 3y - 3z = -4.3
\]
\[5x - 4y - z = -6.1
\]
\[x + z = -0.7
\]

\[-2x - 3y + 2z = 3
\]

615. \[
-x + 2y + 4z = -5
\]
\[-2y + 5z = -3
\]

For the following exercises, write a system of equations to solve each problem. Solve the system of equations.

616. Students were asked to bring their favorite fruit to class. 90% of the fruits consisted of banana, apple, and oranges. If oranges were half as popular as bananas and apples were 5% more popular than bananas, what are the percentages of each individual fruit?

617. A sorority held a bake sale to raise money and sold brownies and chocolate chip cookies. They priced the brownies at $2 and the chocolate chip cookies at $1. They raised $250 and sold 175 items. How many brownies and how many cookies were sold?

Section 9.8
For the following exercises, find the determinant.

618. \[
\begin{vmatrix}
100 & 0 \\
0 & 0
\end{vmatrix}
\]
For the following exercises, use Cramer’s Rule to solve the linear systems of equations.

622. \[ \begin{align*}
4x - 2y &= 23 \\
-5x - 10y &= -35
\end{align*} \]

623. \[ \begin{align*}
0.2x - 0.1y &= 0 \\
-0.3x + 0.3y &= 2.5
\end{align*} \]

624. \[ \begin{align*}
-0.5x + 0.1y &= 0.3 \\
-0.25x + 0.05y &= 0.15
\end{align*} \]

625. \[ \begin{align*}
x + 6y + 3z &= 4 \\
2x + y + 2z &= 3 \\
3x - 2y + z &= 0
\end{align*} \]

626. \[ \begin{align*}
4x - 3y + 5z &= -\frac{5}{2} \\
7x - 9y - 3z &= \frac{3}{2} \\
x - 5y - 5z &= \frac{5}{2}
\end{align*} \]

627. \[ \begin{align*}
\frac{3}{10}x - \frac{1}{5}y - \frac{3}{10}z &= -\frac{1}{50} \\
\frac{1}{10}x - \frac{1}{10}y - \frac{1}{2}z &= -\frac{9}{50} \\
\frac{2}{5}x - \frac{1}{2}y - \frac{3}{5}z &= -\frac{1}{5}
\end{align*} \]

CHAPTER 9 PRACTICE TEST

Is the following ordered pair a solution to the system of equations?

594. \[ \begin{align*}
-5x - y &= 12 \\
x + 4y &= 9
\end{align*} \] with \((-3, 3)\)

For the following exercises, solve the systems of linear and nonlinear equations using substitution or elimination. Indicate if no solution exists.

595. \[ \begin{align*}
\frac{1}{2}x - \frac{1}{3}y &= 4 \\
\frac{3}{2}x - y &= 0
\end{align*} \]
596. \[
\begin{align*}
\frac{1}{2}x - 4y &= 4 \\
2x + 16y &= 2
\end{align*}
\]

597. \[
\begin{align*}
5x - y &= 1 \\
-10x + 2y &= -2
\end{align*}
\]

\[
4x - 6y - 2z = \frac{1}{10}
\]

598. \[
\begin{align*}
x - 7y + 5z &= -\frac{1}{4} \\
3x + 6y - 9z &= \frac{6}{5}
\end{align*}
\]

\[
x + z = 20
\]

599. \[
\begin{align*}
x + y + z &= 20 \\
x + 2y + z &= 10
\end{align*}
\]

\[
5x - 4y - 3z = 0
\]

600. \[
\begin{align*}
2x + y + 2z &= 0 \\
x - 6y - 7z &= 0
\end{align*}
\]

\[
y = x^2 + 2x - 3
\]

601. \[
y = x - 1
\]

602. \[
\begin{align*}
y^2 + x^2 &= 25 \\
y^2 - 2x^2 &= 1
\end{align*}
\]

For the following exercises, graph the following inequalities.

603. \[
y < x^2 + 9
\]

604. \[
\begin{align*}
x^2 + y^2 &> 4 \\
y &< x^2 + 1
\end{align*}
\]

For the following exercises, write the partial fraction decomposition.

605. \[
\frac{-8x - 30}{x^2 + 10x + 25}
\]

606. \[
\frac{13x + 2}{(3x + 1)^2}
\]

607. \[
\frac{x^4 - x^3 + 2x - 1}{x(x^2 + 1)^2}
\]

For the following exercises, perform the given matrix operations.

608. \[
\begin{bmatrix}
5 & 4 & 9 \\
-2 & 3 & -6 \\
4 & -12 & -8
\end{bmatrix}
\]
609. \[
\begin{bmatrix}
1 & 4 & -7 \\
-2 & 9 & 5 \\
12 & 0 & -4
\end{bmatrix}
\begin{bmatrix}
3 & -4 \\
1 & 3 \\
5 & 10
\end{bmatrix}
\]

610. \[
\begin{bmatrix}
\frac{1}{2} & \frac{1}{3} \\
\frac{1}{2} & \frac{1}{5}
\end{bmatrix}
\]

611. \[
\det
\begin{vmatrix}
0 & 0 \\
400 & 4,000
\end{vmatrix}
\]

612. \[
\det
\begin{vmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} \\
0 & \frac{1}{2} & 0
\end{vmatrix}
\]

613. If \( \det(A) = -6 \), what would be the determinant if you switched rows 1 and 3, multiplied the second row by 12, and took the inverse?

\[
14x - 2y + 13z = 140
\]

614. Rewrite the system of linear equations as an augmented matrix. 
\[
\begin{align*}
-2x + 3y - 6z &= -1 \\
x - 5y + 12z &= 11
\end{align*}
\]

615. Rewrite the augmented matrix as a system of linear equations. 
\[
\begin{bmatrix}
1 & 0 & 3 & 12 \\
-2 & 4 & 9 & -5 \\
-6 & 1 & 2 & 8
\end{bmatrix}
\]

For the following exercises, use Gaussian elimination to solve the systems of equations.

616. \[
\begin{align*}
x - 6y &= 4 \\
2x - 12y &= 0
\end{align*}
\]

\[
2x + y + z = -3
\]

617. \[
\begin{align*}
x - 2y + 3z &= 6 \\
x - y - z &= 6
\end{align*}
\]

For the following exercises, use the inverse of a matrix to solve the systems of equations.

618. \[
\begin{align*}
4x - 5y &= -50 \\
-x + 2y &= 80
\end{align*}
\]

\[
\frac{1}{100}x - \frac{3}{100}y + \frac{1}{20}z = -49
\]

619. \[
\begin{align*}
3 \frac{1}{100}x - \frac{7}{100}y - \frac{1}{100}z &= 13 \\
9 \frac{1}{100}x - \frac{9}{100}y - \frac{9}{100}z &= 99
\end{align*}
\]

For the following exercises, use Cramer’s Rule to solve the systems of equations.

620. \[
\begin{align*}
200x - 300y &= 2 \\
400x + 715y &= 4
\end{align*}
\]
\[
0.1x + 0.1y - 0.1z = -1.2 \\
0.1x - 0.2y + 0.4z = -1.2 \\
0.5x - 0.3y + 0.8z = -5.9
\]

For the following exercises, solve using a system of linear equations.

**622.** A factory producing cell phones has the following cost and revenue functions: \( C(x) = x^2 + 75x + 2,688 \) and \( R(x) = x^2 + 160x \). What is the range of cell phones they should produce each day so there is profit? Round to the nearest number that generates profit.

**623.** A small fair charges $1.50 for students, $1 for children, and $2 for adults. In one day, three times as many children as adults attended. A total of 800 tickets were sold for a total revenue of $1,050. How many of each type of ticket was sold?
10 | ANALYTIC GEOMETRY

10.1 | Introduction to Analytic Geometry

Learning Objectives

10.2 | The Ellipse

Learning Objectives

10.3 | The Hyperbola

Learning Objectives

10.4 | The Parabola

Learning Objectives

10.5 | Rotation of Axes

Learning Objectives

10.6 | Conic Sections in Polar Coordinates

Learning Objectives
Introduction

A lottery winner has some big decisions to make regarding what to do with the winnings. Buy a villa in Saint Barthélemy? A luxury convertible? A cruise around the world?

The likelihood of winning the lottery is slim, but we all love to fantasize about what we could buy with the winnings. One of the first things a lottery winner has to decide is whether to take the winnings in the form of a lump sum or as a series of regular payments, called an annuity, over the next 30 years or so.

This decision is often based on many factors, such as tax implications, interest rates, and investment strategies. There are also personal reasons to consider when making the choice, and one can make many arguments for either decision. However, most lottery winners opt for the lump sum.
In this chapter, we will explore the mathematics behind situations such as these. We will take an in-depth look at annuities. We will also look at the branch of mathematics that would allow us to calculate the number of ways to choose lottery numbers and the probability of winning.

11.1 | Sequences and Their Notation

Learning Objectives

In this section, you will:

11.1.1 Write the terms of a sequence defined by an explicit formula.
11.1.2 Write the terms of a sequence defined by a recursive formula.
11.1.3 Use factorial notation.

A video game company launches an exciting new advertising campaign. They predict the number of online visits to their website, or hits, will double each day. The model they are using shows 2 hits the first day, 4 hits the second day, 8 hits the third day, and so on. See Table 11.1.

<table>
<thead>
<tr>
<th>Day</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>…</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hits</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>…</td>
</tr>
</tbody>
</table>

Table 11.1

If their model continues, how many hits will there be at the end of the month? To answer this question, we’ll first need to know how to determine a list of numbers written in a specific order. In this section, we will explore these kinds of ordered lists.

Writing the Terms of a Sequence Defined by an Explicit Formula

One way to describe an ordered list of numbers is as a sequence. A sequence is a function whose domain is a subset of the counting numbers. The sequence established by the number of hits on the website is

\[ \{2, 4, 8, 16, 32, \ldots \} . \]  

The ellipsis (…) indicates that the sequence continues indefinitely. Each number in the sequence is called a term. The first five terms of this sequence are 2, 4, 8, 16, and 32.

Listing all of the terms for a sequence can be cumbersome. For example, finding the number of hits on the website at the end of the month would require listing out as many as 31 terms. A more efficient way to determine a specific term is by writing a formula to define the sequence.

One type of formula is an explicit formula, which defines the terms of a sequence using their position in the sequence. Explicit formulas are helpful if we want to find a specific term of a sequence without finding all of the previous terms. We can use the formula to find the \( n \)th term of the sequence, where \( n \) is any positive number. In our example, each number in the sequence is double the previous number, so we can use powers of 2 to write a formula for the \( n \)th term.

\[ \{2, 4, 8, 16, 32, \ldots, ?, \ldots\} \]

\[ \{2^1, 2^2, 2^3, 2^4, 2^5, \ldots, 2^n \ldots\} \]

The first term of the sequence is \( 2^1 = 2 \), the second term is \( 2^2 = 4 \), the third term is \( 2^3 = 8 \), and so on. The \( n \)th term of the sequence can be found by raising 2 to the \( n \)th power. An explicit formula for a sequence is named by a lower case letter \( a, b, c \ldots \) with the subscript \( n \). The explicit formula for this sequence is

\[ a_n = 2^n . \]  

(11.2)
Now that we have a formula for the $n$th term of the sequence, we can answer the question posed at the beginning of this section. We were asked to find the number of hits at the end of the month, which we will take to be 31 days. To find the number of hits on the last day of the month, we need to find the 31st term of the sequence. We will substitute 31 for $n$ in the formula.

$$a_{31} = 2^{31} = 2,147,483,648$$

If the doubling trend continues, the company will get 2,147,483,648 hits on the last day of the month. That is over 2.1 billion hits! The huge number is probably a little unrealistic because it does not take consumer interest and competition into account. It does, however, give the company a starting point from which to consider business decisions.

Another way to represent the sequence is by using a table. The first five terms of the sequence and the $n$th term of the sequence are shown in Table 11.2.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$th term of the sequence, $a_n$</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>$2^n$</td>
</tr>
</tbody>
</table>

Graphing provides a visual representation of the sequence as a set of distinct points. We can see from the graph in Figure 11.2 that the number of hits is rising at an exponential rate. This particular sequence forms an exponential function.

Lastly, we can write this particular sequence as

$$[2, 4, 8, 16, 32, \ldots, 2^n, \ldots].$$

A sequence that continues indefinitely is called an infinite sequence. The domain of an infinite sequence is the set of counting numbers. If we consider only the first 10 terms of the sequence, we could write

$$[2, 4, 8, 16, 32, \ldots, 2^n, \ldots, 1024].$$

This sequence is called a finite sequence because it does not continue indefinitely.

**Sequence**

A sequence is a function whose domain is the set of positive integers. A finite sequence is a sequence whose domain consists of only the first $n$ positive integers. The numbers in a sequence are called terms. The variable $a$ with a number subscript is used to represent the terms in a sequence and to indicate the position of the term in the sequence.
We call \( a_1 \) the first term of the sequence, \( a_2 \) the second term of the sequence, \( a_3 \) the third term of the sequence, and so on. The term \( a_n \) is called the \textit{nth term of the sequence}, or the general term of the sequence. An \textit{explicit formula} defines the \textit{nth} term of a sequence using the position of the term. A sequence that continues indefinitely is an \textit{infinite sequence}.

Does a sequence always have to begin with \( a_1 \)?

No. In certain problems, it may be useful to define the initial term as \( a_0 \) instead of \( a_1 \). In these problems, the domain of the function includes 0.

Given an explicit formula, write the first \( n \) terms of a sequence.

1. Substitute each value of \( n \) into the formula. Begin with \( n = 1 \) to find the first term, \( a_1 \).
2. To find the second term, \( a_2 \), use \( n = 2 \).
3. Continue in the same manner until you have identified all \( n \) terms.

Example 11.1

Writing the Terms of a Sequence Defined by an Explicit Formula

Write the first five terms of the sequence defined by the explicit formula \( a_n = -3n + 8 \).

**Solution**

**Analysis**

The sequence values can be listed in a table. A table, such as Table 11.2, is a convenient way to input the function into a graphing utility.

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_n )</td>
<td>5</td>
<td>2</td>
<td>−1</td>
<td>−4</td>
<td>−7</td>
</tr>
</tbody>
</table>

A graph can be made from this table of values. From the graph in Figure 11.2, we can see that this sequence represents a linear function, but notice the graph is not continuous because the domain is over the positive integers only.
Write the first five terms of the sequence defined by the explicit formula $t_n = 5n - 4$.

**Investigating Alternating Sequences**

Sometimes sequences have terms that are alternate. In fact, the terms may actually alternate in sign. The steps to finding terms of the sequence are the same as if the signs did not alternate. However, the resulting terms will not show increase or decrease as $n$ increases. Let’s take a look at the following sequence.

\[
\{2, -4, 6, -8\} \tag{11.7}
\]

Notice the first term is greater than the second term, the second term is less than the third term, and the third term is greater than the fourth term. This trend continues forever. Do not rearrange the terms in numerical order to interpret the sequence.

**Given an explicit formula with alternating terms, write the first $n$ terms of a sequence.**

1. Substitute each value of $n$ into the formula. Begin with $n = 1$ to find the first term, $a_1$. The sign of the term is given by the $(-1)^n$ in the explicit formula.

2. To find the second term, $a_2$, use $n = 2$.

3. Continue in the same manner until you have identified all $n$ terms.

**Example 11.2**

**Writing the Terms of an Alternating Sequence Defined by an Explicit Formula**

Write the first five terms of the sequence.
The graph of this function, shown in Figure 11.2, looks different from the ones we have seen previously in this section because the terms of the sequence alternate between positive and negative values.

**In Example 11.2, does the \((-1)^n\) account for the oscillations of signs?**

Yes, the power might be \(n, n+1, n-1\), and so on, but any odd powers will result in a negative term, and any even power will result in a positive term.

**Investigating Piecewise Explicit Formulas**

We’ve learned that sequences are functions whose domain is over the positive integers. This is true for other types of functions, including some piecewise functions. Recall that a piecewise function is a function defined by multiple subsections. A different formula might represent each individual subsection.

**Given an explicit formula for a piecewise function, write the first \(n\) terms of a sequence**

1. Identify the formula to which \(n = 1\) applies.
2. To find the first term, \(a_1\), use \(n = 1\) in the appropriate formula.
3. Identify the formula to which \(n = 2\) applies.
4. To find the second term, \(a_2\), use \(n = 2\) in the appropriate formula.
5. Continue in the same manner until you have identified all \(n\) terms.
Example 11.3

Writing the Terms of a Sequence Defined by a Piecewise Explicit Formula

Write the first six terms of the sequence.

\[ a_n = \begin{cases} 
  n^2 & \text{if } n \text{ is not divisible by 3} \\
  \frac{n}{3} & \text{if } n \text{ is divisible by 3}
\end{cases} \]

Solution

Analysis

Every third point on the graph shown in Figure 11.2 stands out from the two nearby points. This occurs because the sequence was defined by a piecewise function.

Try It 11.3

Write the first six terms of the sequence.

\[ a_n = \begin{cases} 
  2n^3 & \text{if } n \text{ is odd} \\
  \frac{5n}{2} & \text{if } n \text{ is even}
\end{cases} \]  

Finding an Explicit Formula

Thus far, we have been given the explicit formula and asked to find a number of terms of the sequence. Sometimes, the explicit formula for the \(n\)th term of a sequence is not given. Instead, we are given several terms from the sequence. When this happens, we can work in reverse to find an explicit formula from the first few terms of a sequence. The key to finding an explicit formula is to look for a pattern in the terms. Keep in mind that the pattern may involve alternating terms, formulas for numerators, formulas for denominators, exponents, or bases.

Given the first few terms of a sequence, find an explicit formula for the sequence.

1. Look for a pattern among the terms.
2. If the terms are fractions, look for a separate pattern among the numerators and denominators.
3. Look for a pattern among the signs of the terms.
4. Write a formula for \(a_n\) in terms of \(n\). Test your formula for \(n = 1\), \(n = 2\), and \(n = 3\).

Example 11.4
Writing an Explicit Formula for the \( n \)th Term of a Sequence

Write an explicit formula for the \( n \)th term of each sequence.

a. \[
\left\{ \frac{-2}{11}, \frac{3}{13}, \frac{-4}{15}, \frac{5}{17}, \frac{-6}{19}, \ldots \right\}
\]

b. \[
\left\{ \frac{-2}{25}, \frac{-2}{125}, \frac{-2}{625}, \frac{-2}{3,125}, \frac{-2}{15,625}, \ldots \right\}
\]

c. \[
\left\{ e^4, e^5, e^6, e^7, e^8, \ldots \right\}
\]

Solution

Write an explicit formula for the \( n \)th term of the sequence.

(11.10) \[
\{9, \ -81, 729, \ -6,561, 59,049, \ldots \}
\]

(11.11) \[
\left\{ \frac{-3}{4}, \ -\frac{9}{8}, \ -\frac{27}{12}, \ -\frac{81}{16}, \ -\frac{243}{20}, \ldots \right\}
\]

(11.12) \[
\left\{ \frac{1}{e^2}, \ \frac{1}{e}, \ 1, \ e, \ e^2, \ldots \right\}
\]

Writing the Terms of a Sequence Defined by a Recursive Formula

Sequences occur naturally in the growth patterns of nautilus shells, pinecones, tree branches, and many other natural structures. We may see the sequence in the leaf or branch arrangement, the number of petals of a flower, or the pattern of the chambers in a nautilus shell. Their growth follows the Fibonacci sequence, a famous sequence in which each term can be found by adding the preceding two terms. The numbers in the sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,… Other examples from the natural world that exhibit the Fibonacci sequence are the Calla Lily, which has just one petal, the Black-Eyed Susan with 13 petals, and different varieties of daisies that may have 21 or 34 petals.

Each term of the Fibonacci sequence depends on the terms that come before it. The Fibonacci sequence cannot easily be written using an explicit formula. Instead, we describe the sequence using a recursive formula, a formula that defines the terms of a sequence using previous terms.

A recursive formula always has two parts: the value of an initial term (or terms), and an equation defining \( a_n \) in terms of preceding terms. For example, suppose we know the following:

\[
a_1 = 3 \\
a_n = 2a_{n-1} - 1, \text{ for } n \geq 2
\]

We can find the subsequent terms of the sequence using the first term.

\[
a_1 = 3 \\
a_2 = 2a_1 - 1 = 2(3) - 1 = 5 \\
a_3 = 2a_2 - 1 = 2(5) - 1 = 9 \\
a_4 = 2a_3 - 1 = 2(9) - 1 = 17
\]

So the first four terms of the sequence are \( \{3, \ 5, \ 9, \ 17\} \).

The recursive formula for the Fibonacci sequence states the first two terms and defines each successive term as the sum of the preceding two terms.
\[ a_1 = 1 \]
\[ a_2 = 1 \]
\[ a_n = a_{n-1} + a_{n-2}, \quad \text{for} \quad n \geq 3 \]  \hspace{1cm} (11.15)

To find the tenth term of the sequence, for example, we would need to add the eighth and ninth terms. We were told previously that the eighth and ninth terms are 21 and 34, so
\[ a_{10} = a_9 + a_8 = 34 + 21 = 55 \]  \hspace{1cm} (11.16)

**Recursive Formula**

A **recursive formula** is a formula that defines each term of a sequence using preceding term(s). Recursive formulas must always state the initial term, or terms, of the sequence.

**Q&A**

**Must the first two terms always be given in a recursive formula?**

No. The Fibonacci sequence defines each term using the two preceding terms, but many recursive formulas define each term using only one preceding term. These sequences need only the first term to be defined.

**How To**: Given a recursive formula with only the first term provided, write the first \( n \) terms of a sequence.

1. Identify the initial term, \( a_1 \), which is given as part of the formula. This is the first term.
2. To find the second term, \( a_2 \), substitute the initial term into the formula for \( a_{n-1} \). Solve.
3. To find the third term, \( a_3 \), substitute the second term into the formula. Solve.
4. Repeat until you have solved for the \( n \)th term.

**Example 11.5**

**Writing the Terms of a Sequence Defined by a Recursive Formula**

Write the first five terms of the sequence defined by the recursive formula.
\[ a_1 = 9 \]
\[ a_n = 3a_{n-1} - 20, \quad \text{for} \quad n \geq 2 \]

**Solution**

Following the process described above, we find:
\[ a_2 = 3a_1 - 20 = 3(9) - 20 = 7 \]
\[ a_3 = 3a_2 - 20 = 3(7) - 20 = 1 \]
\[ a_4 = 3a_3 - 20 = 3(1) - 20 = -17 \]
\[ a_5 = 3a_4 - 20 = 3(-17) - 20 = -71 \]

Therefore, the first five terms of the sequence are 9, 7, 1, -17, and -71.

**Try It**

Write the first five terms of the sequence defined by the recursive formula.
\[ a_1 = 2 \]
\[ a_n = 2a_{n-1} + 1, \quad \text{for} \quad n \geq 2 \]  \hspace{1cm} (11.17)

**How To**: Given a recursive formula with two initial terms, write the first \( n \) terms of a sequence.

1. Identify the initial term, \( a_1 \), which is given as part of the formula.
2. Identify the second term, \( a_2 \), which is given as part of the formula.
3. To find the third term, substitute the initial term and the second term into the formula. Evaluate.
4. Repeat until you have evaluated the \( n \)th term.
Example 11.6

**Writing the Terms of a Sequence Defined by a Recursive Formula**

Write the first six terms of the sequence defined by the recursive formula.

\[ a_1 = 1, \quad a_2 = 2, \quad a_n = 3a_{n-1} + 4a_{n-2}, \text{ for } n \geq 3 \]

**Solution**

Write the first six terms of the sequence defined by the recursive formula.

\[ a_1 = 0, \quad a_2 = 1, \quad a_3 = 1, \quad a_n = \frac{a_{n-1}}{a_{n-2}} + a_{n-3}, \text{ for } n \geq 4 \]

**Using Factorial Notation**

The formulas for some sequences include products of consecutive positive integers. **n factorial**, written as \( n! \), is the product of the positive integers from 1 to \( n \). For example,

\[ 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \]
\[ 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \]

An example of formula containing a factorial is \( a_n = (n + 1)! \). The sixth term of the sequence can be found by substituting 6 for \( n \).

\[ a_6 = (6 + 1)! = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040 \]

The factorial of any whole number \( n \) is \( n(n - 1)! \). We can therefore also think of \( 5! \) as \( 5 \cdot 4! \).

**Factorial**

**n factorial** is a mathematical operation that can be defined using a recursive formula. The factorial of \( n \), denoted \( n! \), is defined for a positive integer \( n \) as:

\[ 0! = 1 \]
\[ 1! = 1 \]
\[ n! = n(n - 1)(n - 2) \cdots (2)(1), \text{ for } n \geq 2 \]

The special case \( 0! \) is defined as \( 0! = 1 \).

**Can factorials always be found using a calculator?**

No. Factorials get large very quickly—faster than even exponential functions! When the output gets too large for the calculator, it will not be able to calculate the factorial.
Writing the Terms of a Sequence Using Factorials

Write the first five terms of the sequence defined by the explicit formula \( a_n = \frac{5n}{(n + 2)!} \).

### Solution

#### Analysis

Figure 11.2 shows the graph of the sequence. Notice that, since factorials grow very quickly, the presence of the factorial term in the denominator results in the denominator becoming much larger than the numerator as \( n \) increases. This means the quotient gets smaller and, as the plot of the terms shows, the terms are decreasing and nearing zero.

![Graph of the sequence](image)

**Try 11.9** Write the first five terms of the sequence defined by the explicit formula \( a_n = \frac{(n + 1)!}{2n} \).

Access this online resource for additional instruction and practice with sequences.

- Finding Terms in a Sequence (http://openstaxcollege.org/l/findingterms)
11.1 EXERCISES

Verbal

1. Discuss the meaning of a sequence. If a finite sequence is defined by a formula, what is its domain? What about an infinite sequence?

2. Describe three ways that a sequence can be defined.

3. Is the ordered set of even numbers an infinite sequence? What about the ordered set of odd numbers? Explain why or why not.

4. What happens to the terms $a_n$ of a sequence when there is a negative factor in the formula that is raised to a power that includes $n$? What is the term used to describe this phenomenon?

5. What is a factorial, and how is it denoted? Use an example to illustrate how factorial notation can be beneficial.

Algebraic

For the following exercises, write the first four terms of the sequence.

6. $a_n = 2^n - 2$

7. $a_n = -\frac{16}{n + 1}$

8. $a_n = -(-5)^{n-1}$

9. $a_n = \frac{2^n}{n^3}$

10. $a_n = \frac{2n + 1}{n^3}$

11. $a_n = 1.25 \cdot (-4)^{n-1}$

12. $a_n = -4 \cdot (-6)^{n-1}$

13. $a_n = \frac{n^2}{2n + 1}$

14. $a_n = (-10)^n + 1$

15. $a_n = -\left(\frac{4 \cdot (-5)^{n-1}}{5}\right)$

For the following exercises, write the first eight terms of the piecewise sequence.

16. $a_n = \begin{cases} (-2)^n - 2 & \text{if } n \text{ is even} \\ (3)^n - 1 & \text{if } n \text{ is odd} \end{cases}$

17. $a_n = \begin{cases} \frac{n^2}{2n + 1} & \text{if } n \leq 5 \\ n^2 - 5 & \text{if } n > 5 \end{cases}$

18. $a_n = \begin{cases} (2n + 1)^2 & \text{if } n \text{ is divisible by } 4 \\ \frac{2}{n} & \text{if } n \text{ is not divisible by } 4 \end{cases}$

19.
\[ a_n = \begin{cases} 
-0.6 \cdot 5^{n-1} & \text{if } n \text{ is prime or } 1 \\
2.5 \cdot (-2)^{n-1} & \text{if } n \text{ is composite} 
\end{cases} \]

20. \[ a_n = \begin{cases} 
4(n^2 - 2) & \text{if } n \leq 3 \text{ or } n > 6 \\
\frac{n^2 - 2}{4} & \text{if } 3 < n \leq 6 
\end{cases} \]

For the following exercises, write an explicit formula for each sequence.

21. 4, 7, 12, 19, 28, ...
22. -4, 2, -10, 14, -34, ...
23. 1, 1, 4/3, 2, 16/5, ...
24. 0, \( \frac{1 - e^1}{1 + e^2} \), \( \frac{1 - e^2}{1 + e^3} \), \( \frac{1 - e^3}{1 + e^4} \), \( \frac{1 - e^4}{1 + e^5} \), ...
25. 1, -1/2, 1/4, -1/8, 1/16, ...

For the following exercises, write the first five terms of the sequence.

26. \( a_1 = 9 \), \( a_n = a_{n-1} + n \)
27. \( a_1 = 3 \), \( a_n = (-3)a_{n-1} \)
28. \( a_1 = -4 \), \( a_n = \frac{a_{n-1} + 2n}{a_{n-1} - 1} \)
29. \( a_1 = -1 \), \( a_n = \frac{(-3)^n - 1}{a_{n-1} - 2} \)
30. \( a_1 = -30 \), \( a_n = (2 + a_{n-1})\left(\frac{1}{2}\right)^n \)

For the following exercises, write the first eight terms of the sequence.

31. \( a_1 = \frac{1}{24} \), \( a_2 = 1 \), \( a_n = (2a_{n-2} - 3a_{n-1}) \)
32. \( a_1 = -1 \), \( a_2 = 5 \), \( a_n = a_{n-2}(3 - a_{n-1}) \)
33. \( a_1 = 2 \), \( a_2 = 10 \), \( a_n = \frac{2(a_{n-1} + 2)}{a_{n-2}} \)

For the following exercises, write a recursive formula for each sequence.

34. -2.5, -5, -10, -20, -40, ...
35. -8, -6, -3, 1, 6, ...
36. 2, 4, 12, 48, 240, ...
37. 35, 38, 41, 44, 47, ...
38. 15, 3, \( \frac{3}{5} \), \( \frac{3}{25} \), \( \frac{3}{125} \), ...

\[ a_n = \begin{cases} 
-0.6 \cdot 5^{n-1} & \text{if } n \text{ is prime or } 1 \\
2.5 \cdot (-2)^{n-1} & \text{if } n \text{ is composite} 
\end{cases} \]

\[ a_n = \begin{cases} 
4(n^2 - 2) & \text{if } n \leq 3 \text{ or } n > 6 \\
\frac{n^2 - 2}{4} & \text{if } 3 < n \leq 6 
\end{cases} \]
For the following exercises, evaluate the factorial.

39. 6!
40. \( \frac{12}{6}! \)
41. \( \frac{12!}{6!} \)
42. \( \frac{100!}{99!} \)

For the following exercises, write the first four terms of the sequence.

43. \( a_n = \frac{n!}{n^2} \)
44. \( a_n = \frac{3 \cdot n!}{4 \cdot n!} \)
45. \( a_n = \frac{n!}{n^2 - n - 1} \)
46. \( a_n = \frac{100 \cdot n}{n(n-1)!} \)

Graphical

For the following exercises, graph the first five terms of the indicated sequence.

47. \( a_n = \frac{(-1)^n}{n} + n \)
48. \( a_n = \begin{cases} 
\frac{4 + n}{2n} & \text{if } n \text{ is even} \\
3 + n & \text{if } n \text{ is odd} 
\end{cases} \)
49. \( a_1 = 2, \ a_n = (-a_{n-1} + 1)^2 \)
50. \( a_1 = 1, \ a_n = a_{n-1} + 8 \)
51. \( a_n = \frac{(n + 1)!}{(n - 1)!} \)

For the following exercises, write an explicit formula for the sequence using the first five points shown on the graph.

52. (Graph with data points at (1, 5), (2, 7), (3, 9), (4, 11), and (5, 13))

53. (Additional question or exercise)
For the following exercises, write a recursive formula for the sequence using the first five points shown on the graph.

54.

55.

56.
Follow these steps to evaluate a sequence defined recursively using a graphing calculator:

- On the home screen, key in the value for the initial term \(a_1\) and press [ENTER].
- Enter the recursive formula by keying in all numerical values given in the formula, along with the key strokes [2ND] ANS for the previous term \(a_{n-1}\). Press [ENTER].
- Continue pressing [ENTER] to calculate the values for each successive term.

For the following exercises, use the steps above to find the indicated term or terms for the sequence.

57. Find the first five terms of the sequence \(a_1 = \frac{87}{111}, \quad a_n = \frac{4}{3}a_{n-1} + \frac{12}{37}\). Use the >Frac feature to give fractional results.

58. Find the 15th term of the sequence \(a_1 = 625, \quad a_n = 0.8a_{n-1} + 18\).

59. Find the first five terms of the sequence \(a_1 = 2, \quad a_n = \frac{1}{2}(a_{n-1} - 1) + 1\).

60. Find the first ten terms of the sequence \(a_1 = 8, \quad a_n = \frac{(a_{n-1} + 1)!}{a_{n-1}!}\).

61. Find the tenth term of the sequence \(a_1 = 2, \quad a_n = na_{n-1}\)

Follow these steps to evaluate a finite sequence defined by an explicit formula. Using a TI-84, do the following.

- In the home screen, press [2ND] LIST.
- Scroll over to OPS and choose “seq(“ from the dropdown list. Press [ENTER].
- In the line headed “Expr:” type in the explicit formula, using the [X,T, \(θ, n\)] button for \(n\)
- In the line headed “Variable:” type in the variable used on the previous step.
- In the line headed “start:” key in the value of \(n\) that begins the sequence.
- In the line headed “end:” key in the value of \(n\) that ends the sequence.
- Press [ENTER] 3 times to return to the home screen. You will see the sequence syntax on the screen. Press [ENTER] to see the list of terms for the finite sequence defined. Use the right arrow key to scroll through the list of terms.

Using a TI-83, do the following.

- In the home screen, press [2ND] LIST.
- Scroll over to OPS and choose “seq(“ from the dropdown list. Press [ENTER].
- Enter the items in the order “Expr”, “Variable”, “start”, “end” separated by commas. See the instructions above for the description of each item.
• Press [ENTER] to see the list of terms for the finite sequence defined. Use the right arrow key to scroll through the list of terms.

For the following exercises, use the steps above to find the indicated terms for the sequence. Round to the nearest thousandth when necessary.

62. List the first five terms of the sequence \( a_n = -\frac{28}{9}n + \frac{5}{3} \).

63. List the first six terms of the sequence \( a_n = \frac{n^3 - 3.5n^2 + 4.1n - 1.5}{2.4n} \).

64. List the first five terms of the sequence \( a_n = \frac{15n \cdot (-2)^{n-1}}{47} \).

65. List the first four terms of the sequence \( a_n = 5.7^n + 0.275(n - 1)! \).

66. List the first six terms of the sequence \( a_n = \frac{n!}{n^4} \).

Extensions

67. Consider the sequence defined by \( a_n = -6 - 8n \). Is \( a_n = -421 \) a term in the sequence? Verify the result.

68. What term in the sequence \( a_n = \frac{n^2 + 4n + 4}{2(n + 2)} \) has the value 41? Verify the result.

69. Find a recursive formula for the sequence 1, 0, -1, -1, 0, 1, 0, -1, -1, 0, 1, 1, ... (Hint: find a pattern for \( a_n \) based on the first two terms.)

70. Calculate the first eight terms of the sequences \( a_n = \frac{(n + 2)!}{(n - 1)!} \) and \( b_n = n^3 + 3n^2 + 2n \), and then make a conjecture about the relationship between these two sequences.

71. Prove the conjecture made in the preceding exercise.
11.2 | Arithmetic Sequences

Learning Objectives

In this section, you will:

11.2.1 Find the common difference for an arithmetic sequence.
11.2.2 Write terms of an arithmetic sequence.
11.2.3 Use a recursive formula for an arithmetic sequence.
11.2.4 Use an explicit formula for an arithmetic sequence.

Companies often make large purchases, such as computers and vehicles, for business use. The book-value of these supplies decreases each year for tax purposes. This decrease in value is called depreciation. One method of calculating depreciation is straight-line depreciation, in which the value of the asset decreases by the same amount each year.

As an example, consider a woman who starts a small contracting business. She purchases a new truck for $25,000. After five years, she estimates that she will be able to sell the truck for $8,000. The loss in value of the truck will therefore be $17,000, which is $3,400 per year for five years. The truck will be worth $21,600 after the first year; $18,200 after two years; $14,800 after three years; $11,400 after four years; and $8,000 at the end of five years. In this section, we will consider specific kinds of sequences that will allow us to calculate depreciation, such as the truck’s value.

Finding Common Differences

The values of the truck in the example are said to form an arithmetic sequence because they change by a constant amount each year. Each term increases or decreases by the same constant value called the common difference of the sequence. For this sequence, the common difference is $-3,400$.

The sequence below is another example of an arithmetic sequence. In this case, the constant difference is 3. You can choose any term of the sequence, and add 3 to find the subsequent term.

Arithmetic Sequence

An arithmetic sequence is a sequence that has the property that the difference between any two consecutive terms is a constant. This constant is called the common difference. If $a_1$ is the first term of an arithmetic sequence and $d$ is the common difference, the sequence will be:

$$[a_n] = [a_1, a_1 + d, a_1 + 2d, a_1 + 3d, ...]$$

(11.22)

Example 11.8

Finding Common Differences

Is each sequence arithmetic? If so, find the common difference.

a.  { 1, 2, 4, 8, 16, ... }

b.  { −3, 1, 5, 9, 13, ... }
Solution
Analysis
The graph of each of these sequences is shown in Figure 11.2. We can see from the graphs that, although both sequences show growth, \( a \) is not linear whereas \( b \) is linear. Arithmetic sequences have a constant rate of change so their graphs will always be points on a line.

![Graphs of sequences](image)

If we are told that a sequence is arithmetic, do we have to subtract every term from the following term to find the common difference?

No. If we know that the sequence is arithmetic, we can choose any one term in the sequence, and subtract it from the subsequent term to find the common difference.

11.10 Is the given sequence arithmetic? If so, find the common difference.

\[
\{ 18, 16, 14, 12, 10, \ldots \}
\]

11.11 Is the given sequence arithmetic? If so, find the common difference.

\[
\{ 1, 3, 6, 10, 15, \ldots \}
\]

Writing Terms of Arithmetic Sequences
Now that we can recognize an arithmetic sequence, we will find the terms if we are given the first term and the common difference. The terms can be found by beginning with the first term and adding the common difference repeatedly. In addition, any term can also be found by plugging in the values of \( n \) and \( d \) into formula below.

\[
a_n = a_1 + (n - 1)d
\]

Given the first term and the common difference of an arithmetic sequence, find the first several terms.

**How To:**
1. Add the common difference to the first term to find the second term.
2. Add the common difference to the second term to find the third term.
3. Continue until all of the desired terms are identified.
4. Write the terms separated by commas within brackets.

Example 11.9
Writing Terms of Arithmetic Sequences

Write the first five terms of the arithmetic sequence with \( a_1 = 17 \) and \( d = -3 \).

Solution

- **Analysis**
  - As expected, the graph of the sequence consists of points on a line as shown in Figure 11.2.

```
11.12 List the first five terms of the arithmetic sequence with \( a_1 = 1 \) and \( d = 5 \).
```

Given any the first term and any other term in an arithmetic sequence, find a given term.

1. Substitute the values given for \( a_1 \), \( a_n \), \( n \) into the formula \( a_n = a_1 + (n - 1)d \) to solve for \( d \).

2. Find a given term by substituting the appropriate values for \( a_1 \), \( n \), and \( d \) into the formula \( a_n = a_1 + (n - 1)d \).

**Example 11.10**

Writing Terms of Arithmetic Sequences

Given \( a_1 = 8 \) and \( a_4 = 14 \), find \( a_5 \).

Solution

- **Analysis**
  - Notice that the common difference is added to the first term once to find the second term, twice to find the third term, three times to find the fourth term, and so on. The tenth term could be found by adding the common difference to the first term nine times or by using the equation \( a_n = a_1 + (n - 1)d \).

```
11.13 Given \( a_3 = 7 \) and \( a_5 = 17 \), find \( a_2 \).
```
Using Recursive Formulas for Arithmetic Sequences

Some arithmetic sequences are defined in terms of the previous term using a recursive formula. The formula provides an algebraic rule for determining the terms of the sequence. A recursive formula allows us to find any term of an arithmetic sequence using a function of the preceding term. Each term is the sum of the previous term and the common difference. For example, if the common difference is 5, then each term is the previous term plus 5. As with any recursive formula, the first term must be given.

\[ a_n = a_{n-1} + d \quad n \geq 2 \quad (11.26) \]

**Recursive Formula for an Arithmetic Sequence**

The recursive formula for an arithmetic sequence with common difference \( d \) is:

\[ a_n = a_{n-1} + d \quad n \geq 2 \quad (11.27) \]

**How To:** Given an arithmetic sequence, write its recursive formula.

1. Subtract any term from the subsequent term to find the common difference.
2. State the initial term and substitute the common difference into the recursive formula for arithmetic sequences.

**Example 11.11**

**Writing a Recursive Formula for an Arithmetic Sequence**

Write a recursive formula for the arithmetic sequence.

\[ \{-18, -7, 4, 15, 26, \ldots\} \]

**Solution**

**Analysis**

We see that the common difference is the slope of the line formed when we graph the terms of the sequence, as shown in Figure 11.2. The growth pattern of the sequence shows the constant difference of 11 units.

Do we have to subtract the first term from the second term to find the common difference?

No. We can subtract any term in the sequence from the subsequent term. It is, however, most common to subtract the first term from the second term because it is often the easiest method of finding the common difference.

**Try It** Write a recursive formula for the arithmetic sequence.

\[ \{25, 37, 49, 61, \ldots\} \]
Using Explicit Formulas for Arithmetic Sequences

We can think of an arithmetic sequence as a function on the domain of the natural numbers; it is a linear function because it has a constant rate of change. The common difference is the constant rate of change, or the slope of the function. We can construct the linear function if we know the slope and the vertical intercept.

$\displaystyle a_n = a_1 + d(n - 1)$  \hspace{1cm} (11.29)

To find the $y$-intercept of the function, we can subtract the common difference from the first term of the sequence. Consider the following sequence.

\[ \{200, 150, 100, 50, 0, \ldots\} \]

The common difference is $-50$, so the sequence represents a linear function with a slope of $-50$. To find the $y$-intercept, we subtract $-50$ from $200$: $200 - (-50) = 200 + 50 = 250$. You can also find the $y$-intercept by graphing the function and determining where a line that connects the points would intersect the vertical axis. The graph is shown in Figure 11.3.

![Graph of an arithmetic sequence](image)

Recall the slope-intercept form of a line is $y = mx + b$. When dealing with sequences, we use $a_n$ in place of $y$ and $n$ in place of $x$. If we know the slope and vertical intercept of the function, we can substitute them for $m$ and $b$ in the slope-intercept form of a line. Substituting $-50$ for the slope and $250$ for the vertical intercept, we get the following equation:

$\displaystyle a_n = -50n + 250$  \hspace{1cm} (11.30)

We do not need to find the vertical intercept to write an explicit formula for an arithmetic sequence. Another explicit formula for this sequence is $a_n = 200 - 50(n - 1)$, which simplifies to $a_n = -50n + 250$.

### Explicit Formula for an Arithmetic Sequence

An explicit formula for the $n$th term of an arithmetic sequence is given by

$\displaystyle a_n = a_1 + d(n - 1)$  \hspace{1cm} (11.31)

**How To:**

Given the first several terms for an arithmetic sequence, write an explicit formula.

1. Find the common difference, $a_2 - a_1$.
2. Substitute the common difference and the first term into $a_n = a_1 + d(n - 1)$.

### Example 11.12
Writing the $n$th Term Explicit Formula for an Arithmetic Sequence

Write an explicit formula for the arithmetic sequence.

\[ \{2, 12, 22, 32, 42, \ldots \} \]

Solution

Analysis

The graph of this sequence, represented in Figure 11.3, shows a slope of 10 and a vertical intercept of $-8$.

Finding the Number of Terms in a Finite Arithmetic Sequence

Explicit formulas can be used to determine the number of terms in a finite arithmetic sequence. We need to find the common difference, and then determine how many times the common difference must be added to the first term to obtain the final term of the sequence.

How To: Given the first three terms and the last term of a finite arithmetic sequence, find the total number of terms.

1. Find the common difference $d$.
2. Substitute the common difference and the first term into $a_n = a_1 + d(n - 1)$.
3. Substitute the last term for $a_n$ and solve for $n$.

Example 11.13

Finding the Number of Terms in a Finite Arithmetic Sequence

Find the number of terms in the finite arithmetic sequence.

\[ \{8, 1, -6, \ldots, -41\} \]
11.16 Find the number of terms in the finite arithmetic sequence.
\{6, 11, 16, ..., 56\} \quad (11.33)

**Solving Application Problems with Arithmetic Sequences**

In many application problems, it often makes sense to use an initial term of \(a_0\) instead of \(a_1\). In these problems, we alter the explicit formula slightly to account for the difference in initial terms. We use the following formula:
\[a_n = a_0 + dn\] \quad (11.34)

**Example 11.14**

**Solving Application Problems with Arithmetic Sequences**

A five-year old child receives an allowance of $1 each week. His parents promise him an annual increase of $2 per week.

a. Write a formula for the child’s weekly allowance in a given year.

b. What will the child’s allowance be when he is 16 years old?

**Solution**

11.17 A woman decides to go for a 10-minute run every day this week and plans to increase the time of her daily run by 4 minutes each week. Write a formula for the time of her run after \(n\) weeks. How long will her daily run be 8 weeks from today?

Access this online resource for additional instruction and practice with arithmetic sequences.
- **Arithmetic Sequences** (http://openstaxcollege.org/l/arithmeticseq)
11.2 EXERCISES

Verbal

72. What is an arithmetic sequence?

73. How is the common difference of an arithmetic sequence found?

74. How do we determine whether a sequence is arithmetic?

75. What are the main differences between using a recursive formula and using an explicit formula to describe an arithmetic sequence?

76. Describe how linear functions and arithmetic sequences are similar. How are they different?

Algebraic

For the following exercises, find the common difference for the arithmetic sequence provided.

77. \{5, 11, 17, 23, 29, \ldots\}

78. \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\right\}

For the following exercises, determine whether the sequence is arithmetic. If so find the common difference.

79. \{11.4, 9.3, 7.2, 5.1, 3, \ldots\}

80. \{4, 16, 64, 256, 1024, \ldots\}

For the following exercises, write the first five terms of the arithmetic sequence given the first term and common difference.

81. \(a_1 = -25, d = -9\)

82. \(a_1 = 0, d = \frac{2}{3}\)

For the following exercises, write the first five terms of the arithmetic series given two terms.

83. \(a_1 = 17, a_7 = -31\)

84. \(a_{13} = -60, a_{33} = -160\)

For the following exercises, find the specified term for the arithmetic sequence given the first term and common difference.

85. First term is 3, common difference is 4, find the 5th term.

86. First term is 4, common difference is 5, find the 4th term.

87. First term is 5, common difference is 6, find the 8th term.

88. First term is 6, common difference is 7, find the 6th term.

89. First term is 7, common difference is 8, find the 7th term.

For the following exercises, find the first term given two terms from an arithmetic sequence.

90. Find the first term or \(a_1\) of an arithmetic sequence if \(a_6 = 12\) and \(a_{14} = 28\).

91. Find the first term or \(a_1\) of an arithmetic sequence if \(a_7 = 21\) and \(a_{15} = 42\).

92. Find the first term or \(a_1\) of an arithmetic sequence if \(a_8 = 40\) and \(a_{23} = 115\).
93. Find the first term or $a_1$ of an arithmetic sequence if $a_9 = 54$ and $a_{17} = 102$.

94. Find the first term or $a_1$ of an arithmetic sequence if $a_{11} = 11$ and $a_{21} = 16$.

For the following exercises, find the specified term given two terms from an arithmetic sequence.

95. $a_1 = 33$ and $a_7 = -15$. Find $a_4$.

96. $a_5 = -17.1$ and $a_{10} = -15.7$. Find $a_{21}$.

For the following exercises, use the recursive formula to write the first five terms of the arithmetic sequence.

97. $a_1 = 39; a_n = a_{n-1} - 3$

98. $a_1 = -19; a_n = a_{n-1} - 1.4$

For the following exercises, write a recursive formula for each arithmetic sequence.

99. $a_n = \{40, 60, 80, \ldots\}$

100. $a_n = \{17, 26, 35, \ldots\}$

101. $a_n = \{-1, 2, 5, \ldots\}$

102. $a_n = \{12, 17, 22, \ldots\}$

103. $a_n = \{-15, -7, 1, \ldots\}$

104. $a_n = \{8.9, 10.3, 11.7, \ldots\}$

105. $a_n = \{-0.52, -1.02, -1.52, \ldots\}$

106. $a_n = \left\{\frac{1}{5}, \frac{9}{20}, \frac{7}{10}, \ldots\right\}$

107. $a_n = \left\{-\frac{1}{2}, -\frac{5}{4}, -2, \ldots\right\}$

108. $a_n = \left\{\frac{1}{6}, -\frac{11}{12}, -2, \ldots\right\}$

For the following exercises, write a recursive formula for the given arithmetic sequence, and then find the specified term.

109. $a_n = \{7, 4, 1, \ldots\}$; Find the 17th term.

110. $a_n = \{4, 11, 18, \ldots\}$; Find the 14th term.

111. $a_n = \{2, 6, 10, \ldots\}$; Find the 12th term.

For the following exercises, use the explicit formula to write the first five terms of the arithmetic sequence.

112. $a_n = 24 - 4n$

113. $a_n = \frac{1}{2}n - \frac{1}{2}$

For the following exercises, write an explicit formula for each arithmetic sequence.

114. $a_n = \{3, 5, 7, \ldots\}$
115. \(a_n = [32, 24, 16, \ldots]\)

116. \(a_n = [-5, 95, 195, \ldots]\)

117. \(a_n = \{-17, -217, -417, \ldots\}\)

118. \(a_n = [1.8, 3.6, 5.4, \ldots]\)

119. \(a_n = [-18.1, -16.2, -14.3, \ldots]\)

120. \(a_n = [15.8, 18.5, 21.2, \ldots]\)

121. \(a_n = \left\{ \frac{1}{3}, -\frac{4}{3}, -3, \ldots \right\}\)

122. \(a_n = \left\{ 0, \frac{1}{3}, \frac{2}{3}, \ldots \right\}\)

123. \(a_n = \left\{ -5, -\frac{10}{3}, -\frac{5}{3}, \ldots \right\}\)

For the following exercises, find the number of terms in the given finite arithmetic sequence.

124. \(a_n = [3, -4, -11, \ldots, -60]\)

125. \(a_n = [1.2, 1.4, 1.6, \ldots, 3.8]\)

126. \(a_n = \left\{ \frac{1}{2}, 2, \frac{7}{2}, \ldots, 8 \right\}\)

**Graphical**

For the following exercises, determine whether the graph shown represents an arithmetic sequence.

127.
128.
For the following exercises, use the information provided to graph the first 5 terms of the arithmetic sequence.

129. \( a_1 = 0, \ d = 4 \)

130. \( a_1 = 9; \ a_n = a_{n-1} - 10 \)

131. \( a_n = -12 + 5n \)

**Technology**

For the following exercises, follow the steps to work with the arithmetic sequence \( a_n = 3n - 2 \) using a graphing calculator:

- Press \([\text{MODE}]\)
  - Select SEQ in the fourth line
  - Select DOT in the fifth line
  - Press \([\text{ENTER}]\)

- Press \([\text{Y=}]\)
  - \( n\text{Min} \) is the first counting number for the sequence. Set \( n\text{Min} = 1 \)
  - \( u(n) \) is the pattern for the sequence. Set \( u(n) = 3n - 2 \)
  - \( u(n\text{Min}) \) is the first number in the sequence. Set \( u(n\text{Min}) = 1 \)

- Press \([\text{2ND}]\) then \([\text{WINDOW}]\) to go to \([\text{TBLSET}]\)
  - Set \( \text{TblStart} = 1 \)
  - Set \( \Delta \text{Tbl} = 1 \)
  - Set Indpnt: Auto and Depend: Auto

- Press \([\text{2ND}]\) then \([\text{GRAPH}]\) to go to the \([\text{TABLE}]\)

132. What are the first seven terms shown in the column with the heading \( u(n) \)?

133. Use the scroll-down arrow to scroll to \( n = 50 \). What value is given for \( u(n) \)?

134. Press \([\text{WINDOW}]\). Set \( n\text{Min} = 1, n\text{Max} = 5, x\text{Min} = 0, x\text{Max} = 6, y\text{Min} = -1, \) and \( y\text{Max} = 14 \). Then press \([\text{GRAPH}]\). Graph the sequence as it appears on the graphing calculator.

For the following exercises, follow the steps given above to work with the arithmetic sequence \( a_n = \frac{1}{2}n + 5 \) using a graphing calculator.

135. What are the first seven terms shown in the column with the heading \( u(n) \) in the \([\text{TABLE}]\) feature?

136. Graph the sequence as it appears on the graphing calculator. Be sure to adjust the \([\text{WINDOW}]\) settings as needed.

**Extensions**

137. Give two examples of arithmetic sequences whose 4th terms are 9.

138. Give two examples of arithmetic sequences whose 10th terms are 206.

139. Find the 5th term of the arithmetic sequence \( \{9b, 5b, b, \ldots\} \).

140. Find the 11th term of the arithmetic sequence \( \{3a - 2b, a + 2b, -a + 6b \ldots\} \).

141. At which term does the sequence \( \{5.4, 14.5, 23.6, \ldots\} \) exceed 151?

142. At which term does the sequence \( \{\frac{17}{3}, \frac{31}{6}, \frac{14}{3}, \ldots\} \) begin to have negative values?
143. For which terms does the finite arithmetic sequence \( \{\frac{5}{2}, \frac{19}{8}, \frac{9}{4}, ..., \frac{1}{8}\} \) have integer values?

144. Write an arithmetic sequence using a recursive formula. Show the first 4 terms, and then find the 31\(^{st}\) term.

145. Write an arithmetic sequence using an explicit formula. Show the first 4 terms, and then find the 28\(^{th}\) term.
11.3 | Geometric Sequences

Learning Objectives

In this section, you will:

11.3.1 Find the common ratio for a geometric sequence.
11.3.2 List the terms of a geometric sequence.
11.3.3 Use a recursive formula for a geometric sequence.
11.3.4 Use an explicit formula for a geometric sequence.

Many jobs offer an annual cost-of-living increase to keep salaries consistent with inflation. Suppose, for example, a recent college graduate finds a position as a sales manager earning an annual salary of $26,000. He is promised a 2% cost of living increase each year. His annual salary in any given year can be found by multiplying his salary from the previous year by 102%. His salary will be $26,520 after one year; $27,050.40 after two years; $27,591.41 after three years; and so on. When a salary increases by a constant rate each year, the salary grows by a constant factor. In this section, we will review sequences that grow in this way.

Finding Common Ratios

The yearly salary values described form a geometric sequence because they change by a constant factor each year. Each term of a geometric sequence increases or decreases by a constant factor called the common ratio. The sequence below is an example of a geometric sequence because each term increases by a constant factor of 6. Multiplying any term of the sequence by the common ratio 6 generates the subsequent term.

\[
\{1, 6, 36, 216, 1296, \ldots \}
\]

**Definition of a Geometric Sequence**

A geometric sequence is one in which any term divided by the previous term is a constant. This constant is called the common ratio of the sequence. The common ratio can be found by dividing any term in the sequence by the previous term. If \(a_1\) is the initial term of a geometric sequence and \(r\) is the common ratio, the sequence will be

\[
\{a_1, \; a_1 r, \; a_1 r^2, \; a_1 r^3, \; \ldots \}.
\]

**How To:** Given a set of numbers, determine if they represent a geometric sequence.

1. Divide each term by the previous term.
2. Compare the quotients. If they are the same, a common ratio exists and the sequence is geometric.

**Example 11.15**

**Finding Common Ratios**

Is the sequence geometric? If so, find the common ratio.

a. 1, 2, 4, 8, 16, ...

b. 48, 12, 4, 2, ...

**Solution**
Solution

Analysis

The graph of each sequence is shown in Figure 11.3. It seems from the graphs that both (a) and (b) appear to have the form of the graph of an exponential function in this viewing window. However, we know that (a) is geometric and so this interpretation holds, but (b) is not.

If you are told that a sequence is geometric, do you have to divide every term by the previous term to find the common ratio?

No. If you know that the sequence is geometric, you can choose any one term in the sequence and divide it by the previous term to find the common ratio.

Is the sequence geometric? If so, find the common ratio.

11.18 Is the sequence geometric? If so, find the common ratio.

5, 10, 15, 20, ...

11.19 Is the sequence geometric? If so, find the common ratio.

100, 20, 4, $\frac{4}{5}$, ...

Writing Terms of Geometric Sequences

Now that we can identify a geometric sequence, we will learn how to find the terms of a geometric sequence if we are given the first term and the common ratio. The terms of a geometric sequence can be found by beginning with the first term and multiplying by the common ratio repeatedly. For instance, if the first term of a geometric sequence is $a_1 = -2$ and the common ratio is $r = 4$, we can find subsequent terms by multiplying $-2 \cdot 4$ to get $-8$ then multiplying the result $-8 \cdot 4$ to get $-32$ and so on.

$a_1 = -2$

$a_2 = (-2 \cdot 4) = -8$

$a_3 = (-8 \cdot 4) = -32$

$a_4 = (-32 \cdot 4) = -128$

The first four terms are $\{-2, -8, -32, -128\}$.
Given the first term and the common factor, find the first four terms of a geometric sequence.

1. Multiply the initial term, \(a_1\), by the common ratio to find the next term, \(a_2\).
2. Repeat the process, using \(a_n = a_2\) to find \(a_3\) and then \(a_3\) to find \(a_4\), until all four terms have been identified.
3. Write the terms separated by commas within brackets.

Example 11.16

**Writing the Terms of a Geometric Sequence**

List the first four terms of the geometric sequence with \(a_1 = 5\) and \(r = -2\).

**Solution**

11.20 List the first five terms of the geometric sequence with \(a_1 = 18\) and \(r = \frac{1}{3}\).

**Using Recursive Formulas for Geometric Sequences**

A recursive formula allows us to find any term of a geometric sequence by using the previous term. Each term is the product of the common ratio and the previous term. For example, suppose the common ratio is 9. Then each term is nine times the previous term. As with any recursive formula, the initial term must be given.

**Recursive Formula for a Geometric Sequence**

The recursive formula for a geometric sequence with common ratio \(r\) and first term \(a_1\) is

\[
a_n = r a_{n-1}, \quad n \geq 2
\]  

(11.39)

**Example 11.17**

**Using Recursive Formulas for Geometric Sequences**

Write a recursive formula for the following geometric sequence.

\{6, 9, 13.5, 20.25, ...\}

**Solution**

**Analysis**
The sequence of data points follows an exponential pattern. The common ratio is also the base of an exponential function as shown in Figure 11.3.

**Do we have to divide the second term by the first term to find the common ratio?**

No. We can divide any term in the sequence by the previous term. It is, however, most common to divide the second term by the first term because it is often the easiest method of finding the common ratio.

**11.21** Write a recursive formula for the following geometric sequence.

\[
\left\{2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \ldots \right\}
\]

**Using Explicit Formulas for Geometric Sequences**

Because a geometric sequence is an exponential function whose domain is the set of positive integers, and the common ratio is the base of the function, we can write explicit formulas that allow us to find particular terms.

\[
a_n = a_1 r^{n-1}
\]

Let's take a look at the sequence \(\{18, 36, 72, 144, 288, \ldots\}\). This is a geometric sequence with a common ratio of 2 and an exponential function with a base of 2. An explicit formula for this sequence is

\[
a_n = 18 \cdot 2^{n-1}
\]

The graph of the sequence is shown in Figure 11.4.
Explicit Formula for a Geometric Sequence

The $n$th term of a geometric sequence is given by the explicit formula:

$$a_n = a_1 r^{n-1}$$  \hfill (11.43)

Example 11.18

Writing Terms of Geometric Sequences Using the Explicit Formula

Given a geometric sequence with $a_1 = 3$ and $a_4 = 24$, find $a_2$.

Solution

Analysis

The common ratio is multiplied by the first term once to find the second term, twice to find the third term, three times to find the fourth term, and so on. The tenth term could be found by multiplying the first term by the common ratio nine times or by multiplying by the common ratio raised to the ninth power.

Example 11.19

Writing an Explicit Formula for the $n$th Term of a Geometric Sequence

Write an explicit formula for the $n$th term of the following geometric sequence.

$$\{2, 10, 50, 250, \ldots\}$$

Solution

Example 11.20

Solving Application Problems with Geometric Sequences

In real-world scenarios involving arithmetic sequences, we may need to use an initial term of $a_0$ instead of $a_1$. In these problems, we can alter the explicit formula slightly by using the following formula:

$$a_n = a_0 r^n$$  \hfill (11.45)
In 2013, the number of students in a small school is 284. It is estimated that the student population will increase by 4% each year.

a. Write a formula for the student population.

b. Estimate the student population in 2020.

\[ P_n = P_0 \times (1 + r)^n \]

where:
- \( P_n \) is the number of students in year \( n \)
- \( P_0 = 284 \) is the initial number of students
- \( r = 0.04 \) is the annual growth rate
- \( n \) is the number of years

Estimation for 2020:

\[ P_{2020} = 284 \times (1 + 0.04)^7 \]

\[ P_{2020} \approx 408 \]

A business starts a new website. Initially the number of hits is 293 due to the curiosity factor. The business estimates the number of hits will increase by 2.6% per week.

a. Write a formula for the number of hits.

b. Estimate the number of hits in 5 weeks.

\[ H_n = H_0 \times (1 + \frac{r}{100})^{n \times 4} \]

where:
- \( H_n \) is the number of hits in week \( n \)
- \( H_0 = 293 \) is the initial number of hits
- \( r = 2.6 \) is the weekly growth rate
- \( n \) is the number of weeks

Estimation for 5 weeks:

\[ H_5 = 293 \times (1 + \frac{2.6}{100})^{5 \times 4} \]

\[ H_5 \approx 365 \]
11.3 EXERCISES

Verbal
146. What is a geometric sequence?
147. How is the common ratio of a geometric sequence found?
148. What is the procedure for determining whether a sequence is geometric?
149. What is the difference between an arithmetic sequence and a geometric sequence?
150. Describe how exponential functions and geometric sequences are similar. How are they different?

Algebraic
For the following exercises, find the common ratio for the geometric sequence.
151. 1, 3, 9, 27, 81, ...
152. −0.125, 0.25, −0.5, 1, −2, ...
153. −2, −\frac{1}{2}, −\frac{1}{8}, −\frac{1}{32}, −\frac{1}{128}, ...

For the following exercises, determine whether the sequence is geometric. If so, find the common ratio.
154. −6, −12, −24, −48, −96, ...
155. 5, 5.2, 5.4, 5.6, 5.8, ...
156. −1, \frac{1}{2}, −\frac{1}{4}, \frac{1}{8}, −\frac{1}{16}, ...
157. 6, 8, 11, 15, 20, ...
158. 0.8, 4, 20, 100, 500, ...

For the following exercises, write the first five terms of the geometric sequence, given the first term and common ratio.
159. \(a_1 = 8, \ r = 0.3\)
160. \(a_1 = 5, \ r = \frac{1}{5}\)

For the following exercises, write the first five terms of the geometric sequence, given any two terms.
161. \(a_7 = 64, \ a_{10} = 512\)
162. \(a_6 = 25, \ a_8 = 6.25\)

For the following exercises, find the specified term for the geometric sequence, given the first term and common ratio.
163. The first term is 2, and the common ratio is 3. Find the 5th term.
164. The first term is 16 and the common ratio is −\frac{1}{3}. Find the 4th term.

For the following exercises, find the specified term for the geometric sequence, given the first four terms.
165. \(a_n = \{-1, 2, -4, 8, \ldots\}\). Find \(a_{12}\).
\[ a_n = \left\{ -2, \frac{2}{3}, -\frac{2}{9}, \frac{2}{27}, \ldots \right\}. \] Find \( a_7 \).

For the following exercises, write the first five terms of the geometric sequence.

167. \( a_1 = -486, \quad a_n = -\frac{1}{3}a_{n-1} \)

168. \( a_1 = 7, \quad a_n = 0.2a_{n-1} \)

For the following exercises, write a recursive formula for each geometric sequence.

169. \( a_n = \{-1, 5, -25, 125, \ldots\} \)

170. \( a_n = \{-32, -16, -8, -4, \ldots\} \)

171. \( a_n = \{14, 56, 224, 896, \ldots\} \)

172. \( a_n = \{10, -3, 0.9, -0.27, \ldots\} \)

173. \( a_n = \{0.61, 1.83, 5.49, 16.47, \ldots\} \)

174. \( a_n = \left\{\frac{3}{5}, \frac{1}{10}, \frac{1}{60}, \frac{1}{360}, \ldots\right\} \)

175. \( a_n = \left\{-2, \frac{4}{3}, -\frac{8}{9}, \frac{16}{27}, \ldots\right\} \)

176. \( a_n = \left\{\frac{1}{512}, -\frac{1}{128}, \frac{1}{32}, -\frac{1}{8}, \ldots\right\} \)

For the following exercises, write the first five terms of the geometric sequence.

177. \( a_n = -4 \cdot 5^{n-1} \)

178. \( a_n = 12 \cdot \left(-\frac{1}{2}\right)^{n-1} \)

For the following exercises, write an explicit formula for each geometric sequence.

179. \( a_n = \{-2, -4, -8, -16, \ldots\} \)

180. \( a_n = \{1, 3, 9, 27, \ldots\} \)

181. \( a_n = \{-4, -12, -36, -108, \ldots\} \)

182. \( a_n = \{0.8, -4, 20, -100, \ldots\} \)

183. \( a_n = \{-1.25, -5, -20, -80, \ldots\} \)

184. \( a_n = \left\{-1, -\frac{4}{5}, -\frac{16}{25}, -\frac{64}{125}, \ldots\right\} \)

185. \( a_n = \left\{2, \frac{1}{3}, \frac{1}{18}, \frac{1}{108}, \ldots\right\} \)

186. \( a_n = \{3, -1, \frac{1}{3}, -\frac{1}{9}, \ldots\} \)

For the following exercises, find the specified term for the geometric sequence given.

187. \( \ldots \)
Let \( a_1 = 4 \), \( a_n = -3a_{n-1} \). Find \( a_8 \).

188. Let \( a_n = -\left(\frac{1}{3}\right)^{n-1} \). Find \( a_{12} \).

For the following exercises, find the number of terms in the given finite geometric sequence.

189. \( a_n = \{ -1, 3, -9, ..., 2187 \} \)

190. \( a_n = \left\{ 2, 1, \frac{1}{2}, ..., \frac{1}{1024} \right\} \)

**Graphical**

For the following exercises, determine whether the graph shown represents a geometric sequence.

191.

192.
For the following exercises, use the information provided to graph the first five terms of the geometric sequence.

193. \( a_1 = 1, \quad r = \frac{1}{2} \)

194. \( a_1 = 3, \quad a_n = 2a_{n-1} \)

195. \( a_n = 27 \cdot 0.3^{n-1} \)

Extensions

196. Use recursive formulas to give two examples of geometric sequences whose 3rd terms are 200.

197. Use explicit formulas to give two examples of geometric sequences whose 7th terms are 1024.

198. Find the 5th term of the geometric sequence \( \{b, 4b, 16b, \ldots\} \).

199. Find the 7th term of the geometric sequence \( \{64a(-b), 32a(-3b), 16a(-9b), \ldots\} \).

200. At which term does the sequence \( \{10, 12, 14.4, 17.28, \ldots\} \) exceed 100?

201. At which term does the sequence \( \left\{ \frac{1}{2187}, \frac{1}{729}, \frac{1}{243}, \frac{1}{81}, \ldots \right\} \) begin to have integer values?

202. For which term does the geometric sequence \( a_n = -36\left(\frac{2}{3}\right)^{n-1} \) first have a non-integer value?

203. Use the recursive formula to write a geometric sequence whose common ratio is an integer. Show the first four terms, and then find the 10th term.

204. Use the explicit formula to write a geometric sequence whose common ratio is a decimal number between 0 and 1. Show the first 4 terms, and then find the 8th term.

205. Is it possible for a sequence to be both arithmetic and geometric? If so, give an example.
11.4 | Series and Their Notations

Learning Objectives

11.4.1 Use summation notation.
11.4.2 Use the formula for the sum of the first n terms of an arithmetic series.
11.4.3 Use the formula for the sum of the first n terms of a geometric series.
11.4.4 Use the formula for the sum of an infinite geometric series.
11.4.5 Solve annuity problems.

A couple decides to start a college fund for their daughter. They plan to invest $50 in the fund each month. The fund pays 6% annual interest, compounded monthly. How much money will they have saved when their daughter is ready to start college in 6 years? In this section, we will learn how to answer this question. To do so, we need to consider the amount of money invested and the amount of interest earned.

Using Summation Notation

To find the total amount of money in the college fund and the sum of the amounts deposited, we need to add the amounts deposited each month and the amounts earned monthly. The sum of the terms of a sequence is called a series. Consider, for example, the following series.

\[ 3 + 7 + 11 + 15 + 19 + \ldots \]  

The nth partial sum of a series is the sum of a finite number of consecutive terms beginning with the first term. The notation \( S_n \) represents the partial sum.

\[
\begin{align*}
S_1 &= 3 \\
S_2 &= 3 + 7 = 10 \\
S_3 &= 3 + 7 + 11 = 21 \\
S_4 &= 3 + 7 + 11 + 15 = 36
\end{align*}
\]

Summation notation is used to represent series. Summation notation is often known as sigma notation because it uses the Greek capital letter sigma, \( \Sigma \), to represent the sum. Summation notation includes an explicit formula and specifies the first and last terms in the series. An explicit formula for each term of the series is given to the right of the sigma. A variable called the index of summation is written below the sigma. The index of summation is set equal to the lower limit of summation, which is the number used to generate the first term in the series. The number above the sigma, called the upper limit of summation, is the number used to generate the last term in a series. See Figure 11.5.

If we interpret the given notation, we see that it asks us to find the sum of the terms in the series \( a_k = 2k \) for \( k = 1 \) through \( k = 5 \). We can begin by substituting the terms for \( k \) and listing out the terms of this series.

\[
\begin{align*}
a_1 &= 2(1) = 2 \\
a_2 &= 2(2) = 4 \\
a_3 &= 2(3) = 6 \\
a_4 &= 2(4) = 8 \\
a_5 &= 2(5) = 10
\end{align*}
\]

We can find the sum of the series by adding the terms:

\[
\sum_{k=1}^{5} 2k = 2 + 4 + 6 + 8 + 10 = 30
\]
Summation Notation

The sum of the first \( n \) terms of a series can be expressed in summation notation as follows:

\[
\sum_{k=1}^{n} a_k
\]  

This notation tells us to find the sum of \( a_k \) from \( k = 1 \) to \( k = n \).

\( k \) is called the **index of summation**, 1 is the **lower limit of summation**, and \( n \) is the **upper limit of summation**.

**Does the lower limit of summation have to be 1?**

No. The lower limit of summation can be any number, but 1 is frequently used. We will look at examples with lower limits of summation other than 1.

**How To:**

Given summation notation for a series, evaluate the value.

1. Identify the lower limit of summation.
2. Identify the upper limit of summation.
3. Substitute each value of \( k \) from the lower limit to the upper limit into the formula.
4. Add to find the sum.

**Example 11.21**

**Using Summation Notation**

Evaluate \( \sum_{k=3}^{7} k^2 \).

**Solution**

Evaluate \( \sum_{k=2}^{5} (3k - 1) \).

**Using the Formula for Arithmetic Series**

Just as we studied special types of sequences, we will look at special types of series. Recall that an arithmetic sequence is a sequence in which the difference between any two consecutive terms is the common difference, \( d \). The sum of the terms of an arithmetic sequence is called an **arithmetic series**. We can write the sum of the first \( n \) terms of an arithmetic series as:

\[
S_n = a_1 + (a_1 + d) + (a_1 + 2d) + ... + (a_n - d) + a_n.
\]  

We can also reverse the order of the terms and write the sum as

\[
S_n = a_n + (a_n - d) + (a_n - 2d) + ... + (a_1 + d) + a_1.
\]  

If we add these two expressions for the sum of the first \( n \) terms of an arithmetic series, we can derive a formula for the sum of the first \( n \) terms of any arithmetic series.
Because there are \( n \) terms in the series, we can simplify this sum to

\[
2S_n = n(a_1 + a_n). \tag{11.54}
\]

We divide by 2 to find the formula for the sum of the first \( n \) terms of an arithmetic series.

\[
S_n = \frac{n(a_1 + a_n)}{2} \tag{11.55}
\]

**Formula for the Sum of the First \( n \) Terms of an Arithmetic Series**

An **arithmetic series** is the sum of the terms of an arithmetic sequence. The formula for the sum of the first \( n \) terms of an arithmetic sequence is

\[
S_n = \frac{n(a_1 + a_n)}{2} \tag{11.56}
\]

**Given terms of an arithmetic series, find the sum of the first \( n \) terms.**

1. Identify \( a_1 \) and \( a_n \).
2. Determine \( n \).
3. Substitute values for \( a_1, a_n, \) and \( n \) into the formula \( S_n = \frac{n(a_1 + a_n)}{2} \).
4. Simplify to find \( S_n \).

**Example 11.22**

**Finding the First \( n \) Terms of an Arithmetic Series**

Find the sum of each arithmetic series.

\[
a. \quad 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32
\]

\[
b. \quad 20 + 15 + 10 + \ldots + -50
\]

\[
c. \quad \sum_{k=1}^{12} \ 3k - 8
\]

**Solution**

Use the formula to find the sum of each arithmetic series.

**Try It** 11.26 \( 1.4 + 1.6 + 1.8 + 2.0 + 2.2 + 2.4 + 2.6 + 2.8 + 3.0 + 3.2 + 3.4 \)

**Try It** 11.27 \( 13 + 21 + 29 + \ldots + 69 \)
Example 11.23

Solving Application Problems with Arithmetic Series

On the Sunday after a minor surgery, a woman is able to walk a half-mile. Each Sunday, she walks an additional quarter-mile. After 8 weeks, what will be the total number of miles she has walked?

Solution

A man earns $100 in the first week of June. Each week, he earns $12.50 more than the previous week. After 12 weeks, how much has he earned?

Using the Formula for Geometric Series

Just as the sum of the terms of an arithmetic sequence is called an arithmetic series, the sum of the terms in a geometric sequence is called a geometric series. Recall that a geometric sequence is a sequence in which the ratio of any two consecutive terms is the common ratio, \( r \). We can write the sum of the first \( n \) terms of a geometric series as

\[
S_n = a_1 + ra_1 + r^2a_1 + \ldots + r^{n-1}a_1.
\]  

(11.57)

Just as with arithmetic series, we can do some algebraic manipulation to derive a formula for the sum of the first \( n \) terms of a geometric series. We will begin by multiplying both sides of the equation by \( r \).

\[
rS_n = ra_1 + r^2a_1 + r^3a_1 + \ldots + r^n a_1
\]

(11.58)

Next, we subtract this equation from the original equation.

\[
S_n = a_1 + ra_1 + r^2a_1 + \ldots + r^{n-1}a_1
\]

\[
rS_n = ra_1 + r^2a_1 + r^3a_1 + \ldots + r^n a_1
\]

\[
(1-r)S_n = a_1 - r^n a_1
\]

(11.59)

Notice that when we subtract, all but the first term of the top equation and the last term of the bottom equation cancel out. To obtain a formula for \( S_n \), divide both sides by \( 1-r \).

\[
S_n = \frac{a_1(1-r^n)}{1-r} \quad r \neq 1
\]

(11.60)

Formula for the Sum of the First \( n \) Terms of a Geometric Series

A geometric series is the sum of the terms in a geometric sequence. The formula for the sum of the first \( n \) terms of a geometric sequence is represented as

\[
S_n = \frac{a_1(1-r^n)}{1-r} \quad r \neq 1
\]

(11.61)
Given a geometric series, find the sum of the first $n$ terms.

1. Identify $a_1$, $r$, and $n$.
2. Substitute values for $a_1$, $r$, and $n$ into the formula $S_n = \frac{a_1(1 - r^n)}{1 - r}$.
3. Simplify to find $S_n$.

Example 11.24

Finding the First $n$ Terms of a Geometric Series

Use the formula to find the indicated partial sum of each geometric series.

a. $S_{11}$ for the series $8 + -4 + 2 + ...$

b. $\sum_{k=1}^{6} 3 \cdot 2^k$

Solution

Use the formula to find the indicated partial sum of each geometric series.

11.30 $S_{20}$ for the series $1,000 + 500 + 250 + ...$

11.31 $\sum_{k=1}^{8} 3^k$

Example 11.25

Solving an Application Problem with a Geometric Series

At a new job, an employee’s starting salary is $26,750. He receives a 1.6% annual raise. Find his total earnings at the end of 5 years.

Solution

11.32 At a new job, an employee’s starting salary is $32,100. She receives a 2% annual raise. How much will she have earned by the end of 8 years?

Using the Formula for the Sum of an Infinite Geometric Series

Thus far, we have looked only at finite series. Sometimes, however, we are interested in the sum of the terms of an infinite sequence rather than the sum of only the first $n$ terms. An infinite series is the sum of the terms of an infinite sequence. An example of an infinite series is $2 + 4 + 6 + 8 + ...$
This series can also be written in summation notation as \( \sum_{k = 1}^{\infty} 2k \), where the upper limit of summation is infinity. Because the terms are not tending to zero, the sum of the series increases without bound as we add more terms. Therefore, the sum of this infinite series is not defined. When the sum is not a real number, we say the series **diverges**.

**Determining Whether the Sum of an Infinite Geometric Series is Defined**

If the terms of an infinite geometric series approach 0, the sum of an infinite geometric series can be defined. The terms in this series approach 0:

\[
1 + 0.2 + 0.04 + 0.008 + 0.0016 + \ldots
\]

The common ratio \( r = 0.2 \). As \( n \) gets very large, the values of \( r^n \) get very small and approach 0. Each successive term affects the sum less than the preceding term. As each succeeding term gets closer to 0, the sum of the terms approaches a finite value. The terms of any infinite geometric series with \(-1 < r < 1\) approach 0; the sum of a geometric series is defined when \(-1 < r < 1\).

**Determining Whether the Sum of an Infinite Geometric Series is Defined**

The sum of an infinite series is defined if the series is geometric and \(-1 < r < 1\).

**How To:**

Given the first several terms of an infinite series, determine if the sum of the series exists.

1. Find the ratio of the second term to the first term.
2. Find the ratio of the third term to the second term.
3. Continue this process to ensure the ratio of a term to the preceding term is constant throughout. If so, the series is geometric.
4. If a common ratio, \( r \), was found in step 3, check to see if \(-1 < r < 1\). If so, the sum is defined. If not, the sum is not defined.

**Example 11.26**

**Determining Whether the Sum of an Infinite Series is Defined**

Determine whether the sum of each infinite series is defined.

a. \( 12 + 8 + 4 + \ldots \)

b. \( \frac{3}{4} + \frac{1}{2} + \frac{1}{3} + \ldots \)

c. \( \sum_{k = 1}^{\infty} 27 \cdot \left(\frac{1}{3}\right)^k \)

d. \( \sum_{k = 1}^{\infty} 5k \)

**Solution**

Determine whether the sum of the infinite series is defined.

\( \frac{1}{3} + \frac{1}{2} + \frac{3}{4} + \frac{9}{8} + \ldots \)
Finding Sums of Infinite Series

When the sum of an infinite geometric series exists, we can calculate the sum. The formula for the sum of an infinite series is related to the formula for the sum of the first $n$ terms of a geometric series.

$$S_n = \frac{a_1 (1 - r^n)}{1 - r} \quad (11.63)$$

We will examine an infinite series with $r = \frac{1}{2}$.

What happens to $r^n$ as $n$ increases?

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$$
$$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$$
$$\left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

The value of $r^n$ decreases rapidly. What happens for greater values of $n$?

$$\left(\frac{1}{2}\right)^{10} = \frac{1}{1,024}$$
$$\left(\frac{1}{2}\right)^{20} = \frac{1}{1,048,576}$$
$$\left(\frac{1}{2}\right)^{30} = \frac{1}{1,073,741,824}$$

As $n$ gets very large, $r^n$ gets very small. We say that, as $n$ increases without bound, $r^n$ approaches 0. As $r^n$ approaches 0, $1 - r^n$ approaches 1. When this happens, the numerator approaches $a_1$. This gives us a formula for the sum of an infinite geometric series.

**Formula for the Sum of an Infinite Geometric Series**

The formula for the sum of an infinite geometric series with $-1 < r < 1$ is

$$S = \frac{a_1}{1 - r} \quad (11.66)$$

**How To:**

**Given an infinite geometric series, find its sum.**

1. Identify $a_1$ and $r$.
2. Confirm that $-1 < r < 1$.
3. Substitute values for $a_1$ and $r$ into the formula, $S = \frac{a_1}{1 - r}$.
4. Simplify to find $S$.

**Example 11.27**

$$24 + (-12) + 6 + (-3) + ...$$

$$\sum_{k=1}^{\infty} 15 \cdot (-0.3)^k$$
Finding the Sum of an Infinite Geometric Series

Find the sum, if it exists, for the following:

a. \( 10 + 9 + 8 + 7 + \ldots \)

b. \( 248.6 + 99.44 + 39.776 + \ldots \)

c. \( \sum_{k=1}^{\infty} 4.374 \cdot \left( -\frac{1}{3} \right)^{k-1} \)

d. \( \sum_{k=1}^{\infty} \frac{1}{9} \cdot \left( \frac{2}{3} \right)^{k} \)

Solution

Example 11.28

Finding an Equivalent Fraction for a Repeating Decimal

Find an equivalent fraction for the repeating decimal \( 0.\overline{3} \).

Solution

Solving Annuity Problems

At the beginning of the section, we looked at a problem in which a couple invested a set amount of money each month into a college fund for six years. An annuity is an investment in which the purchaser makes a sequence of periodic, equal payments. To find the amount of an annuity, we need to find the sum of all the payments and the interest earned. In the example, the couple invests $50 each month. This is the value of the initial deposit. The account paid 6% annual interest, compounded monthly. To find the interest rate per payment period, we need to divide the 6% annual percentage interest (APR) rate by 12. So the monthly interest rate is 0.5%. We can multiply the amount in the account each month by 100.5% to find the value of the account after interest has been added.

We can find the value of the annuity right after the last deposit by using a geometric series with \( a_1 = 50 \) and \( r = 100.5\% = 1.005 \). After the first deposit, the value of the annuity will be $50. Let us see if we can determine the amount in the college fund and the interest earned.
We can find the value of the annuity after \( n \) deposits using the formula for the sum of the first \( n \) terms of a geometric series. In 6 years, there are 72 months, so \( n = 72 \). We can substitute \( a_1 = 50 \), \( r = 1.005 \), and \( n = 72 \) into the formula, and simplify to find the value of the annuity after 6 years.

\[
S_{72} = \frac{50(1 - 1.005^{72})}{1 - 1.005} \approx 4,320.44
\]  

(11.67)

After the last deposit, the couple will have a total of $4,320.44 in the account. Notice, the couple made 72 payments of $50 each for a total of 72(50) = $3,600. This means that because of the annuity, the couple earned $720.44 interest in their college fund.

**Given an initial deposit and an interest rate, find the value of an annuity.**

1. Determine \( a_1 \), the value of the initial deposit.
2. Determine \( n \), the number of deposits.
3. Determine \( r \).
   a. Divide the annual interest rate by the number of times per year that interest is compounded.
   b. Add 1 to this amount to find \( r \).
4. Substitute values for \( a_1 \), \( r \), and \( n \) into the formula for the sum of the first \( n \) terms of a geometric series,

\[
S_n = \frac{a_1(1 - r^n)}{1 - r}
\]

5. Simplify to find \( S_n \), the value of the annuity after \( n \) deposits.

**Example 11.29**

**Solving an Annuity Problem**

A deposit of $100 is placed into a college fund at the beginning of every month for 10 years. The fund earns 9% annual interest, compounded monthly, and paid at the end of the month. How much is in the account right after the last deposit?

**Solution**

At the beginning of each month, $200 is deposited into a retirement fund. The fund earns 6% annual interest, compounded monthly, and paid into the account at the end of the month. How much is in the account if deposits are made for 10 years?

Access these online resources for additional instruction and practice with series.

- Arithmetic Series (http://openstaxcollege.org/l/arithmeticser)
- Geometric Series (http://openstaxcollege.org/l/geometricser)
- Summation Notation (http://openstaxcollege.org/l/sumnotation)
11.4 EXERCISES

Verbal

206. What is an \( n \)th partial sum?

207. What is the difference between an arithmetic sequence and an arithmetic series?

208. What is a geometric series?

209. How is finding the sum of an infinite geometric series different from finding the \( n \)th partial sum?

210. What is an annuity?

Algebraic

For the following exercises, express each description of a sum using summation notation.

211. The sum of terms \( m^2 + 3m \) from \( m = 1 \) to \( m = 5 \)

212. The sum from of \( n = 0 \) to \( n = 4 \) of \( 5n \)

213. The sum of \( 6k - 5 \) from \( k = -2 \) to \( k = 1 \)

214. The sum that results from adding the number 4 five times

For the following exercises, express each arithmetic sum using summation notation.

215. \( 5 + 10 + 15 + 20 + 25 + 30 + 35 + 40 + 45 + 50 \)

216. \( 10 + 18 + 26 + \ldots + 162 \)

217. \( \frac{1}{2} + 1 + \frac{3}{2} + 2 + \ldots + 4 \)

For the following exercises, use the formula for the sum of the first \( n \) terms of each arithmetic sequence.

218. \( \frac{3}{2} + 2 + \frac{5}{2} + 3 + \frac{7}{2} \)

219. \( 19 + 25 + 31 + \ldots + 73 \)

220. \( 3.2 + 3.4 + 3.6 + \ldots + 5.6 \)

For the following exercises, express each geometric sum using summation notation.

221. \( 1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187 \)

222. \( 8 + 4 + 2 + \ldots + 0.125 \)

223. \( -\frac{1}{6} + \frac{1}{12} - \frac{1}{24} + \ldots + \frac{1}{768} \)

For the following exercises, use the formula for the sum of the first \( n \) terms of each geometric sequence, and then state the indicated sum.

224. \( 9 + 3 + 1 + \frac{1}{3} + \frac{1}{9} \)

225. \( \sum_{n=1}^{9} 5 \cdot 2^n - 1 \)

226.
\[ \sum_{a=1}^{11} 64 \cdot 0.2^a - 1 \]

For the following exercises, determine whether the infinite series has a sum. If so, write the formula for the sum. If not, state the reason.

227. \[ 12 + 18 + 24 + 30 + \ldots \]
228. \[ 2 + 1.6 + 1.28 + 1.024 + \ldots \]
229. \[ \sum_{m=1}^{\infty} 4^m - 1 \]
230. \[ \sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^{k-1} \]

**Graphical**

For the following exercises, use the following scenario. Javier makes monthly deposits into a savings account. He opened the account with an initial deposit of $50. Each month thereafter he increased the previous deposit amount by $20.

231. Graph the arithmetic sequence showing one year of Javier’s deposits.
232. Graph the arithmetic series showing the monthly sums of one year of Javier’s deposits.

For the following exercises, use the geometric series \[ \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \].

233. Graph the first 7 partial sums of the series.
234. What number does \( S_n \) seem to be approaching in the graph? Find the sum to explain why this makes sense.

**Numeric**

For the following exercises, find the indicated sum.

235. \[ \sum_{a=1}^{14} a \]
236. \[ \sum_{n=1}^{6} n(n - 2) \]
237. \[ \sum_{k=1}^{17} k^2 \]
238. \[ \sum_{k=1}^{7} 2^k \]

For the following exercises, use the formula for the sum of the first \( n \) terms of an arithmetic series to find the sum.

239. \[ -1.7 + -0.4 + 0.9 + 2.2 + 3.5 + 4.8 \]
240. \[ 6 + \frac{16}{2} + 9 + \frac{21}{2} + 12 + \frac{27}{2} + 15 \]
241. \[ -1 + 3 + 7 + \ldots + 31 \]
\[ \sum_{k=1}^{11} \left( \frac{k}{2} - \frac{1}{2} \right) \]

For the following exercises, use the formula for the sum of the first \( n \) terms of a geometric series to find the partial sum.

243. \( S_6 \) for the series \(-2 - 10 - 50 - 250...\)

244. \( S_7 \) for the series \(0.4 - 2 + 10 - 50...\)

245. \[ \sum_{k=1}^{9} 2^k - 1 \]

246. \[ \sum_{n=1}^{10} -2 \cdot \left( \frac{1}{2} \right)^{n-1} \]

For the following exercises, find the sum of the infinite geometric series.

247. \( 4 + 2 + 1 + \frac{1}{2}... \)

248. \( -1 - \frac{1}{4} - \frac{1}{16} - \frac{1}{64}... \)

249. \[ \sum_{k=1}^{\infty} 3 \cdot \left( \frac{1}{4} \right)^{k-1} \]

250. \[ \sum_{n=1}^{\infty} 4.6 \cdot 0.5^n - 1 \]

For the following exercises, determine the value of the annuity for the indicated monthly deposit amount, the number of deposits, and the interest rate.

251. Deposit amount: $50; total deposits: 60; interest rate: 5%, compounded monthly

252. Deposit amount: $150; total deposits: 24; interest rate: 3%, compounded monthly

253. Deposit amount: $450; total deposits: 60; interest rate: 4.5%, compounded quarterly

254. Deposit amount: $100; total deposits: 120; interest rate: 10%, compounded semi-annually

Extensions

255. The sum of terms \( 50 - k^2 \) from \( k = x \) through 7 is 115. What is \( x \)?

256. Write an explicit formula for \( a_k \) such that \[ \sum_{k=0}^{6} a_k = 189. \] Assume this is an arithmetic series.

257. Find the smallest value of \( n \) such that \[ \sum_{k=1}^{n} (3k - 5) > 100. \]

258. How many terms must be added before the series \(-1 - 3 - 5 - 7.... \) has a sum less than \(-75\)?

259. Write \( 0.\overline{65} \) as an infinite geometric series using summation notation. Then use the formula for finding the sum of an infinite geometric series to convert \( 0.\overline{65} \) to a fraction.

260. The sum of an infinite geometric series is five times the value of the first term. What is the common ratio of the series?
To get the best loan rates available, the Riches want to save enough money to place 20% down on a $160,000 home. They plan to make monthly deposits of $125 in an investment account that offers 8.5% annual interest compounded semi-annually. Will the Riches have enough for a 20% down payment after five years of saving? How much money will they have saved?

Karl has two years to save $10,000 to buy a used car when he graduates. To the nearest dollar, what would his monthly deposits need to be if he invests in an account offering a 4.2% annual interest rate that compounds monthly?

Real-World Applications

Keisha devised a week-long study plan to prepare for finals. On the first day, she plans to study for 1 hour, and each successive day she will increase her study time by 30 minutes. How many hours will Keisha have studied after one week?

A boulder rolled down a mountain, traveling 6 feet in the first second. Each successive second, its distance increased by 8 feet. How far did the boulder travel after 10 seconds?

A scientist places 50 cells in a petri dish. Every hour, the population increases by 1.5%. What will the cell count be after 1 day?

A pendulum travels a distance of 3 feet on its first swing. On each successive swing, it travels \( \frac{3}{4} \) the distance of the previous swing. What is the total distance traveled by the pendulum when it stops swinging?

Rachael deposits $1,500 into a retirement fund each year. The fund earns 8.2% annual interest, compounded monthly. If she opened her account when she was 19 years old, how much will she have by the time she is 55? How much of that amount will be interest earned?
A new company sells customizable cases for tablets and smartphones. Each case comes in a variety of colors and can be personalized for an additional fee with images or a monogram. A customer can choose not to personalize or could choose to have one, two, or three images or a monogram. The customer can choose the order of the images and the letters in the monogram. The company is working with an agency to develop a marketing campaign with a focus on the huge number of options they offer. Counting the possibilities is challenging!

We encounter a wide variety of counting problems every day. There is a branch of mathematics devoted to the study of counting problems such as this one. Other applications of counting include secure passwords, horse racing outcomes, and college scheduling choices. We will examine this type of mathematics in this section.

### Using the Addition Principle

The company that sells customizable cases offers cases for tablets and smartphones. There are 3 supported tablet models and 5 supported smartphone models. The Addition Principle tells us that we can add the number of tablet options to the number of smartphone options to find the total number of options. By the Addition Principle, there are 8 total options, as we can see in Figure 11.6.

**Figure 11.6**

<table>
<thead>
<tr>
<th>The Addition Principle</th>
</tr>
</thead>
<tbody>
<tr>
<td>According to the <strong>Addition Principle</strong>, if one event can occur in $m$ ways and a second event with no common outcomes can occur in $n$ ways, then the first or second event can occur in $m + n$ ways.</td>
</tr>
</tbody>
</table>

**Example 11.30**
Using the Addition Principle

There are 2 vegetarian entrée options and 5 meat entrée options on a dinner menu. What is the total number of entrée options?

Solution

Using the Multiplication Principle

The Multiplication Principle applies when we are making more than one selection. Suppose we are choosing an appetizer, an entrée, and a dessert. If there are 2 appetizer options, 3 entrée options, and 2 dessert options on a fixed-price dinner menu, there are a total of 12 possible choices of one each as shown in the tree diagram in Figure 11.7.

Figure 11.7

The possible choices are:
1. soup, chicken, cake
2. soup, chicken, pudding
3. soup, fish, cake
4. soup, fish, pudding
5. soup, steak, cake
6. soup, steak, pudding
7. salad, chicken, cake
8. salad, chicken, pudding
9. salad, fish, cake
10. salad, fish, pudding
11. salad, steak, cake
12. salad, steak, pudding

We can also find the total number of possible dinners by multiplying.

We could also conclude that there are 12 possible dinner choices simply by applying the Multiplication Principle.

\[
\text{# of appetizer options} \times \text{# of entrée options} \times \text{# of dessert options} = 12
\]
The Multiplication Principle
According to the Multiplication Principle, if one event can occur in \(m\) ways and a second event can occur in \(n\) ways after the first event has occurred, then the two events can occur in \(m \times n\) ways. This is also known as the Fundamental Counting Principle.

Example 11.31

Using the Multiplication Principle
Diane packed 2 skirts, 4 blouses, and a sweater for her business trip. She will need to choose a skirt and a blouse for each outfit and decide whether to wear the sweater. Use the Multiplication Principle to find the total number of possible outfits.

Solution

A restaurant offers a breakfast special that includes a breakfast sandwich, a side dish, and a beverage. There are 3 types of breakfast sandwiches, 4 side dish options, and 5 beverage choices. Find the total number of possible breakfast specials.

Finding the Number of Permutations of \(n\) Distinct Objects
The Multiplication Principle can be used to solve a variety of problem types. One type of problem involves placing objects in order. We arrange letters into words and digits into numbers, line up for photographs, decorate rooms, and more. An ordering of objects is called a permutation.

Finding the Number of Permutations of \(n\) Distinct Objects Using the Multiplication Principle
To solve permutation problems, it is often helpful to draw line segments for each option. That enables us to determine the number of each option so we can multiply. For instance, suppose we have four paintings, and we want to find the number of ways we can hang three of the paintings in order on the wall. We can draw three lines to represent the three places on the wall.

\[
\begin{align*}
\_ \times \_ \times \_ \\
\_ \times \_ \times \_ \\
\_ \times \_ \times \_ \\
\end{align*}
\]

There are four options for the first place, so we write a 4 on the first line.

\[
\begin{align*}
\_ \times \_ \times \_ \\
4 \times \_ \times \_ \\
\end{align*}
\]

After the first place has been filled, there are three options for the second place so we write a 3 on the second line.

\[
\begin{align*}
\_ \times \_ \times \_ \\
4 \times \_ \times 3 \\
\end{align*}
\]

After the second place has been filled, there are two options for the third place so we write a 2 on the third line. Finally, we find the product.

\[
\begin{align*}
\_ \times \_ \times \_ \\
4 \times 3 \times 2 \\
\end{align*}
\]

There are 24 possible permutations of the paintings.
Given \( n \) distinct options, determine how many permutations there are.

1. Determine how many options there are for the first situation.
2. Determine how many options are left for the second situation.
3. Continue until all of the spots are filled.
4. Multiply the numbers together.

Example 11.32

Finding the Number of Permutations Using the Multiplication Principle

At a swimming competition, nine swimmers compete in a race.

a. How many ways can they place first, second, and third?

b. How many ways can they place first, second, and third if a swimmer named Ariel wins first place? (Assume there is only one contestant named Ariel.)

c. How many ways can all nine swimmers line up for a photo?

Solution

Analysis

Note that in part c, we found there were \( 9! \) ways for 9 people to line up. The number of permutations of \( n \) distinct objects can always be found by \( n! \).

A family of five is having portraits taken. Use the Multiplication Principle to find the following.

11.42 How many ways can the family line up for the portrait?

11.43 How many ways can the photographer line up 3 family members?

11.44 How many ways can the family line up for the portrait if the parents are required to stand on each end?

Finding the Number of Permutations of \( n \) Distinct Objects Using a Formula

For some permutation problems, it is inconvenient to use the Multiplication Principle because there are so many numbers to multiply. Fortunately, we can solve these problems using a formula. Before we learn the formula, let’s look at two common notations for permutations. If we have a set of \( n \) objects and we want to choose \( r \) objects from the set in order, we write \( P(n, r) \). Another way to write this is \( n^P_r \), a notation commonly seen on computers and calculators. To calculate \( P(n, r) \), we begin by finding \( n! \), the number of ways to line up all \( n \) objects. We then divide by \( (n – r)! \) to cancel out the \( (n – r) \) items that we do not wish to line up.

Let’s see how this works with a simple example. Imagine a club of six people. They need to elect a president, a vice president, and a treasurer. Six people can be elected president, any one of the five remaining people can be elected vice president, and any of the remaining four people could be elected treasurer. The number of ways this may be done is \( 6 \times 5 \times 4 = 120 \). Using factorials, we get the same result.

\[
\frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!} = 6 \cdot 5 \cdot 4 = 120
\]
There are 120 ways to select 3 officers in order from a club with 6 members. We refer to this as a permutation of 6 taken 3 at a time. The general formula is as follows.

\[ P(n, r) = \frac{n!}{(n-r)!} \]  

(11.70)

Note that the formula stills works if we are choosing all \( n \) objects and placing them in order. In that case we would be dividing by \( (n-n)! \) or 0!, which we said earlier is equal to 1. So the number of permutations of \( n \) objects taken \( n \) at a time is \( \frac{n!}{1} \) or just \( n! \).

**Formula for Permutations of \( n \) Distinct Objects**

Given \( n \) distinct objects, the number of ways to select \( r \) objects from the set in order is

\[ P(n, r) = \frac{n!}{(n-r)!} \]  

(11.71)

**How To:**

Given a word problem, evaluate the possible permutations.

1. Identify \( n \) from the given information.
2. Identify \( r \) from the given information.
3. Replace \( n \) and \( r \) in the formula with the given values.
4. Evaluate.

**Example 11.33**

**Finding the Number of Permutations Using the Formula**

A professor is creating an exam of 9 questions from a test bank of 12 questions. How many ways can she select and arrange the questions?

**Solution**

**Analysis**

We can also use a calculator to find permutations. For this problem, we would enter 15, press the \( _nP_r \) function, enter 12, and then press the equal sign. The \( _nP_r \) function may be located under the MATH menu with probability commands.

**Could we have solved Example 11.33 using the Multiplication Principle?**

Yes. We could have multiplied \( 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \) to find the same answer.

A play has a cast of 7 actors preparing to make their curtain call. Use the permutation formula to find the following.

**Try It**

11.45 How many ways can the 7 actors line up?

11.46 How many ways can 5 of the 7 actors be chosen to line up?
Find the Number of Combinations Using the Formula

So far, we have looked at problems asking us to put objects in order. There are many problems in which we want to select a few objects from a group of objects, but we do not care about the order. When we are selecting objects and the order does not matter, we are dealing with combinations. A selection of \( r \) objects from a set of \( n \) objects where the order does not matter can be written as \( C(n, r) \). Just as with permutations, \( C(n, r) \) can also be written as \( _nC_r \). In this case, the general formula is as follows.

\[
C(n, r) = \frac{n!}{r!(n-r)!}
\]  

(11.72)

An earlier problem considered choosing 3 of 4 possible paintings to hang on a wall. We found that there were 24 ways to select 3 of the 4 paintings in order. But what if we did not care about the order? We would expect a smaller number because selecting paintings 1, 2, 3 would be the same as selecting paintings 2, 3, 1. To find the number of ways to select 3 of the 4 paintings, disregarding the order of the paintings, divide the number of permutations by the number of ways to order 3 paintings. There are \( 3! = 3 \cdot 2 \cdot 1 = 6 \) ways to order 3 paintings. There are \( \frac{24}{6} = 4 \) ways to select 3 of the 4 paintings.

This number makes sense because every time we are selecting 3 paintings, we are not selecting 1 painting. There are 4 paintings we could choose not to select, so there are 4 ways to select 3 of the 4 paintings.

<table>
<thead>
<tr>
<th>Formula for Combinations of ( n ) Distinct Objects</th>
</tr>
</thead>
<tbody>
<tr>
<td>Given ( n ) distinct objects, the number of ways to select ( r ) objects from the set is</td>
</tr>
</tbody>
</table>
| \[
C(n, r) = \frac{n!}{r!(n-r)!}
\]  

(11.73)

**How To:**
1. Identify \( n \) from the given information.
2. Identify \( r \) from the given information.
3. Replace \( n \) and \( r \) in the formula with the given values.
4. Evaluate.

**Example 11.34**

**Finding the Number of Combinations Using the Formula**

A fast food restaurant offers five side dish options. Your meal comes with two side dishes.

a. How many ways can you select your side dishes?

b. How many ways can you select 3 side dishes?

**Solution**

**Analysis**

We can also use a graphing calculator to find combinations. Enter 5, then press \( _nC_r \), enter 3, and then press the equal sign. The \( _nC_r \) function may be located under the MATH menu with probability commands.

**Is it a coincidence that parts (a) and (b) in Example 11.34 have the same answers?**

No. When we choose \( r \) objects from \( n \) objects, we are not choosing \( (n-r) \) objects. Therefore, \( C(n, r) = C(n, n-r) \).
An ice cream shop offers 10 flavors of ice cream. How many ways are there to choose 3 flavors for a banana split?

Finding the Number of Subsets of a Set

We have looked only at combination problems in which we chose exactly \( r \) objects. In some problems, we want to consider choosing every possible number of objects. Consider, for example, a pizza restaurant that offers 5 toppings. Any number of toppings can be ordered. How many different pizzas are possible?

To answer this question, we need to consider pizzas with any number of toppings. There is \( C(5, 0) = 1 \) way to order a pizza with no toppings. There are \( C(5, 1) = 5 \) ways to order a pizza with exactly one topping. If we continue this process, we get

\[
C(5, 0) + C(5, 1) + C(5, 2) + C(5, 3) + C(5, 4) + C(5, 5) = 32
\]

There are 32 possible pizzas. This result is equal to \( 2^5 \).

We are presented with a sequence of choices. For each of the \( n \) objects we have two choices: include it in the subset or not. So for the whole subset we have made \( n \) choices, each with two options. So there are a total of \( 2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 \) possible resulting subsets, all the way from the empty subset, which we obtain when we say “no” each time, to the original set itself, which we obtain when we say “yes” each time.

Formula for the Number of Subsets of a Set

A set containing \( n \) distinct objects has \( 2^n \) subsets.

Example 11.35

Finding the Number of Subsets of a Set

A restaurant offers butter, cheese, chives, and sour cream as toppings for a baked potato. How many different ways are there to order a potato?

Solution

11.48 A sundae bar at a wedding has 6 toppings to choose from. Any number of toppings can be chosen. How many different sundaes are possible?

Finding the Number of Permutations of \( n \) Non-Distinct Objects

We have studied permutations where all of the objects involved were distinct. What happens if some of the objects are indistinguishable? For example, suppose there is a sheet of 12 stickers. If all of the stickers were distinct, there would be \( 12! \) ways to order the stickers. However, 4 of the stickers are identical stars, and 3 are identical moons. Because all of the objects are not distinct, many of the \( 12! \) permutations we counted are duplicates. The general formula for this situation is as follows.

\[
\frac{n!}{r_1!r_2!\ldots r_k!}
\]

In this example, we need to divide by the number of ways to order the 4 stars and the ways to order the 3 moons to find the number of unique permutations of the stickers. There are \( 4! \) ways to order the stars and \( 3! \) ways to order the moon.

\[
\frac{12!}{4!3!} = 3,326,400
\]
There are 3,326,400 ways to order the sheet of stickers.

### Formula for Finding the Number of Permutations of n Non-Distinct Objects

If there are \( n \) elements in a set and \( r_1 \) are alike, \( r_2 \) are alike, \( r_3 \) are alike, and so on through \( r_k \), the number of permutations can be found by

\[
\frac{n!}{r_1!r_2! \ldots r_k!}
\]

(11.77)

### Example 11.36

**Finding the Number of Permutations of \( n \) Non-Distinct Objects**

Find the number of rearrangements of the letters in the word DISTINCT.

**Solution**

Find the number of rearrangements of the letters in the word CARRIER.

**Try it**

Find the number of rearrangements of the letters in the word CARRIER.

Access these online resources for additional instruction and practice with combinations and permutations.

- Combinations (http://openstaxcollege.org/l/combinations)
- Permutations (http://openstaxcollege.org/l/permutations)
11.5 EXERCISES

Verbal

For the following exercises, assume that there are $n$ ways an event $A$ can happen, $m$ ways an event $B$ can happen, and that $A$ and $B$ are non-overlapping.

268. Use the Addition Principle of counting to explain how many ways event $A$ or $B$ can occur.

269. Use the Multiplication Principle of counting to explain how many ways event $A$ and $B$ can occur.

Answer the following questions.

270. When given two separate events, how do we know whether to apply the Addition Principle or the Multiplication Principle when calculating possible outcomes? What conjunctions may help to determine which operations to use?

271. Describe how the permutation of $n$ objects differs from the permutation of choosing $r$ objects from a set of $n$ objects. Include how each is calculated.

272. What is the term for the arrangement that selects $r$ objects from a set of $n$ objects when the order of the $r$ objects is not important? What is the formula for calculating the number of possible outcomes for this type of arrangement?

Numeric

For the following exercises, determine whether to use the Addition Principle or the Multiplication Principle. Then perform the calculations.

273. Let the set $A = \{-5, -3, -1, 2, 3, 4, 5, 6\}$. How many ways are there to choose a negative or an even number from $A$?

274. Let the set $B = \{-23, -16, -7, -2, 20, 36, 48, 72\}$. How many ways are there to choose a positive or an odd number from $A$?

275. How many ways are there to pick a red ace or a club from a standard card playing deck?

276. How many ways are there to pick a paint color from 5 shades of green, 4 shades of blue, or 7 shades of yellow?

277. How many outcomes are possible from tossing a pair of coins?

278. How many outcomes are possible from tossing a coin and rolling a 6-sided die?

279. How many two-letter strings—the first letter from $A$ and the second letter from $B$ — can be formed from the sets $A = \{b, c, d\}$ and $B = \{a, e, i, o, u\}$?

280. How many ways are there to construct a string of 3 digits if numbers can be repeated?

281. How many ways are there to construct a string of 3 digits if numbers cannot be repeated?

For the following exercises, compute the value of the expression.

282. $P(5, 2)$

283. $P(8, 4)$

284. $P(3, 3)$

285. $P(9, 6)$

286. $P(11, 5)$

287. $C(8, 5)$

288. $C(12, 4)$
For the following exercises, find the number of subsets in each given set.

292. \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}

293. \{a, b, c, \ldots, z\}

294. A set containing 5 distinct numbers, 4 distinct letters, and 3 distinct symbols

295. The set of even numbers from 2 to 28

296. The set of two-digit numbers between 1 and 100 containing the digit 0

For the following exercises, find the distinct number of arrangements.

297. The letters in the word “juggernaut”

298. The letters in the word “academia”

299. The letters in the word “academia” that begin and end in “a”

300. The symbols in the string #,#,#,@,@,$,$,$,%,%,%

301. The symbols in the string #,#,#,@,@,$,$,$,%,%,% that begin and end with “%”

Extensions

302. The set, \(S\) consists of 900,000,000 whole numbers, each being the same number of digits long. How many digits long is a number from \(S\)? (Hint: use the fact that a whole number cannot start with the digit 0.)

303. The number of 5-element subsets from a set containing \(n\) elements is equal to the number of 6-element subsets from the same set. What is the value of \(n\)? (Hint: the order in which the elements for the subsets are chosen is not important.)

304. Can \(C(n, r)\) ever equal \(P(n, r)\)? Explain.

305. Suppose a set \(A\) has 2,048 subsets. How many distinct objects are contained in \(A\)?

306. How many arrangements can be made from the letters of the word “mountains” if all the vowels must form a string?

Real-World Applications

307. A family consisting of 2 parents and 3 children is to pose for a picture with 2 family members in the front and 3 in the back.
   a. How many arrangements are possible with no restrictions?
   b. How many arrangements are possible if the parents must sit in the front?
   c. How many arrangements are possible if the parents must be next to each other?

308. A cell phone company offers 6 different voice packages and 8 different data packages. Of those, 3 packages include both voice and data. How many ways are there to choose either voice or data, but not both?

309. In horse racing, a “trifecta” occurs when a bettor wins by selecting the first three finishers in the exact order (1st place, 2nd place, and 3rd place). How many different trifectas are possible if there are 14 horses in a race?

310. A wholesale T-shirt company offers sizes small, medium, large, and extra-large in organic or non-organic cotton and colors white, black, gray, blue, and red. How many different T-shirts are there to choose from?

311. Hector wants to place billboard advertisements throughout the county for his new business. How many ways can Hector choose 15 neighborhoods to advertise in if there are 30 neighborhoods in the county?
312. An art store has 4 brands of paint pens in 12 different colors and 3 types of ink. How many paint pens are there to choose from?

313. How many ways can a committee of 3 freshmen and 4 juniors be formed from a group of 8 freshmen and 11 juniors?

314. How many ways can a baseball coach arrange the order of 9 batters if there are 15 players on the team?

315. A conductor needs 5 cellists and 5 violinists to play at a diplomatic event. To do this, he ranks the orchestra’s 10 cellists and 16 violinists in order of musical proficiency. What is the ratio of the total cellist rankings possible to the total violinist rankings possible?

316. A motorcycle shop has 10 choppers, 6 bobbers, and 5 café racers—different types of vintage motorcycles. How many ways can the shop choose 3 choppers, 5 bobbers, and 2 café racers for a weekend showcase?

317. A skateboard shop stocks 10 types of board decks, 3 types of trucks, and 4 types of wheels. How many different skateboards can be constructed?

318. Just-For-Kicks Sneaker Company offers an online customizing service. How many ways are there to design a custom pair of Just-For-Kicks sneakers if a customer can choose from a basic shoe up to 11 customizable options?

319. A car wash offers the following optional services to the basic wash: clear coat wax, triple foam polish, undercarriage wash, rust inhibitor, wheel brightener, air freshener, and interior shampoo. How many washes are possible if any number of options can be added to the basic wash?

320. Susan bought 20 plants to arrange along the border of her garden. How many distinct arrangements can she make if the plants are comprised of 6 tulips, 6 roses, and 8 daisies?

321. How many unique ways can a string of Christmas lights be arranged from 9 red, 10 green, 6 white, and 12 gold color bulbs?
11.6 | Binomial Theorem

Learning Objectives

In this section, you will:

11.6.1 Apply the Binomial Theorem.

A polynomial with two terms is called a binomial. We have already learned to multiply binomials and to raise binomials to powers, but raising a binomial to a high power can be tedious and time-consuming. In this section, we will discuss a shortcut that will allow us to find \((x + y)^n\) without multiplying the binomial by itself \(n\) times.

Identifying Binomial Coefficients

In Counting Principles (http://legacy.cnx.org/content/m10365/latest/), we studied combinations. In the shortcut to finding \((x + y)^n\), we will need to use combinations to find the coefficients that will appear in the expansion of the binomial. In this case, we use the notation \(\binom{n}{r}\) instead of \(C(n, r)\), but it can be calculated in the same way. So

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\]  (11.78)

The combination \(\binom{n}{r}\) is called a binomial coefficient. An example of a binomial coefficient is \(\binom{5}{2} = C(5, 2) = 10\).

Binomial Coefficients

If \(n\) and \(r\) are integers greater than or equal to 0 with \(n \geq r\), then the binomial coefficient is

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\]  (11.79)

Q&A

Is a binomial coefficient always a whole number?

Yes. Just as the number of combinations must always be a whole number, a binomial coefficient will always be a whole number.

Example 11.37

Finding Binomial Coefficients

Find each binomial coefficient.

a. \(\binom{5}{3}\)

b. \(\binom{9}{2}\)

c. \(\binom{9}{7}\)

Solution

Analysis
Notice that we obtained the same result for parts (b) and (c). If you look closely at the solution for these two parts, you will see that you end up with the same two factorials in the denominator, but the order is reversed, just as with combinations.

\[ \binom{n}{r} = \binom{n}{n-r} \]

**11.50** Find each binomial coefficient.

- a. \( \binom{7}{3} \)
- b. \( \binom{11}{4} \)

**Using the Binomial Theorem**

When we expand \((x + y)^n\) by multiplying, the result is called a **binomial expansion**, and it includes binomial coefficients. If we wanted to expand \((x + y)^{52}\), we might multiply \((x + y)\) by itself fifty-two times. This could take hours! If we examine some simple binomial expansions, we can find patterns that will lead us to a shortcut for finding more complicated binomial expansions.

\[
(x + y)^2 = x^2 + 2xy + y^2 \\
(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \\
(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4
\]

First, let’s examine the exponents. With each successive term, the exponent for \(x\) decreases and the exponent for \(y\) increases. The sum of the two exponents is \(n\) for each term.

Next, let’s examine the coefficients. Notice that the coefficients increase and then decrease in a symmetrical pattern. The coefficients follow a pattern:

\[
\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}.
\]

These patterns lead us to the **Binomial Theorem**, which can be used to expand any binomial.

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k
\]

Another way to see the coefficients is to examine the expansion of a binomial in general form, \(x + y\), to successive powers 1, 2, 3, and 4.

\[
(x + y)^1 = x + y \\
(x + y)^2 = x^2 + 2xy + y^2 \\
(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \\
(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4
\]

Can you guess the next expansion for the binomial \((x + y)^5\)?
Figure 11.8

See Figure 11.8, which illustrates the following:

- There are \( n + 1 \) terms in the expansion of \((x + y)^n\).
- The degree (or sum of the exponents) for each term is \( n \).
- The powers on \( x \) begin with \( n \) and decrease to 0.
- The powers on \( y \) begin with 0 and increase to \( n \).
- The coefficients are symmetric.

To determine the expansion on \((x + y)^5\), we see \( n = 5 \), thus, there will be 5+1 = 6 terms. Each term has a combined degree of 5. In descending order for powers of \( x \), the pattern is as follows:

- Introduce \( x^5 \), and then for each successive term reduce the exponent on \( x \) by 1 until \( x^0 = 1 \) is reached.
- Introduce \( y^0 = 1 \), and then increase the exponent on \( y \) by 1 until \( y^5 \) is reached.

\[
(x + y)^5 = x^5, \ x^4y, \ x^3y^2, \ x^2y^3, \ xyz^4, \ y^5
\]  

(11.84)

The next expansion would be

\[
(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5
\]  

(11.85)

But where do those coefficients come from? The binomial coefficients are symmetric. We can see these coefficients in an array known as Pascal’s Triangle, shown in Figure 11.9.

Figure 11.9

To generate Pascal’s Triangle, we start by writing a 1. In the row below, row 2, we write two 1’s. In the 3\textsuperscript{rd} row, flank the ends of the rows with 1’s, and add 1 + 1 to find the middle number, 2. In the \( n \)th row, flank the ends of the row with 1’s. Each element in the triangle is the sum of the two elements immediately above it.
To see the connection between Pascal’s Triangle and binomial coefficients, let us revisit the expansion of the binomials in general form.

\[
\begin{align*}
\binom{n}{k} & = 1 \\
(x + y)^0 & = 1 \\
(x + y)^1 & = x + y \\
(x + y)^2 & = x^2 + 2xy + y^2 \\
(x + y)^3 & = x^3 + 3x^2y + 3xy^2 + y^3 \\
(x + y)^4 & = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \\
(x + y)^5 & = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5
\end{align*}
\]

The Binomial Theorem

The **Binomial Theorem** is a formula that can be used to expand any binomial.

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k
\]

\[
= x^n + \left(\binom{n}{1}\right)x^{n-1}y + \left(\binom{n}{2}\right)x^{n-2}y^2 + \ldots + \left(\binom{n}{n-1}\right)x^1y^{n-1} + y^n
\]

**How To**

Given a binomial, write it in expanded form.

1. Determine the value of \( n \) according to the exponent.
2. Evaluate the \( k = 0 \) through \( k = n \) using the Binomial Theorem formula.

**Example 11.38**

**Expanding a Binomial**

Write in expanded form.

a. \((x + y)^5\)

b. \((3x - y)^4\)

**Solution**

**Analysis**

Notice the alternating signs in part b. This happens because \((-y)\) raised to odd powers is negative, but \((-y)\) raised to even powers is positive. This will occur whenever the binomial contains a subtraction sign.

**Try It**

11.51 Write in expanded form.

a. \((x - y)^5\)

b. \((2x + 5y)^3\)
Using the Binomial Theorem to Find a Single Term

Expanding a binomial with a high exponent such as \((x + 2y)^{16}\) can be a lengthy process.

Sometimes we are interested only in a certain term of a binomial expansion. We do not need to fully expand a binomial to find a single specific term.

Note the pattern of coefficients in the expansion of \((x + y)^5\).

\[
(x + y)^5 = x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + y^5
\]  

(11.87)

The second term is \(\binom{5}{1}x^4y\). The third term is \(\binom{5}{2}x^3y^2\). We can generalize this result.

\[
\binom{n}{r}x^{n-r}y^r
\]

(11.88)

**The \((r+1)\)th Term of a Binomial Expansion**

The \((r + 1)\)th term of the binomial expansion of \((x + y)^n\) is:

\[
\binom{n}{r}x^{n-r}y^r
\]

(11.89)

**Given a binomial, write a specific term without fully expanding.**

1. Determine the value of \(n\) according to the exponent.
2. Determine \((r + 1)\).
3. Determine \(r\).
4. Replace \(r\) in the formula for the \((r + 1)\)th term of the binomial expansion.

**Example 11.39**

**Writing a Given Term of a Binomial Expansion**

Find the tenth term of \((x + 2y)^{16}\) without fully expanding the binomial.

**Solution**

**Example 11.39**

Find the sixth term of \((3x - y)^9\) without fully expanding the binomial.

Access these online resources for additional instruction and practice with binomial expansion.

- The Binomial Theorem (http://openstaxcollege.org/l/binomialtheorem)
- Binomial Theorem Example (http://openstaxcollege.org/l/btexample)
11.6 EXERCISES

Verbal

322. What is a binomial coefficient, and how it is calculated?

323. What role do binomial coefficients play in a binomial expansion? Are they restricted to any type of number?

324. What is the Binomial Theorem and what is its use?

325. When is it an advantage to use the Binomial Theorem? Explain.

Algebraic

For the following exercises, evaluate the binomial coefficient.

326. \( \binom{6}{2} \)

327. \( \binom{5}{3} \)

328. \( \binom{7}{4} \)

329. \( \binom{9}{7} \)

330. \( \binom{10}{9} \)

331. \( \binom{25}{11} \)

332. \( \binom{17}{6} \)

333. \( \binom{200}{199} \)

For the following exercises, use the Binomial Theorem to expand each binomial.

334. \((4a - b)^3\)

335. \((5a + 2)^3\)

336. \((3a + 2b)^3\)

337. \((2x + 3y)^4\)

338. \((4x + 2y)^5\)

339. \((3x - 2y)^4\)

340. \((4x - 3y)^5\)

341. \((\frac{1}{x} + 3y)^5\)

342. \((x^{-1} + 2y^{-1})^4\)
343. \((x - y)^5\)

For the following exercises, use the Binomial Theorem to write the first three terms of each binomial.

344. \((a + b)^{17}\)
345. \((x - 1)^{18}\)
346. \((a - 2b)^{15}\)
347. \((x - 2y)^8\)
348. \((3a + b)^{20}\)
349. \((2a + 4b)^7\)
350. \((x^3 - b)^8\)

For the following exercises, find the indicated term of each binomial without fully expanding the binomial.

351. The fourth term of \((2x - 3y)^4\)
352. The fourth term of \((3x - 2y)^5\)
353. The third term of \((6x - 3y)^7\)
354. The eighth term of \((7 + 5y)^{14}\)
355. The seventh term of \((a + b)^{11}\)
356. The fifth term of \((x - y)^7\)
357. The tenth term of \((x - 1)^{12}\)
358. The ninth term of \((a - 3b^2)^{11}\)
359. The fourth term of \(\left(x^3 - \frac{1}{2}\right)^{10}\)
360. The eighth term of \(\left(\frac{y}{2} + \frac{2}{x}\right)^9\)

**Graphical**

For the following exercises, use the Binomial Theorem to expand the binomial \(f(x) = (x + 3)^4\). Then find and graph each indicated sum on one set of axes.

361. Find and graph \(f_1(x)\), such that \(f_1(x)\) is the first term of the expansion.
362. Find and graph \(f_2(x)\), such that \(f_2(x)\) is the sum of the first two terms of the expansion.
363. Find and graph \(f_3(x)\), such that \(f_3(x)\) is the sum of the first three terms of the expansion.
364. Find and graph \(f_4(x)\), such that \(f_4(x)\) is the sum of the first four terms of the expansion.
365. Find and graph \( f_5(x) \), such that \( f_5(x) \) is the sum of the first five terms of the expansion.

**Extensions**

366. In the expansion of \((5x + 3y)^n\), each term has the form \( \binom{n}{k} a^{n-k} b^k \), where \( k \) successively takes on the value 0, 1, 2, ..., \( n \). If \( \binom{n}{k} = \left( \begin{array}{c} 7 \\ 2 \end{array} \right) \), what is the corresponding term?

367. In the expansion of \((a + b)^n\), the coefficient of \( a^{n-k} b^k \) is the same as the coefficient of which other term?

368. Consider the expansion of \((x + b)^{40}\). What is the exponent of \( b \) in the \( k \)th term?

369. Find \( \binom{n}{k} + \binom{n}{k-1} \) and write the answer as a binomial coefficient in the form \( \binom{n}{k} \). Prove it. Hint: Use the fact that, for any integer \( p \), such that \( p \geq 1 \), \( p! = p(p - 1)! \).

370. Which expression cannot be expanded using the Binomial Theorem? Explain.

- \((x^2 - 2x + 1)\)
- \((\sqrt{a} + 4\sqrt{a} - 5)^8\)
- \((x^3 + 2y^2 - z)^5\)
- \((3x^2 + \sqrt[3]{2}y^3)^{12}\)
11.7 Probability

Learning Objectives

In this section, you will:

11.7.1 Construct probability models.
11.7.2 Compute probabilities of equally likely outcomes.
11.7.3 Compute probabilities of the union of two events.
11.7.4 Use the complement rule to find probabilities.
11.7.5 Compute probability using counting theory.

Figure 11.10 An example of a “spaghetti model,” which can be used to predict possible paths of a tropical storm.[1]

Residents of the Southeastern United States are all too familiar with charts, known as spaghetti models, such as the one in Figure 11.10. They combine a collection of weather data to predict the most likely path of a hurricane. Each colored line represents one possible path. The group of squiggly lines can begin to resemble strands of spaghetti, hence the name. In this section, we will investigate methods for making these types of predictions.

Constructing Probability Models

Suppose we roll a six-sided number cube. Rolling a number cube is an example of an experiment, or an activity with an observable result. The numbers on the cube are possible results, or outcomes, of this experiment. The set of all possible outcomes of an experiment is called the sample space of the experiment. The sample space for this experiment is \{1, 2, 3, 4, 5, 6\}. An event is any subset of a sample space.

The likelihood of an event is known as probability. The probability of an event \( p \) is a number that always satisfies \( 0 \leq p \leq 1 \), where 0 indicates an impossible event and 1 indicates a certain event. A probability model is a mathematical description of an experiment listing all possible outcomes and their associated probabilities. For instance, if there is a 1% chance of winning a raffle and a 99% chance of losing the raffle, a probability model would look much like Table 11.3.

1. The figure is for illustrative purposes only and does not model any particular storm.
<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Winning the raffle</td>
<td>1%</td>
</tr>
<tr>
<td>Losing the raffle</td>
<td>99%</td>
</tr>
</tbody>
</table>

Table 11.3

The sum of the probabilities listed in a probability model must equal 1, or 100%.

**How To:** Given a probability event where each event is equally likely, construct a probability model.

1. Identify every outcome.
2. Determine the total number of possible outcomes.
3. Compare each outcome to the total number of possible outcomes.

**Example 11.40**

**Constructing a Probability Model**

Construct a probability model for rolling a single, fair die, with the event being the number shown on the die.

**Solution**

Do probabilities always have to be expressed as fractions?

No. Probabilities can be expressed as fractions, decimals, or percents. Probability must always be a number between 0 and 1, inclusive of 0 and 1.

**Try it** 11.53 Construct a probability model for tossing a fair coin.

**Computing Probabilities of Equally Likely Outcomes**

Let $S$ be a sample space for an experiment. When investigating probability, an event is any subset of $S$. When the outcomes of an experiment are all equally likely, we can find the probability of an event by dividing the number of outcomes in the event by the total number of outcomes in $S$. Suppose a number cube is rolled, and we are interested in finding the probability of the event “rolling a number less than or equal to 4.” There are 4 possible outcomes in the event and 6 possible outcomes in $S$, so the probability of the event is $\frac{4}{6} = \frac{2}{3}$.

**Computing the Probability of an Event with Equally Likely Outcomes**

The probability of an event $E$ in an experiment with sample space $S$ with equally likely outcomes is given by

$$P(E) = \frac{\text{number of elements in } E}{\text{number of elements in } S} = \frac{n(E)}{n(S)}$$

where $E$ is a subset of $S$, so it is always true that $0 \leq P(E) \leq 1$. 

This content is available for free at http://legacy.cnx.org/content/col11667/1.2
Example 11.41

Computing the Probability of an Event with Equally Likely Outcomes

A number cube is rolled. Find the probability of rolling an odd number.

Solution

Try It

11.54 A number cube is rolled. Find the probability of rolling a number greater than 2.

Computing the Probability of the Union of Two Events

We are often interested in finding the probability that one of multiple events occurs. Suppose we are playing a card game, and we will win if the next card drawn is either a heart or a king. We would be interested in finding the probability of the next card being a heart or a king. The union of two events \( E \) and \( F \), written \( E \cup F \), is the event that occurs if either or both events occur.

\[
P(E \cup F) = P(E) + P(F) - P(E \cap F)
\]  

(11.91)

Suppose the spinner in Figure 11.11 is spun. We want to find the probability of spinning orange or spinning a \( b \).

![Figure 11.11](image)

There are a total of 6 sections, and 3 of them are orange. So the probability of spinning orange is \( \frac{3}{6} = \frac{1}{2} \). There are a total of 6 sections, and 2 of them have a \( b \). So the probability of spinning a \( b \) is \( \frac{2}{6} = \frac{1}{3} \). If we added these two probabilities, we would be counting the sector that is both orange and a \( b \) twice. To find the probability of spinning an orange or a \( b \), we need to subtract the probability that the sector is both orange and has a \( b \).

\[
\frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}
\]  

(11.92)

The probability of spinning orange or a \( b \) is \( \frac{2}{3} \).
Probability of the Union of Two Events

The probability of the union of two events \( E \) and \( F \) (written \( E \cup F \)) equals the sum of the probability of \( E \) and the probability of \( F \) minus the probability of \( E \) and \( F \) occurring together (which is called the intersection of \( E \) and \( F \) and is written as \( E \cap F \)).

\[
P(E \cup F) = P(E) + P(F) - P(E \cap F)
\]

(11.93)

Example 11.42

Computing the Probability of the Union of Two Events

A card is drawn from a standard deck. Find the probability of drawing a heart or a 7.

Solution

A card is drawn from a standard deck. Find the probability of drawing a red card or an ace.

Computing the Probability of Mutually Exclusive Events

Suppose the spinner in Figure 11.11 is spun again, but this time we are interested in the probability of spinning an orange or a \( d \). There are no sectors that are both orange and contain a \( d \), so these two events have no outcomes in common. Events are said to be mutually exclusive events when they have no outcomes in common. Because there is no overlap, there is nothing to subtract, so the general formula is

\[
P(E \cup F) = P(E) + P(F)
\]

(11.94)

Notice that with mutually exclusive events, the intersection of \( E \) and \( F \) is the empty set. The probability of spinning an orange is \( \frac{3}{6} = \frac{1}{2} \) and the probability of spinning a \( d \) is \( \frac{1}{6} \). We can find the probability of spinning an orange or a \( d \) simply by adding the two probabilities.

\[
P(E \cup F) = P(E) + P(F)
= \frac{1}{2} + \frac{1}{6}
= \frac{2}{3}
\]

The probability of spinning an orange or a \( d \) is \( \frac{2}{3} \).

Probability of the Union of Mutually Exclusive Events

The probability of the union of two mutually exclusive events \( E \) and \( F \) is given by

\[
P(E \cup F) = P(E) + P(F)
\]

(11.96)
Given a set of events, compute the probability of the union of mutually exclusive events.

1. Determine the total number of outcomes for the first event.
2. Find the probability of the first event.
3. Determine the total number of outcomes for the second event.
4. Find the probability of the second event.
5. Add the probabilities.

Example 11.43

Computing the Probability of the Union of Mutually Exclusive Events

A card is drawn from a standard deck. Find the probability of drawing a heart or a spade.

Solution

A card is drawn from a standard deck. Find the probability of drawing an ace or a king.

Using the Complement Rule to Compute Probabilities

We have discussed how to calculate the probability that an event will happen. Sometimes, we are interested in finding the probability that an event will not happen. The complement of an event $E$, denoted $E'$, is the set of outcomes in the sample space that are not in $E$. For example, suppose we are interested in the probability that a horse will lose a race. If event $W$ is the horse winning the race, then the complement of event $W$ is the horse losing the race.

To find the probability that the horse loses the race, we need to use the fact that the sum of all probabilities in a probability model must be 1.

\[
P(E') = 1 - P(E)
\] \hspace{1cm} (11.97)

The probability of the horse winning added to the probability of the horse losing must be equal to 1. Therefore, if the probability of the horse winning the race is $\frac{1}{9}$, the probability of the horse losing the race is simply

\[
1 - \frac{1}{9} = \frac{8}{9}
\] \hspace{1cm} (11.98)

The Complement Rule

The probability that the complement of an event will occur is given by

\[
P(E') = 1 - P(E)
\] \hspace{1cm} (11.99)

Example 11.44

Using the Complement Rule to Calculate Probabilities

Two six-sided number cubes are rolled.

a. Find the probability that the sum of the numbers rolled is less than or equal to 3.

b. Find the probability that the sum of the numbers rolled is greater than 3.
Computing Probability Using Counting Theory

Many interesting probability problems involve counting principles, permutations, and combinations. In these problems, we will use permutations and combinations to find the number of elements in events and sample spaces. These problems can be complicated, but they can be made easier by breaking them down into smaller counting problems.

Assume, for example, that a store has 8 cellular phones and that 3 of those are defective. We might want to find the probability that a couple purchasing 2 phones receives 2 phones that are not defective. To solve this problem, we need to calculate all of the ways to select 2 phones that are not defective as well as all of the ways to select 2 phones. There are 5 phones that are not defective, so there are \( \binom{5}{2} \) ways to select 2 phones that are not defective. There are 8 phones, so there are \( \binom{8}{2} \) ways to select 2 phones. The probability of selecting 2 phones that are not defective is:

\[
\frac{\text{ways to select 2 phones that are not defective}}{\text{ways to select 2 phones}} = \frac{\binom{5}{2}}{\binom{8}{2}} = \frac{10}{28} = \frac{5}{14} \tag{11.100}
\]

Example 11.45

Computing Probability Using Counting Theory

A child randomly selects 5 toys from a bin containing 3 bunnies, 5 dogs, and 6 bears.

a. Find the probability that only bears are chosen.
b. Find the probability that 2 bears and 3 dogs are chosen.
c. Find the probability that at least 2 dogs are chosen.

Solution

11.58 A child randomly selects 3 gumballs from a container holding 4 purple gumballs, 8 yellow gumballs, and 2 green gumballs.

a. Find the probability that all 3 gumballs selected are purple.
b. Find the probability that no yellow gumballs are selected.
c. Find the probability that at least 1 yellow gumball is selected.

Access these online resources for additional instruction and practice with probability.

- Introduction to Probability (http://openstaxcollege.org/l/introprob)
- Determining Probability (http://openstaxcollege.org/l/determineprob)
11.7 EXERCISES

Verbal

371. What term is used to express the likelihood of an event occurring? Are there restrictions on its values? If so, what are they? If not, explain.

372. What is a sample space?

373. What is an experiment?

374. What is the difference between events and outcomes? Give an example of both using the sample space of tossing a coin 50 times.

375. The union of two sets is defined as a set of elements that are present in at least one of the sets. How is this similar to the definition used for the union of two events from a probability model? How is it different?

Numeric

For the following exercises, use the spinner shown in Figure 11.12 to find the probabilities indicated.

![Figure 11.12]

376. Landing on red

377. Landing on a vowel

378. Not landing on blue

379. Landing on purple or a vowel

380. Landing on blue or a vowel

381. Landing on green or blue

382. Landing on yellow or a consonant

383. Not landing on yellow or a consonant

For the following exercises, two coins are tossed.

384. What is the sample space?

385. Find the probability of tossing two heads.

386. Find the probability of tossing exactly one tail.

387. Find the probability of tossing at least one tail.

For the following exercises, four coins are tossed.

388. What is the sample space?
389. Find the probability of tossing exactly two heads.
390. Find the probability of tossing exactly three heads.
391. Find the probability of tossing four heads or four tails.
392. Find the probability of tossing all tails.
393. Find the probability of tossing not all tails.
394. Find the probability of tossing exactly two heads or at least two tails.
395. Find the probability of tossing either two heads or three heads.

For the following exercises, one card is drawn from a standard deck of 52 cards. Find the probability of drawing the following:

396. A club
397. A two
398. Six or seven
399. Red six
400. An ace or a diamond
401. A non-ace
402. A heart or a non-jack

For the following exercises, two dice are rolled, and the results are summed.

403. Construct a table showing the sample space of outcomes and sums.
404. Find the probability of rolling a sum of 3.
405. Find the probability of rolling at least one four or a sum of 8.
406. Find the probability of rolling an odd sum less than 9.
407. Find the probability of rolling a sum greater than or equal to 15.
408. Find the probability of rolling a sum less than 15.
409. Find the probability of rolling a sum less than 6 or greater than 9.
410. Find the probability of rolling a sum between 6 and 9, inclusive.
411. Find the probability of rolling a sum of 5 or 6.
412. Find the probability of rolling any sum other than 5 or 6.

For the following exercises, a coin is tossed, and a card is pulled from a standard deck. Find the probability of the following:

413. A head on the coin or a club
414. A tail on the coin or red ace
415. A head on the coin or a face card
416. No aces

For the following exercises, use this scenario: a bag of M&Ms contains 12 blue, 6 brown, 10 orange, 8 yellow, 8 red, and 4 green M&Ms. Reaching into the bag, a person grabs 5 M&Ms.
417. What is the probability of getting all blue M&Ms?
418. What is the probability of getting 4 blue M&Ms?
419. What is the probability of getting 3 blue M&Ms?
420. What is the probability of getting no brown M&Ms?

**Extensions**

Use the following scenario for the exercises that follow: In the game of Keno, a player starts by selecting 20 numbers from the numbers 1 to 80. After the player makes his selections, 20 winning numbers are randomly selected from numbers 1 to 80. A win occurs if the player has correctly selected 3, 4, or 5 of the 20 winning numbers. (Round all answers to the nearest hundredth of a percent.)

421. What is the percent chance that a player selects exactly 3 winning numbers?
422. What is the percent chance that a player selects exactly 4 winning numbers?
423. What is the percent chance that a player selects all 5 winning numbers?
424. What is the percent chance of winning?
425. How much less is a player’s chance of selecting 3 winning numbers than the chance of selecting either 4 or 5 winning numbers?

**Real-World Applications**

Use this data for the exercises that follow: In 2013, there were roughly 317 million citizens in the United States, and about 40 million were elderly (aged 65 and over).\(^{[2]}\)

426. If you meet a U.S. citizen, what is the percent chance that the person is elderly? (Round to the nearest tenth of a percent.)
427. If you meet five U.S. citizens, what is the percent chance that exactly one is elderly? (Round to the nearest tenth of a percent.)
428. If you meet five U.S. citizens, what is the percent chance that three are elderly? (Round to the nearest tenth of a percent.)
429. If you meet five U.S. citizens, what is the percent chance that four are elderly? (Round to the nearest thousandth of a percent.)
430. It is predicted that by 2030, one in five U.S. citizens will be elderly. How much greater will the chances of meeting an elderly person be at that time? What policy changes do you foresee if these statistics hold true?

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CHAPTER 11 REVIEW

KEY TERMS

Addition Principle: if one event can occur in \( m \) ways and a second event with no common outcomes can occur in \( n \) ways, then the first or second event can occur in \( m + n \) ways.

annuity: an investment in which the purchaser makes a sequence of periodic, equal payments.

arithmetic sequence: a sequence in which the difference between any two consecutive terms is a constant.

arithmetic series: the sum of the terms in an arithmetic sequence.

Binomial Theorem: a formula that can be used to expand any binomial.

binomial coefficient: the number of ways to choose \( r \) objects from \( n \) objects where order does not matter; equivalent to \( \binom{n}{r} \) denoted \( \binom{n}{r} \).

binomial expansion: the result of expanding \((x + y)^n\) by multiplying.

combination: a selection of objects in which order does not matter.

common difference: the difference between any two consecutive terms in an arithmetic sequence.

common ratio: the ratio between any two consecutive terms in a geometric sequence.

complement of an event: the set of outcomes in the sample space that are not in the event \( E \).

diverge: a series is said to diverge if the sum is not a real number.

event: any subset of a sample space.

experiment: an activity with an observable result.

explicit formula: a formula that defines each term of a sequence in terms of its position in the sequence.

Fundamental Counting Principle: if one event can occur in \( m \) ways and a second event can occur in \( n \) ways after the first event has occurred, then the two events can occur in \( m \times n \) ways; also known as the Multiplication Principle.

finite sequence: a function whose domain consists of a finite subset of the positive integers \( \{1, 2, \ldots, n\} \) for some positive integer \( n \).

geometric sequence: a sequence in which the ratio of a term to a previous term is a constant.

geometric series: the sum of the terms in a geometric sequence.

index of summation: in summation notation, the variable used in the explicit formula for the terms of a series and written below the sigma with the lower limit of summation.

infinite sequence: a function whose domain is the set of positive integers.

infinite series: the sum of the terms in an infinite sequence.

lower limit of summation: the number used in the explicit formula to find the first term in a series.

Multiplication Principle: if one event can occur in \( m \) ways and a second event can occur in \( n \) ways after the first event has occurred, then the two events can occur in \( m \times n \) ways; also known as the Fundamental Counting Principle.

mutually exclusive events: events that have no outcomes in common.

n factorial: the product of all the positive integers from 1 to \( n \).
**nth partial sum**: the sum of the first \( n \) terms of a sequence

**nth term of a sequence**: a formula for the general term of a sequence

**outcomes**: the possible results of an experiment

**permutation**: a selection of objects in which order matters

**probability model**: a mathematical description of an experiment listing all possible outcomes and their associated probabilities

**probability**: a number from 0 to 1 indicating the likelihood of an event

**recursive formula**: a formula that defines each term of a sequence using previous term(s)

**sample space**: the set of all possible outcomes of an experiment

**sequence**: a function whose domain is a subset of the positive integers

**series**: the sum of the terms in a sequence

**summation notation**: a notation for series using the Greek letter sigma; it includes an explicit formula and specifies the first and last terms in the series

**term**: a number in a sequence

**union of two events**: the event that occurs if either or both events occur

**upper limit of summation**: the number used in the explicit formula to find the last term in a series

### KEY EQUATIONS

<table>
<thead>
<tr>
<th>Formula for a factorial</th>
<th>( 0! = 1 )</th>
<th>( 1! = 1 )</th>
<th>( n! = n(n - 1)(n - 2) \cdots (2)(1) ), for ( n \geq 2 )</th>
</tr>
</thead>
</table>

**Table 11.4**

<table>
<thead>
<tr>
<th>Recursive formula for nth term of a geometric sequence</th>
<th>( a_1 = \frac{1}{24} ), ( a_2 = 1 ), ( a_n = (2a_{n-2} - 3a_{n-1}) )</th>
</tr>
</thead>
</table>

**Table 11.5**

<table>
<thead>
<tr>
<th>Recursive formula for nth term of an arithmetic sequence</th>
<th>( a_n = (-1)^n(n + 1) )</th>
<th>( 2n + 9 )</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Explicit formulas for nth term of an arithmetic sequence</th>
<th>( a_n = a_1 + d(n - 1) )</th>
<th>( a_n = a_0 + dn )</th>
</tr>
</thead>
</table>

**Table 11.6**
sum of the first \( n \) terms of an arithmetic series
\[ S_n = \frac{n(a_1 + a_n)}{2} \]

sum of the first \( n \) terms of a geometric series
\[ S_n = \frac{a_1(1 - r^n)}{1 - r} \cdot r \neq 1 \]

sum of an infinite geometric series with \(-1 < r < 1\)
\[ S_n = \frac{a_1}{1 - r} \cdot r \neq 1 \]

**Table 11.7**

number of permutations of \( n \) distinct objects taken \( r \) at a time
\[ P(n, r) = \frac{n!}{(n-r)!} \]

number of combinations of \( n \) distinct objects taken \( r \) at a time
\[ C(n, r) = \frac{n!}{r!(n-r)!} \]

number of permutations of \( n \) non-distinct objects
\[ \frac{n!}{r_1!r_2! \ldots r_k!} \]

**Table 11.8**

Binomial Theorem
\[ (x + y)^n \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \]

\((r + 1)th\) term of a binomial expansion
\[ \binom{n}{r} x^{n-r} y^r \]

**Table 11.9**

**KEY CONCEPTS**

11.1 Sequences and Their Notation
- A sequence is a list of numbers, called terms, written in a specific order.
- Explicit formulas define each term of a sequence using the position of the term. See Example 11.1, Example 11.2, and Example 11.3.
- An explicit formula for the \( n \)th term of a sequence can be written by analyzing the pattern of several terms. See Example 11.4.
- Recursive formulas define each term of a sequence using previous terms.
- Recursive formulas must state the initial term, or terms, of a sequence.
- A set of terms can be written by using a recursive formula. See Example 11.5 and Example 11.6.
A factorial is a mathematical operation that can be defined recursively. The factorial of \( n \) is the product of all integers from 1 to \( n \) See Example 11.7.

### 11.2 Arithmetic Sequences

- An arithmetic sequence is a sequence where the difference between any two consecutive terms is a constant.
- The constant between two consecutive terms is called the common difference.
- The common difference is the number added to any one term of an arithmetic sequence that generates the subsequent term. See Example 11.8.
- The terms of an arithmetic sequence can be found by beginning with the initial term and adding the common difference repeatedly. See Example 11.9 and Example 11.10.
- A recursive formula for an arithmetic sequence with common difference \( d \) is given by \( a_n = a_{n-1} + d \), \( n \geq 2 \). See Example 11.11.
- As with any recursive formula, the initial term of the sequence must be given.
- An explicit formula for an arithmetic sequence with common difference \( d \) is given by \( a_n = a_1 + d(n - 1) \). See Example 11.12.
- An explicit formula can be used to find the number of terms in a sequence. See Example 11.13.
- In application problems, we sometimes alter the explicit formula slightly to \( a_n = a_0 + dn \). See Example 11.14.

### 11.3 Geometric Sequences

- A geometric sequence is a sequence in which the ratio between any two consecutive terms is a constant.
- The constant ratio between two consecutive terms is called the common ratio.
- The common ratio can be found by dividing any term in the sequence by the previous term. See Example 11.15.
- The terms of a geometric sequence can be found by beginning with the first term and multiplying by the common ratio repeatedly. See Example 11.16 and Example 11.18.
- A recursive formula for a geometric sequence with common ratio \( r \) is given by \( a_n = ra_{n-1} \) for \( n \geq 2 \).
- As with any recursive formula, the initial term of the sequence must be given. See Example 11.17.
- An explicit formula for a geometric sequence with common ratio \( r \) is given by \( a_n = a_1 r^{n-1} \). See Example 11.19.
- In application problems, we sometimes alter the explicit formula slightly to \( a_n = a_0 r^n \). See Example 11.20.

### 11.4 Series and Their Notations

- The sum of the terms in a sequence is called a series.
- A common notation for series is called summation notation, which uses the Greek letter sigma to represent the sum. See Example 11.21.
- The sum of the terms in an arithmetic sequence is called an arithmetic series.
- The sum of the first \( n \) terms of an arithmetic series can be found using a formula. See Example 11.22 and Example 11.23.
- The sum of the terms in a geometric sequence is called a geometric series.
- The sum of the first \( n \) terms of a geometric series can be found using a formula. See Example 11.24 and Example 11.25.
- The sum of an infinite series exists if the series is geometric with \(-1 < r < 1\).
- If the sum of an infinite series exists, it can be found using a formula. See Example 11.26, Example 11.27, and Example 11.28.
- An annuity is an account into which the investor makes a series of regularly scheduled payments. The value of an annuity can be found using geometric series. See Example 11.29.
11.5 Counting Principles

• If one event can occur in $m$ ways and a second event with no common outcomes can occur in $n$ ways, then the first or second event can occur in $m + n$ ways. See Example 11.30.

• If one event can occur in $m$ ways and a second event can occur in $n$ ways after the first event has occurred, then the two events can occur in $m \times n$ ways. See Example 11.31.

• A permutation is an ordering of $n$ objects.

• If we have a set of $n$ objects and we want to choose $r$ objects from the set in order, we write $P(n, r)$.

• Permutation problems can be solved using the Multiplication Principle or the formula for $P(n, r)$. See Example 11.32 and Example 11.33.

• A selection of objects where the order does not matter is a combination.

• Given $n$ distinct objects, the number of ways to select $r$ objects from the set is $C(n, r)$ and can be found using a formula. See Example 11.34.

• A set containing $n$ distinct objects has $2^n$ subsets. See Example 11.35.

• For counting problems involving non-distinct objects, we need to divide to avoid counting duplicate permutations. See Example 11.36.

11.6 Binomial Theorem

• \[ \binom{n}{r} \] is called a binomial coefficient and is equal to $C(n, r)$. See Example 11.37.

• The Binomial Theorem allows us to expand binomials without multiplying. See Example 11.38.

• We can find a given term of a binomial expansion without fully expanding the binomial. See Example 11.39.

11.7 Probability

• Probability is always a number between 0 and 1, where 0 means an event is impossible and 1 means an event is certain.

• The probabilities in a probability model must sum to 1. See Example 11.40.

• When the outcomes of an experiment are all equally likely, we can find the probability of an event by dividing the number of outcomes in the event by the total number of outcomes in the sample space for the experiment. See Example 11.41.

• To find the probability of the union of two events, we add the probabilities of the two events and subtract the probability that both events occur simultaneously. See Example 11.42.

• To find the probability of the union of two mutually exclusive events, we add the probabilities of each of the events. See Example 11.43.

• The probability of the complement of an event is the difference between 1 and the probability that the event occurs. See Example 11.44.

• In some probability problems, we need to use permutations and combinations to find the number of elements in events and sample spaces. See Example 11.45.

CHAPTER 11 REVIEW EXERCISES

m10361 (http://legacy.cnx.org/content/m10361/latest/)

489. Write the first four terms of the sequence defined by the recursive formula $a_1 = 2, a_n = a_{n-1} + n$.

490. Evaluate \[ \frac{6!}{(5-3)!3!} \].

491. Write the first four terms of the sequence defined by the explicit formula $a_n = 10^n + 3$. 
492. Write the first four terms of the sequence defined by the explicit formula $a_n = \frac{n!}{n(n + 1)}$.

m10362 (http://legacy.cnx.org/content/m10362/latest/)

493. Is the sequence $\frac{4}{7}, \frac{47}{21}, \frac{82}{21}, \frac{39}{7}$ ... arithmetic? If so, find the common difference.

494. Is the sequence $2, 4, 8, 16, ...$ arithmetic? If so, find the common difference.

495. An arithmetic sequence has the first term $a_1 = 18$ and common difference $d = -8$. What are the first five terms?

496. An arithmetic sequence has terms $a_2 = 11.7$ and $a_8 = -14.6$. What is the first term?

497. Write a recursive formula for the arithmetic sequence $-20, -10, 0, 10, ...$

498. Write a recursive formula for the arithmetic sequence $0, -\frac{1}{2}, -1, -\frac{3}{2}, ...$, and then find the 31st term.

499. Write an explicit formula for the arithmetic sequence $\frac{7}{8}, \frac{29}{24}, \frac{37}{24}, \frac{15}{8}, ...$

500. How many terms are in the finite arithmetic sequence $12, 20, 28, ..., 172$?

m10363 (http://legacy.cnx.org/content/m10363/latest/)

501. Find the common ratio for the geometric sequence $2.5, 5, 10, 20, ...$

502. Is the sequence $4, 16, 28, 40, ...$ geometric? If so find the common ratio. If not, explain why.

503. A geometric sequence has terms $a_7 = 16,384$ and $a_9 = 262,144$ What are the first five terms?

504. A geometric sequence has the first term $a_1 = -3$ and common ratio $r = \frac{4}{7}$. What is the 8th term?

505. What are the first five terms of the geometric sequence $a_1 = 3, a_n = 4 \cdot a_{n-1}$?

506. Write a recursive formula for the geometric sequence $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, ...$

507. Write an explicit formula for the geometric sequence $-\frac{1}{3}, -\frac{1}{15}, -\frac{1}{45}, -\frac{1}{135}, ...$

508. How many terms are in the finite geometric sequence $-5, -\frac{5}{3}, -\frac{5}{9}, ..., -\frac{5}{39,049}$?

m10364 (http://legacy.cnx.org/content/m10364/latest/)

509. Use summation notation to write the sum of terms $\frac{1}{2}m + 5$ from $m = 0$ to $m = 5$.

510. Use summation notation to write the sum that results from adding the number 13 twenty times.

511. Use the formula for the sum of the first $n$ terms of an arithmetic series to find the sum of the first eleven terms of the arithmetic series $2.5, 4, 5.5, ...$.
512. A ladder has 15 tapered rungs, the lengths of which increase by a common difference. The first rung is 5 inches long, and the last rung is 20 inches long. What is the sum of the lengths of the rungs?

513. Use the formula for the sum of the first n terms of a geometric series to find $S_9$ for the series 12, 6, $\frac{3}{2}$, ... 

514. The fees for the first three years of a hunting club membership are given in Table 11.11. If fees continue to rise at the same rate, how much will the total cost be for the first ten years of membership?

<table>
<thead>
<tr>
<th>Year</th>
<th>Membership Fees</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1500</td>
</tr>
<tr>
<td>2</td>
<td>$1950</td>
</tr>
<tr>
<td>3</td>
<td>$2535</td>
</tr>
</tbody>
</table>

Table 11.11

515. Find the sum of the infinite geometric series $\sum_{k=1}^{\infty} 45 \cdot \left( -\frac{1}{3} \right)^{k-1}$.

516. A ball has a bounce-back ratio of $\frac{3}{5}$ the height of the previous bounce. Write a series representing the total distance traveled by the ball, assuming it was initially dropped from a height of 5 feet. What is the total distance? (Hint: the total distance the ball travels on each bounce is the sum of the heights of the rise and the fall.)

517. Alejandro deposits $80 of his monthly earnings into an annuity that earns 6.25% annual interest, compounded monthly. How much money will he have saved after 5 years?

518. The twins Sarah and Scott both opened retirement accounts on their 21st birthday. Sarah deposits $4,800.00 each year, earning 5.5% annual interest, compounded monthly. Scott deposits $3,600.00 each year, earning 8.5% annual interest, compounded monthly. Which twin will earn the most interest by the time they are 55 years old? How much more?

m10365 (http://legacy.cnx.org/content/m10365/latest/)

519. How many ways are there to choose a number from the set \{ $-10, -6, 4, 10, 12, 18, 24, 32$ \} that is divisible by either 4 or 6?

520. In a group of 20 musicians, 12 play piano, 7 play trumpet, and 2 play both piano and trumpet. How many musicians play either piano or trumpet?

521. How many ways are there to construct a 4-digit code if numbers can be repeated?

522. A palette of water color paints has 3 shades of green, 3 shades of blue, 2 shades of red, 2 shades of yellow, and 1 shade of black. How many ways are there to choose one shade of each color?

523. Calculate $P(18, 4)$.

524. In a group of 5 freshman, 10 sophomores, 3 juniors, and 2 seniors, how many ways can a president, vice president, and treasurer be elected?

525. Calculate $C(15, 6)$. 

This content is available for free at http://legacy.cnx.org/content/col11667/1.2
526. A coffee shop has 7 Guatemalan roasts, 4 Cuban roasts, and 10 Costa Rican roasts. How many ways can the shop choose 2 Guatemalan, 2 Cuban, and 3 Costa Rican roasts for a coffee tasting event?

527. How many subsets does the set \{1, 3, 5, \ldots, 99\} have?

528. A day spa charges a basic day rate that includes use of a sauna, pool, and showers. For an extra charge, guests can choose from the following additional services: massage, body scrub, manicure, pedicure, facial, and straight-razor shave. How many ways are there to order additional services at the day spa?

529. How many distinct ways can the word DEADWOOD be arranged?

530. How many distinct rearrangements of the letters of the word DEADWOOD are there if the arrangement must begin and end with the letter D?

531. Evaluate the binomial coefficient \( \binom{23}{8} \).

532. Use the Binomial Theorem to expand \( \left(3x + \frac{1}{2}y\right)^6 \).

533. Use the Binomial Theorem to write the first three terms of \( (2a + b)^{17} \).

534. Find the fourth term of \( (3a^2 - 2b)^{11} \) without fully expanding the binomial.

535. Construct a table showing the sample space.

536. What is the probability that a roll includes a 2?

537. What is the probability of rolling a pair?

538. What is the probability that a roll includes a 2 or results in a pair?

539. What is the probability that a roll doesn’t include a 2 or result in a pair?

540. What is the probability of rolling a 5 or a 6?

541. What is the probability that a roll includes neither a 5 nor a 6?

542. What is the percent chance that all the children attending the party prefer soda?

543. What is the percent chance that at least one of the children attending the party prefers milk?

544. What is the percent chance that exactly 3 of the children attending the party prefer soda?

545. What is the percent chance that exactly 3 of the children attending the party prefer milk?
CHAPTER 11 PRACTICE TEST

488. Write the first four terms of the sequence defined by the recursive formula $a = -14$, $a_n = \frac{2 + a_{n-1}}{2}$.

489. Write the first four terms of the sequence defined by the explicit formula $a_n = \frac{n^2 - n - 1}{n!}$.

490. Is the sequence 0.3, 1.2, 2.1, 3, … arithmetic? If so find the common difference.

491. An arithmetic sequence has the first term $a_1 = -4$ and common difference $d = -\frac{4}{3}$. What is the 6th term?

492. Write a recursive formula for the arithmetic sequence $-2, -\frac{7}{2}, -5, -\frac{13}{2}, \ldots$ and then find the 22nd term.

493. Write an explicit formula for the arithmetic sequence 15.6, 15, 14.4, 13.8, … and then find the 32nd term.

494. Is the sequence $-2, -1, -\frac{1}{2}, -\frac{1}{4}, \ldots$ geometric? If so find the common ratio. If not, explain why.

495. What is the 11th term of the geometric sequence $-1.5, -3, -6, -12, \ldots$?

496. Write a recursive formula for the geometric sequence $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$

497. Write an explicit formula for the geometric sequence $4, -\frac{4}{3}, \frac{4}{9}, -\frac{4}{27}, \ldots$

498. Use summation notation to write the sum of terms $3k^2 - \frac{5}{6}k$ from $k = -3$ to $k = 15$.

499. A community baseball stadium has 10 seats in the first row, 13 seats in the second row, 16 seats in the third row, and so on. There are 56 rows in all. What is the seating capacity of the stadium?

500. Use the formula for the sum of the first $n$ terms of a geometric series to find $\sum_{k=1}^{7} -0.2 \cdot (-5)^{k-1}$.

501. Find the sum of the infinite geometric series $\sum_{k=1}^{\infty} \frac{1}{3} \cdot \left(-\frac{1}{5}\right)^{k-1}$.

502. Rachael deposits $3,600 into a retirement fund each year. The fund earns 7.5% annual interest, compounded monthly. If she opened her account when she was 20 years old, how much will she have by the time she’s 55? How much of that amount was interest earned?

503. In a competition of 50 professional ballroom dancers, 22 compete in the fox-trot competition, 18 compete in the tango competition, and 6 compete in both the fox-trot and tango competitions. How many dancers compete in the fox-trot or tango competitions?

504. A buyer of a new sedan can custom order the car by choosing from 5 different exterior colors, 3 different interior colors, 2 sound systems, 3 motor designs, and either manual or automatic transmission. How many choices does the buyer have?

505. To allocate annual bonuses, a manager must choose his top four employees and rank them first to fourth. In how many ways can he create the “Top-Four” list out of the 32 employees?
506. A rock group needs to choose 3 songs to play at the annual Battle of the Bands. How many ways can they choose their set if have 15 songs to pick from?

507. A self-serve frozen yogurt shop has 8 candy toppings and 4 fruit toppings to choose from. How many ways are there to top a frozen yogurt?

508. How many distinct ways can the word EVANESCENCE be arranged if the anagram must end with the letter E?

509. Use the Binomial Theorem to expand \( \left( \frac{3}{2}x - \frac{1}{2}y \right)^{\frac{5}{2}} \).

510. Find the seventh term of \( \left( x^2 - \frac{1}{2} \right)^{13} \) without fully expanding the binomial.

For the following exercises, use the spinner in Figure 11.13.

511. Construct a probability model showing each possible outcome and its associated probability. (Use the first letter for colors.)

512. What is the probability of landing on an odd number?

513. What is the probability of landing on blue?

514. What is the probability of landing on blue or an odd number?

515. What is the probability of landing on anything other than blue or an odd number?

516. A bowl of candy holds 16 peppermint, 14 butterscotch, and 10 strawberry flavored candies. Suppose a person grabs a handful of 7 candies. What is the percent chance that exactly 3 are butterscotch? (Show calculations and round to the nearest tenth of a percent.)
The eight-time world champion and winner of six Olympic gold medals in sprinting, Usain Bolt has truly earned his nickname as the “fastest man on Earth.” Also known as the “lightning bolt,” he set the track on fire by running at a top speed of 27.79 mph—the fastest time ever recorded by a human runner.

Like the fastest land animal, a cheetah, Bolt does not run at his top speed at every instant. How then, do we approximate his speed at any given instant? We will find the answer to this and many related questions in this chapter.
12.1 | Finding Limits: Numerical and Graphical Approaches

Learning Objectives

In this section, you will:

12.1.1 Understand limit notation.
12.1.2 Find a limit using a graph.
12.1.3 Find a limit using a table.

Intuitively, we know what a limit is. A car can go only so fast and no faster. A trash can might hold 33 gallons and no more. It is natural for measured amounts to have limits. What, for instance, is the limit to the height of a woman? The tallest woman on record was Jinlian Zeng from China, who was 8 ft 1 in.\(^1\) Is this the limit of the height to which women can grow? Perhaps not, but there is likely a limit that we might describe in inches if we were able to determine what it was.

To put it mathematically, the function whose input is a woman and whose output is a measured height in inches has a limit. In this section, we will examine numerical and graphical approaches to identifying limits.

Understanding Limit Notation

We have seen how a sequence can have a limit, a value that the sequence of terms moves toward as the number of terms increases. For example, the terms of the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$  \hspace{1cm} (12.1)

gets closer and closer to 0. A sequence is one type of function, but functions that are not sequences can also have limits. We can describe the behavior of the function as the input values get close to a specific value. If the limit of a function \(f(x) = L\), then as the input \(x\) gets closer and closer to \(a\), the output \(y\)-coordinate gets closer and closer to \(L\). We say that the output “approaches” \(L\).

Figure 12.2 provides a visual representation of the mathematical concept of limit. As the input value \(x\) approaches \(a\), the output value \(f(x)\) approaches \(L\).

\[\lim_{{x \to a}} f(x) = L.\]  \hspace{1cm} (12.2)

---

This notation indicates that as \( x \) approaches \( a \) both from the left of \( x = a \) and the right of \( x = a \), the output value approaches \( L \).

Consider the function

\[
f(x) = \frac{x^2 - 6x - 7}{x - 7}.
\]

We can factor the function as shown.

\[
f(x) = \frac{(x - 7)(x + 1)}{x - 7}
\]

Cancel like factors in numerator and denominator.

\[
f(x) = x + 1, \quad x \neq 7
\]

Simplify.

Notice that \( x \) cannot be 7, or we would be dividing by 0, so 7 is not in the domain of the original function. In order to avoid changing the function when we simplify, we set the same condition, \( x \neq 7 \), for the simplified function. We can represent the function graphically as shown in Figure 12.3.

![Graph of the function](image)

**Figure 12.3** Because 7 is not allowed as an input, there is no point at \( x = 7 \).

What happens at \( x = 7 \) is completely different from what happens at points close to \( x = 7 \) on either side. The notation

\[
\lim_{x \to 7} f(x) = 8
\]

indicates that as the input \( x \) approaches 7 from either the left or the right, the output approaches 8. The output can get as close to 8 as we like if the input is sufficiently near 7.

What happens at \( x = 7 \)? When \( x = 7 \), there is no corresponding output. We write this as

\[
f(7) \text{ does not exist.}
\]

This notation indicates that 7 is not in the domain of the function. We had already indicated this when we wrote the function as

\[
f(x) = x + 1, \quad x \neq 7.
\]

Notice that the limit of a function can exist even when \( f(x) \) is not defined at \( x = a \). Much of our subsequent work will be determining limits of functions as \( x \) nears \( a \), even though the output at \( x = a \) does not exist.
The Limit of a Function

A quantity \( L \) is the limit of a function \( f(x) \) as \( x \) approaches \( a \) if, as the input values of \( x \) approach \( a \) (but do not equal \( a \)), the corresponding output values of \( f(x) \) get closer to \( L \). Note that the value of the limit is not affected by the output value of \( f(x) \) at \( a \). Both \( a \) and \( L \) must be real numbers. We write it as

\[
\lim_{x \to a} f(x) = L
\]

Example 12.1

Understanding the Limit of a Function

For the following limit, define \( a \), \( f(x) \), and \( L \).

\[
\lim_{x \to 2} (3x + 5) = 11
\]

Solution

Analysis
Recall that \( y = 3x + 5 \) is a line with no breaks. As the input values approach 2, the output values will get close to 11. This may be phrased with the equation \( \lim_{x \to 2} (3x + 5) = 11 \), which means that as \( x \) nears 2 (but is not exactly 2), the output of the function \( f(x) = 3x + 5 \) gets as close as we want to \( 3(2) + 5 \), or 11, which is the limit \( L \), as we take values of \( x \) sufficiently near 2 but not at \( x = 2 \).

For the following limit, define \( a \), \( f(x) \), and \( L \).

\[
\lim_{x \to 5} \left(2x^2 - 4\right) = 46
\]

Understanding Left-Hand Limits and Right-Hand Limits

We can approach the input of a function from either side of a value—from the left or the right. Figure 12.4 shows the values of

\[
f(x) = x + 1, \quad x \neq 7
\]

as described earlier and depicted in Figure 12.3.

<table>
<thead>
<tr>
<th>( x )</th>
<th>Values of ( x ) approach 7 from the left (( x &lt; 7 ))</th>
<th>( x = 7 )</th>
<th>Values of ( x ) approach 7 from the right (( x &gt; 7 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>6.9</td>
<td>6.99</td>
<td>6.999</td>
</tr>
<tr>
<td></td>
<td>7.9</td>
<td>7.99</td>
<td>7.999</td>
</tr>
</tbody>
</table>

Figure 12.4

Values described as “from the left” are less than the input value 7 and would therefore appear to the left of the value on a number line. The input values that approach 7 from the left in Figure 12.4 are 6.9, 6.99, and 6.999. The corresponding outputs are 7.9, 7.99, and 7.999. These values are getting closer to 8. The limit of values of \( f(x) \) as \( x \) approaches from the left is known as the left-hand limit. For this function, 8 is the left-hand limit of the function \( f(x) = x + 1, \quad x \neq 7 \) as \( x \) approaches 7.
Values described as “from the right” are greater than the input value 7 and would therefore appear to the right of the value on a number line. The input values that approach 7 from the right in Figure 12.4 are 7.1, 7.01, and 7.001. These values are getting closer to 8. The limit of values of \( f(x) \) as \( x \) approaches the right is known as the right-hand limit. For this function, 8 is also the right-hand limit of the function \( f(x) = x + 1, \ x \neq 7 \) as \( x \) approaches 7.

**Figure 12.4** shows that we can get the output of the function within a distance of 0.1 from 8 by using an input within a distance of 0.1 from 7. In other words, we need an input \( x \) within the interval \( 6.9 < x < 7.1 \) to produce an output value of \( f(x) \) within the interval \( 7.9 < f(x) < 8.1 \).

We also see that we can get output values of \( f(x) \) successively closer to 8 by selecting input values closer to 7. In fact, we can obtain output values within any specified interval if we choose appropriate input values.

**Figure 12.5** provides a visual representation of the left- and right-hand limits of the function. From the graph of \( f(x) \), we observe the output can get infinitesimally close to \( L = 8 \) as \( x \) approaches 7 from the left and as \( x \) approaches 7 from the right.

To indicate the left-hand limit, we write

\[
\lim_{x \to 7^-} f(x) = 8. \tag{12.11}
\]

To indicate the right-hand limit, we write

\[
\lim_{x \to 7^+} f(x) = 8. \tag{12.12}
\]

---

**Left- and Right-Hand Limits**

The **left-hand limit** of a function \( f(x) \) as \( x \) approaches \( a \) from the left is equal to \( L \), denoted by

\[
\lim_{x \to a^-} f(x) = L. \tag{12.13}
\]

The values of \( f(x) \) can get as close to the limit \( L \) as we like by taking values of \( x \) sufficiently close to \( a \) such that \( x < a \) and \( x \neq a \).

The **right-hand limit** of a function \( f(x) \), as \( x \) approaches \( a \) from the right, is equal to \( L \), denoted by

\[
\lim_{x \to a^+} f(x) = L. \tag{12.14}
\]
The values of \( f(x) \) can get as close to the limit \( L \) as we like by taking values of \( x \) sufficiently close to \( a \) but greater than \( a \). Both \( a \) and \( L \) are real numbers.

**Understanding Two-Sided Limits**

In the previous example, the left-hand limit and right-hand limit as \( x \) approaches \( a \) are equal. If the left- and right-hand limits are equal, we say that the function \( f(x) \) has a two-sided limit as \( x \) approaches \( a \). More commonly, we simply refer to a two-sided limit as a limit. If the left-hand limit does not equal the right-hand limit, or if one of them does not exist, we say the limit does not exist.

### The Two-Sided Limit of Function as \( x \) Approaches \( a \)

The limit of a function \( f(x) \), as \( x \) approaches \( a \), is equal to \( L \), that is,

\[
\lim_{x \to a} f(x) = L
\]

if and only if

\[
\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).
\]

In other words, the left-hand limit of a function \( f(x) \) as \( x \) approaches \( a \) is equal to the right-hand limit of the same function as \( x \) approaches \( a \). If such a limit exists, we refer to the limit as a two-sided limit. Otherwise we say the limit does not exist.

**Finding a Limit Using a Graph**

To visually determine if a limit exists as \( x \) approaches \( a \), we observe the graph of the function when \( x \) is very near to \( x = a \). In Figure 12.6 we observe the behavior of the graph on both sides of \( a \).

![Figure 12.6](http://legacy.cnx.org/content/col11667/1.2)

To determine if a left-hand limit exists, we observe the branch of the graph to the left of \( x = a \), but near \( x = a \). This is where \( x < a \). We see that the outputs are getting close to some real number \( L \) so there is a left-hand limit.

To determine if a right-hand limit exists, observe the branch of the graph to the right of \( x = a \), but near \( x = a \). This is where \( x > a \). We see that the outputs are getting close to some real number \( L \), so there is a right-hand limit.

If the left-hand limit and the right-hand limit are the same, as they are in Figure 12.6, then we know that the function has a two-sided limit. Normally, when we refer to a “limit,” we mean a two-sided limit, unless we call it a one-sided limit.
Finally, we can look for an output value for the function \( f(x) \) when the input value \( x \) is equal to \( a \). The coordinate pair of the point would be \( (a, f(a)) \). If such a point exists, then \( f(a) \) has a value. If the point does not exist, as in Figure 12.6, then we say that \( f(a) \) does not exist.

**Given a function \( f(x) \), use a graph to find the limits and a function value as \( x \) approaches \( a \).**

1. Examine the graph to determine whether a left-hand limit exists.
2. Examine the graph to determine whether a right-hand limit exists.
3. If the two one-sided limits exist and are equal, then there is a two-sided limit—what we normally call a “limit.”
4. If there is a point at \( x = a \), then \( f(a) \) is the corresponding function value.

**Example 12.2**

**Finding a Limit Using a Graph**

a. Determine the following limits and function value for the function \( f \) shown in Figure 12.7.

i. \( \lim_{x \to 2^-} f(x) \)

ii. \( \lim_{x \to 2^+} f(x) \)

iii. \( \lim_{x \to 2} f(x) \)

iv. \( f(2) \)

---

b. Determine the following limits and function value for the function \( f \) shown in Figure 12.8.

i. \( \lim_{x \to 2^-} f(x) \)

ii. \( \lim_{x \to 2^+} f(x) \)
iii. \( \lim_{x \to 2} f(x) \)

iv. \( f(2) \)

**Solution**

**Try It**

12.2 Using the graph of the function \( y = f(x) \) shown in Figure 12.9, estimate the following limits.

a. \( \lim_{x \to 0^-} f(x) \)  
b. \( \lim_{x \to 0^+} f(x) \)  
c. \( \lim_{x \to 0} f(x) \)

d. \( \lim_{x \to 2^-} f(x) \)  
e. \( \lim_{x \to 2^+} f(x) \)  
f. \( \lim_{x \to 2} f(x) \)

g. \( \lim_{x \to 4^-} f(x) \)  
h. \( \lim_{x \to 4^+} f(x) \)  
i. \( \lim_{x \to 4} f(x) \)

**Figure 12.9**
**Finding a Limit Using a Table**

Creating a table is a way to determine limits using numeric information. We create a table of values in which the input values of \( x \) approach \( a \) from both sides. Then we determine if the output values get closer and closer to some real value, the limit \( L \).

Let’s consider an example using the following function:

\[
\lim_{x \to 5} \left( \frac{x^3 - 125}{x - 5} \right)
\]  

(12.17)

To create the table, we evaluate the function at values close to \( x = 5 \). We use some input values less than 5 and some values greater than 5 as in Figure 12.10. The table values show that when \( x > 5 \) but nearing 5, the corresponding output gets close to 75. When \( x > 5 \) but nearing 5, the corresponding output also gets close to 75.

<table>
<thead>
<tr>
<th>( x )</th>
<th>4.90</th>
<th>4.99</th>
<th>4.999</th>
<th>5.00</th>
<th>5.001</th>
<th>5.01</th>
<th>5.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>73.51</td>
<td>74.8501</td>
<td>74.985001</td>
<td>Undefined</td>
<td>75.015001</td>
<td>75.1501</td>
<td>76.51</td>
</tr>
</tbody>
</table>

Figure 12.10

Because

\[
\lim_{x \to 5^-} f(x) = 75 = \lim_{x \to 5^+} f(x),
\]

(12.18)

then

\[
\lim_{x \to 5} f(x) = 75.
\]

(12.19)

Remember that \( f(5) \) does not exist.

**How To:** **Given a function \( f \), use a table to find the limit as \( x \) approaches \( a \) and the value of \( f(a) \), if it exists.**

1. Choose several input values that approach \( a \) from both the left and right. Record them in a table.
2. Evaluate the function at each input value. Record them in the table.
3. Determine if the table values indicate a left-hand limit and a right-hand limit.
4. If the left-hand and right-hand limits exist and are equal, there is a two-sided limit.
5. Replace \( x \) with \( a \) to find the value of \( f(a) \).

**Example 12.3**

**Finding a Limit Using a Table**

Numerically estimate the limit of the following expression by setting up a table of values on both sides of the limit.

\[
\lim_{x \to 0} \left( \frac{5\sin(x)}{3x} \right)
\]
Is it possible to check our answer using a graphing utility?

Yes. We previously used a table to find a limit of 75 for the function \( f(x) = \frac{x^3 - 125}{x - 5} \) as \( x \) approaches 5. To check, we graph the function on a viewing window as shown in Figure 12.11. A graphical check shows both branches of the graph of the function get close to the output 75 as \( x \) nears 5. Furthermore, we can use the ‘trace’ feature of a graphing calculator. By approaching \( x = 5 \) we may numerically observe the corresponding outputs getting close to 75.

Figure 12.11

12.3 Numerically estimate the limit of the following function by making a table:

\[
\lim_{x \to 0} \left( \frac{20 \sin(x)}{4x} \right)
\]

(12.20)

Is one method for determining a limit better than the other?

No. Both methods have advantages. Graphing allows for quick inspection. Tables can be used when graphical utilities aren’t available, and they can be calculated to a higher precision than could be seen with an unaided eye inspecting a graph.

Example 12.4

Using a Graphing Utility to Determine a Limit

With the use of a graphing utility, if possible, determine the left- and right-hand limits of the following function as \( x \) approaches 0. If the function has a limit as \( x \) approaches 0, state it. If not, discuss why there is no limit.

\[ f(x) = 3 \sin \left( \frac{\pi}{x} \right) \]

Solution

12.4 Numerically estimate the following limit: \( \lim_{x \to 0} \left( \sin \left( \frac{2}{x} \right) \right) \).

Access these online resources for additional instruction and practice with finding limits.

- Introduction to Limits (http://openstaxcollege.org/l/introtolimits)
- Formal Definition of a Limit (http://openstaxcollege.org/l/formaldeflimit)
12.1 EXERCISES

Verbal

1. Explain the difference between a value at \( x = a \) and the limit as \( x \) approaches \( a \).

2. Explain why we say a function does not have a limit as \( x \) approaches \( a \) if, as \( x \) approaches \( a \), the left-hand limit is not equal to the right-hand limit.

Graphical

For the following exercises, estimate the functional values and the limits from the graph of the function \( f \) provided in Figure 12.12.

3. \( \lim_{x \to -2^-} f(x) \)

4. \( \lim_{x \to -2^+} f(x) \)

5. \( \lim_{x \to -2} f(x) \)

6. \( f(-2) \)

7. \( \lim_{x \to -1^-} f(x) \)

8. \( \lim_{x \to 1^+} f(x) \)

9. \( \lim_{x \to 1} f(x) \)

10. \( f(1) \)

11. \( \lim_{x \to 4^-} f(x) \)

12. \( \lim_{x \to 4^+} f(x) \)
13. \( \lim_{x \to 4} f(x) \)

14. \( f(4) \)

For the following exercises, draw the graph of a function from the functional values and limits provided.

15. \( \lim_{x \to 0^-} f(x) = 2, \lim_{x \to 0^+} f(x) = -3, \lim_{x \to 2} f(x) = 2, f(0) = 4, f(2) = -1, f(-3) \) does not exist.

16. \( \lim_{x \to 0^-} f(x) = 0, \lim_{x \to 0^+} f(x) = -2, \lim_{x \to 0} f(x) = 3, f(2) = 5, f(0) \)

17. \( \lim_{x \to 2^-} f(x) = 2, \lim_{x \to 2^+} f(x) = -3, \lim_{x \to 0} f(x) = 5, f(0) = 1, f(1) = 0 \)

18. \( \lim_{x \to 3^-} f(x) = 0, \lim_{x \to 3^+} f(x) = 5, \lim_{x \to 5} f(x) = 0, f(5) = 4, f(3) \) does not exist.

19. \( \lim_{x \to 4} f(x) = 6, \lim_{x \to 6^-} f(x) = -1, \lim_{x \to 0} f(x) = 5, f(4) = 6, f(2) = 6 \)

20. \( \lim_{x \to -3} f(x) = 2, \lim_{x \to 1^-} f(x) = -2, \lim_{x \to -3} f(x) = -4, f(-3) = 0, f(0) = 0 \)

21. \( \lim_{x \to \pi} f(x) = \pi^2, \lim_{x \to \pi^-} f(x) = \frac{\pi}{2}, \lim_{x \to 1} f(x) = 0, f(\pi) = \sqrt{\pi}, f(0) \) does not exist.

For the following exercises, use a graphing calculator to determine the limit to 5 decimal places as \( x \) approaches 0.

22. \( f(x) = (1 + x) \frac{1}{x} \)

23. \( g(x) = (1 + x) \frac{2}{x} \)

24. \( h(x) = (1 + x) \frac{3}{x} \)

25. \( i(x) = (1 + x) \frac{4}{x} \)

26. \( j(x) = (1 + x) \frac{5}{x} \)

27. Based on the pattern you observed in the exercises above, make a conjecture as to the limit of \( f(x) = (1 + x)^{\frac{6}{x}}, \) \( g(x) = (1 + x)^{\frac{7}{x}}, \) and \( h(x) = (1 + x)^{\frac{8}{x}}. \)

For the following exercises, use a graphing utility to find graphical evidence to determine the left- and right-hand limits of the function given as \( x \) approaches \( a. \) If the function has a limit as \( x \) approaches \( a, \) state it. If not, discuss why there is no limit.

28. \( f(x) = \begin{cases} |x| - 1, & \text{if } x \neq 1 \\ x^3, & \text{if } x = 1 \end{cases}, \quad a = 1 \)

29. \( g(x) = \begin{cases} \frac{1}{x + 1}, & \text{if } x = -2 \\ (x + 1)^2, & \text{if } x \neq -2 \end{cases}, \quad a = -2 \)
Numeric

For the following exercises, use numerical evidence to determine whether the limit exists at \( x = a \). If not, describe the behavior of the graph of the function near \( x = a \). Round answers to two decimal places.

30. \( f(x) = \frac{x^2 - 4x}{16 - x^2}; \ a = 4 \)

31. \( f(x) = \frac{x^2 - x - 6}{x^2 - 9}; \ a = 3 \)

32. \( f(x) = \frac{x^2 - 6x - 7}{x^2 - 7x}; \ a = 7 \)

33. \( f(x) = \frac{x^2 - 1}{x^2 - 3x + 2}; \ a = 1 \)

34. \( f(x) = \frac{1 - x^2}{x^2 - 3x + 2}; \ a = 1 \)

35. \( f(x) = \frac{10 - 10x^2}{x^2 - 3x + 2}; \ a = 1 \)

36. \( f(x) = \frac{x}{6x^2 - 5x - 6}; \ a = \frac{3}{2} \)

37. \( f(x) = \frac{x}{4x^2 + 4x + 1}; \ a = -\frac{1}{2} \)

38. \( f(x) = \frac{2}{x - 4}; \ a = 4 \)

For the following exercises, use a calculator to estimate the limit by preparing a table of values. If there is no limit, describe the behavior of the function as \( x \) approaches the given value.

39. \( \lim_{x \to 0} \frac{7 \tan x}{3x} \)

40. \( \lim_{x \to 4} \frac{x^2}{x - 4} \)

41. \( \lim_{x \to 0} \frac{2 \sin x}{4 \tan x} \)

For the following exercises, use a graphing utility to find numerical or graphical evidence to determine the left and right-hand limits of the function given as \( x \) approaches \( a \). If the function has a limit as \( x \) approaches \( a \), state it. If not, discuss why there is no limit.

42. \( \lim_{x \to 0} e^{\frac{1}{x}} \)

43. \( \lim_{x \to 0} e^{\frac{-1}{x^2}} \)

44. \( \lim_{x \to 0} \frac{|x|}{x} \)

45. \( \lim_{x \to -1} \frac{|x + 1|}{x + 1} \)
46. \( \lim_{x \to 5} \frac{|x - 5|}{5 - x} \)

47. \( \lim_{x \to -1} \frac{1}{(x + 1)^2} \)

48. \( \lim_{x \to 1} \frac{1}{(x - 1)^3} \)

49. \( \lim_{x \to 0} \frac{5}{1 - e^{\frac{x^2}{2}}} \)

50. Use numerical and graphical evidence to compare and contrast the limits of two functions whose formulas appear similar: \( f(x) = \left| \frac{1 - x}{x} \right| \) and \( g(x) = \left| \frac{1 + x}{x} \right| \) as \( x \) approaches 0. Use a graphing utility, if possible, to determine the left- and right-hand limits of the functions \( f(x) \) and \( g(x) \) as \( x \) approaches 0. If the functions have a limit as \( x \) approaches 0, state it. If not, discuss why there is no limit.

**Extensions**

51. According to the Theory of Relativity, the mass \( m \) of a particle depends on its velocity \( v \). That is

\[
m = \frac{m_0}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}}
\]

(12.21)

where \( m_0 \) is the mass when the particle is at rest and \( c \) is the speed of light. Find the limit of the mass, \( m \), as \( v \) approaches \( c^- \).

52. Allow the speed of light, \( c \), to be equal to 1.0. If the mass, \( m \), is 1, what occurs to \( m \) as \( v \to c \)? Using the values listed in Table 12.1, make a conjecture as to what the mass is as \( v \) approaches 1.00.

<table>
<thead>
<tr>
<th>( v )</th>
<th>0.5</th>
<th>0.9</th>
<th>0.95</th>
<th>0.99</th>
<th>0.999</th>
<th>0.99999</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>1.15</td>
<td>2.29</td>
<td>3.20</td>
<td>7.09</td>
<td>22.36</td>
<td>223.61</td>
</tr>
</tbody>
</table>

Table 12.1
12.2 | Finding Limits: Properties of Limits

Learning Objectives

In this section, you will:

12.2.1 Find the limit of a sum, a difference, and a product.
12.2.2 Find the limit of a polynomial.
12.2.3 Find the limit of a power or a root.
12.2.4 Find the limit of a quotient.

Consider the rational function

\[ f(x) = \frac{x^2 - 6x - 7}{x - 7}. \]  \hspace{1cm} \text{(12.22)}

The function can be factored as follows:

\[ f(x) = \frac{(x - 7)(x + 1)}{x - 7}, \text{ which gives us } f(x) = x + 1, \ x \neq 7. \]  \hspace{1cm} \text{(12.23)}

Does this mean the function \( f \) is the same as the function \( g(x) = x + 1 \)?

The answer is no. Function \( f \) does not have \( x = 7 \) in its domain, but \( g \) does. Graphically, we observe there is a hole in the graph of \( f(x) \) at \( x = 7 \), as shown in Figure 12.13 and no such hole in the graph of \( g(x) \), as shown in Figure 12.14.

Figure 12.13  The graph of function \( f \) contains a break at \( x = 7 \) and is therefore not continuous at \( x = 7 \).
So, do these two different functions also have different limits as \( x \) approaches 7?

Not necessarily. Remember, in determining a limit of a function as \( x \) approaches \( a \), what matters is whether the output approaches a real number as we get close to \( x = a \). The existence of a limit does not depend on what happens when \( x \) equals \( a \).

Look again at Figure 12.13 and Figure 12.14. Notice that in both graphs, as \( x \) approaches 7, the output values approach 8. This means

\[
\lim_{x \to 7} f(x) = \lim_{x \to 7} g(x).
\] 

(12.24)

Remember that when determining a limit, the concern is what occurs near \( x = a \), not at \( x = a \). In this section, we will use a variety of methods, such as rewriting functions by factoring, to evaluate the limit. These methods will give us formal verification for what we formerly accomplished by intuition.

**Finding the Limit of a Sum, a Difference, and a Product**

Graphing a function or exploring a table of values to determine a limit can be cumbersome and time-consuming. When possible, it is more efficient to use the properties of limits, which is a collection of theorems for finding limits.

Knowing the properties of limits allows us to compute limits directly. We can add, subtract, multiply, and divide the limits of functions as if we were performing the operations on the functions themselves to find the limit of the result. Similarly, we can find the limit of a function raised to a power by raising the limit to that power. We can also find the limit of the root of a function by taking the root of the limit. Using these operations on limits, we can find the limits of more complex functions by finding the limits of their simpler component functions.

<table>
<thead>
<tr>
<th>Properties of Limits</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let ( a, k, A, ) and ( B ) represent real numbers, and ( f ) and ( g ) be functions, such that ( \lim_{x \to a} f(x) = A ) and ( \lim_{x \to a} g(x) = B ). For limits that exist and are finite, the properties of limits are summarized in <strong>Table 12.2</strong>.</td>
</tr>
</tbody>
</table>

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This content is available for free at http://legacy.cnx.org/content/col11667/1.2
<table>
<thead>
<tr>
<th><strong>Constant, ( k )</strong></th>
<th>( \lim_{x \to a} k = k )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Constant times a function</strong></td>
<td>( \lim_{x \to a} [k \cdot f(x)] = k \lim_{x \to a} f(x) = kA )</td>
</tr>
<tr>
<td><strong>Sum of functions</strong></td>
<td>( \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = A + B )</td>
</tr>
<tr>
<td><strong>Difference of functions</strong></td>
<td>( \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = A - B )</td>
</tr>
<tr>
<td><strong>Product of functions</strong></td>
<td>( \lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = A \cdot B )</td>
</tr>
<tr>
<td><strong>Quotient of functions</strong></td>
<td>( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{A}{B} \quad B \neq 0 )</td>
</tr>
<tr>
<td><strong>Function raised to an exponent</strong></td>
<td>( \lim_{x \to a} [f(x)]^n = \left[ \lim_{x \to a} f(x) \right]^n = A^n ), where ( n ) is a positive integer</td>
</tr>
<tr>
<td><strong>( n )th root of a function, where ( n ) is a positive integer</strong></td>
<td>( \lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{A} )</td>
</tr>
<tr>
<td><strong>Polynomial function</strong></td>
<td>( \lim_{x \to a} p(x) = p(a) )</td>
</tr>
</tbody>
</table>

**Table 12.2**

---

**Example 12.5**

**Evaluating the Limit of a Function Algebraically**

Evaluate \( \lim_{x \to 3} (2x + 5) \).

**Solution**

Evaluate the following limit: \( \lim_{x \to \frac{1}{2}} (-2x + 2) \).

---

**Finding the Limit of a Polynomial**

Not all functions or their limits involve simple addition, subtraction, or multiplication. Some may include polynomials. Recall that a polynomial is an expression consisting of the sum of two or more terms, each of which consists of a constant and a variable raised to a nonnegative integral power. To find the limit of a polynomial function, we can find the limits of
the individual terms of the function, and then add them together. Also, the limit of a polynomial function as $x$ approaches $a$ is equivalent to simply evaluating the function for $a$.

**Given a function containing a polynomial, find its limit.**

1. Use the properties of limits to break up the polynomial into individual terms.
2. Find the limits of the individual terms.
3. Add the limits together.
4. Alternatively, evaluate the function for $a$.

---

**Example 12.6**

**Evaluating the Limit of a Function Algebraically**

Evaluate $\lim_{x \to 3} (5x^2)$.

**Solution**

Evaluate $\lim_{x \to 4} (x^3 - 5)$.

**Example 12.7**

**Evaluating the Limit of a Polynomial Algebraically**

Evaluate $\lim_{x \to 5} (2x^3 - 3x + 1)$.

**Solution**

Evaluate the following limit: $\lim_{x \to -1} (x^4 - 4x^3 + 5)$.

**Finding the Limit of a Power or a Root**

When a limit includes a power or a root, we need another property to help us evaluate it. The square of the limit of a function equals the limit of the square of the function; the same goes for higher powers. Likewise, the square root of the limit of a function equals the limit of the square root of the function; the same holds true for higher roots.

**Example 12.8**

**Evaluating a Limit of a Power**

Evaluate $\lim_{x \to 2} (3x + 1)^5$.  

This content is available for free at http://legacy.cnx.org/content/col11667/1.2
Evaluate the following limit: \( \lim_{x \to -4} (10x + 36)^3 \).

If we can’t directly apply the properties of a limit, for example in \( \lim_{x \to 2} \frac{x^2 + 6x + 8}{x - 2} \), can we still determine the limit of the function as \( x \) approaches \( a \)?

Yes. Some functions may be algebraically rearranged so that one can evaluate the limit of a simplified equivalent form of the function.

**Finding the Limit of a Quotient**

Finding the limit of a function expressed as a quotient can be more complicated. We often need to rewrite the function algebraically before applying the properties of a limit. If the denominator evaluates to 0 when we apply the properties of a limit directly, we must rewrite the quotient in a different form. One approach is to write the quotient in factored form and simplify.

**Given the limit of a function in quotient form, use factoring to evaluate it.**

1. Factor the numerator and denominator completely.
2. Simplify by dividing any factors common to the numerator and denominator.
3. Evaluate the resulting limit, remembering to use the correct domain.

**Example 12.9**

**Evaluating the Limit of a Quotient by Factoring**

Evaluate \( \lim_{x \to 2} \left( \frac{x^2 - 6x + 8}{x - 2} \right) \).

**Solution**

**Analysis**

When the limit of a rational function cannot be evaluated directly, factored forms of the numerator and denominator may simplify to a result that can be evaluated.

Notice, the function

\[ f(x) = \frac{x^2 - 6x + 8}{x - 2} \]

is equivalent to the function

\[ f(x) = x - 4, \quad x \neq 2. \]

Notice that the limit exists even though the function is not defined at \( x = 2 \).

Evaluate the following limit: \( \lim_{x \to 7} \left( \frac{x^2 - 11x + 28}{7 - x} \right) \).
Example 12.10

Evaluating the Limit of a Quotient by Finding the LCD

Evaluate $\lim_{x \to 5} \left( \frac{\frac{1}{x} - \frac{1}{5}}{x - 5} \right)$.

Solution

Analysis

When determining the limit of a rational function that has terms added or subtracted in either the numerator or denominator, the first step is to find the common denominator of the added or subtracted terms; then, convert both terms to have that denominator, or simplify the rational function by multiplying numerator and denominator by the least common denominator. Then check to see if the resulting numerator and denominator have any common factors.

12.10  Evaluate $\lim_{x \to -5} \left( \frac{\frac{1}{5} + \frac{1}{x}}{\frac{1}{10} + 2x} \right)$.

How to:

Given a limit of a function containing a root, use a conjugate to evaluate.

1. If the quotient as given is not in indeterminate $(\frac{0}{0})$ form, evaluate directly.
2. Otherwise, rewrite the sum (or difference) of two quotients as a single quotient, using the least common denominator (LCD).
3. If the numerator includes a root, rationalize the numerator; multiply the numerator and denominator by the conjugate of the numerator. Recall that $a \pm \sqrt{b}$ are conjugates.
4. Simplify.
5. Evaluate the resulting limit.

Example 12.11

Evaluating a Limit Containing a Root Using a Conjugate

Evaluate $\lim_{x \to 0} \left( \frac{\sqrt{25 - x} - 5}{x} \right)$.

Solution

Analysis

When determining a limit of a function with a root as one of two terms where we cannot evaluate directly, think about multiplying the numerator and denominator by the conjugate of the terms.

12.11  Evaluate the following limit: $\lim_{h \to 0} \left( \frac{\sqrt{16 - h} - h}{h} \right)$.
Example 12.12

Evaluating the Limit of a Quotient of a Function by Factoring

Evaluate \( \lim_{x \to 4} \frac{4 - x}{(\sqrt{x} - 2)} \).

Solution

Analysis

Multiplying by a conjugate would expand the numerator; look instead for factors in the numerator. Four is a perfect square so that the numerator is in the form \( a^2 - b^2 \) and may be factored as \((a + b)(a - b)\).

Try It 12.12 Evaluate the following limit: \( \lim_{x \to 3} \frac{x - 3}{(\sqrt{x} - \sqrt{3})} \).

Given a quotient with absolute values, evaluate its limit.

1. Try factoring or finding the LCD.
2. If the limit cannot be found, choose several values close to and on either side of the input where the function is undefined.
3. Use the numeric evidence to estimate the limits on both sides.

Example 12.13

Evaluating the Limit of a Quotient with Absolute Values

Evaluate \( \lim_{x \to 7} \frac{|x - 7|}{x - 7} \).

Solution

Try It 12.13 Evaluate \( \lim_{x \to 6^+} \frac{6 - x}{|x - 6|} \).

Access the following online resource for additional instruction and practice with properties of limits.

- Determine a Limit Analytically (http://openstaxcollege.org/l/limitanalytic)
12.2 EXERCISES

Verbal

53. Give an example of a type of function $f$ whose limit, as $x$ approaches $a$, is $f(a)$.

54. When direct substitution is used to evaluate the limit of a rational function as $x$ approaches $a$ and the result is $f(a) = \frac{0}{0}$, does this mean that the limit of $f$ does not exist?

55. What does it mean to say the limit of $f(x)$, as $x$ approaches $c$, is undefined?

Algebraic

For the following exercises, evaluate the limits algebraically.

56. $\lim_{x \to 0} (3)$

57. $\lim_{x \to 2} \left( -\frac{5x}{x^2 - 1} \right)$

58. $\lim_{x \to 2} \left( \frac{x^2 - 5x + 6}{x + 2} \right)$

59. $\lim_{x \to 2} \left( \frac{x^2 - 9}{x - 3} \right)$

60. $\lim_{x \to -1} \left( \frac{x^2 - 2x - 3}{x + 1} \right)$

61. $\lim_{x \to \frac{3}{2}} \left( \frac{6x^2 - 17x + 12}{2x - 3} \right)$

62. $\lim_{x \to -\frac{7}{2}} \left( \frac{8x^2 + 18x - 35}{2x + 7} \right)$

63. $\lim_{x \to 3} \left( \frac{x^2 - 9}{x - 3} \right)$

64. $\lim_{x \to -3} \left( \frac{-7x^4 - 21x^3}{-12x^4 + 108x^2} \right)$

65. $\lim_{x \to 3} \left( \frac{x^2 + 2x - 3}{x - 3} \right)$

66. $\lim_{h \to 0} \left( \frac{(3 + h)^3 - 27}{h} \right)$

67. $\lim_{h \to 0} \left( \frac{(2 - h)^3 - 8}{h} \right)$

68. $\lim_{h \to 0} \left( \frac{(h + 3)^2 - 9}{h} \right)$

69. $\lim_{h \to 0} \left( \frac{\sqrt{5} - h - \sqrt{5}}{h} \right)$
70. \( \lim_{x \to 0} \left( \frac{\sqrt[3]{3} - x - \sqrt[3]{3}}{x} \right) \)

71. \( \lim_{x \to 9} \left( \frac{x^2 - 81}{3 - \sqrt{x}} \right) \)

72. \( \lim_{x \to 1} \left( \frac{\sqrt{x} - x^2}{1 - \sqrt{x}} \right) \)

73. \( \lim_{x \to 0} \left( \frac{x}{\sqrt{1 + 2x - 1}} \right) \)

74. \( \lim_{x \to \frac{3}{2}} \left( \frac{x^2 - 1}{4x - 1} \right) \)

75. \( \lim_{x \to 4} \left( \frac{x^3 - 64}{x^2 - 16} \right) \)

76. \( \lim_{x \to 2} \left( \frac{|x - 2|}{x - 2} \right) \)

77. \( \lim_{x \to 2^+} \left( \frac{|x - 2|}{x - 2} \right) \)

78. \( \lim_{x \to 2} \left( \frac{|x - 2|}{x - 2} \right) \)

79. \( \lim_{x \to 4^-} \left( \frac{|x - 4|}{4 - x} \right) \)

80. \( \lim_{x \to 4^+} \left( \frac{|x - 4|}{4 - x} \right) \)

81. \( \lim_{x \to 4} \left( \frac{|x - 4|}{4 - x} \right) \)

82. \( \lim_{x \to 2} \left( \frac{-8 + 6x - x^2}{x - 2} \right) \)

For the following exercise, use the given information to evaluate the limits: \( \lim_{x \to c} f(x) = 3, \lim_{x \to c} g(x) = 5 \)

83. \( \lim_{x \to c} \left[ 2f(x) + \sqrt{g(x)} \right] \)

84. \( \lim_{x \to c} \left[ 3f(x) + \sqrt{g(x)} \right] \)

85. \( \lim_{x \to c} \frac{f(x)}{g(x)} \)

For the following exercises, evaluate the following limits.

86. \( \lim_{x \to 2} \cos(\pi x) \)

87. \( \lim_{x \to 2} \sin(\pi x) \)

88. \( \lim_{x \to 2} \frac{\sin(\frac{2}{x})}{x} \)
89. \[ f(x) = \begin{cases} 2x^2 + 2x + 1, & x \leq 0; \\ x - 3, & x > 0 \end{cases} \quad \lim_{x \to 0^+} f(x) \]

90. \[ f(x) = \begin{cases} 2x^2 + 2x + 1, & x \leq 0; \\ x - 3, & x > 0 \end{cases} \quad \lim_{x \to 0^-} f(x) \]

91. \[ f(x) = \begin{cases} 2x^2 + 2x + 1, & x \leq 0; \\ x - 3, & x > 0 \end{cases} \quad \lim_{x \to 0} f(x) \]

92. \[ \lim_{x \to 4} \frac{\sqrt{x + 5} - 3}{x - 4} \]

93. \[ \lim_{x \to 2^+} (2x - [x]) \]

94. \[ \lim_{x \to 2^+} \frac{\sqrt{x + 7} - 3}{x - 2x^2 - x - 2} \]

95. \[ \lim_{x \to 3^+} \frac{x^2}{x^2 - 9} \]

For the following exercises, find the average rate of change \( \frac{f(x + h) - f(x)}{h} \).

96. \( f(x) = x + 1 \)

97. \( f(x) = 2x^2 - 1 \)

98. \( f(x) = x^2 + 3x + 4 \)

99. \( f(x) = x^2 + 4x - 100 \)

100. \( f(x) = 3x^2 + 1 \)

101. \( f(x) = \cos(x) \)

102. \( f(x) = 2x^3 - 4x \)

103. \( f(x) = \frac{1}{x} \)

104. \( f(x) = \frac{1}{x^2} \)

105. \( f(x) = \sqrt{x} \)

**Graphical**

106. Find an equation that could be represented by **Figure 12.15**.
107. Find an equation that could be represented by Figure 12.16.

For the following exercises, refer to Figure 12.17.

108. What is the right-hand limit of the function as $x$ approaches 0?

109. What is the left-hand limit of the function as $x$ approaches 0?
Real-World Applications

110. The position function \( s(t) = -16t^2 + 144t \) gives the position of a projectile as a function of time. Find the average velocity (average rate of change) on the interval \([1, 2]\).

111. The height of a projectile is given by \( s(t) = -64t^2 + 192t \). Find the average rate of change of the height from \( t = 1 \) second to \( t = 1.5 \) seconds.

112. The amount of money in an account after \( t \) years compounded continuously at 4.25% interest is given by the formula \( A = A_0 e^{0.0425t} \), where \( A_0 \) is the initial amount invested. Find the average rate of change of the balance of the account from \( t = 1 \) year to \( t = 2 \) years if the initial amount invested is $1,000.00.
Arizona is known for its dry heat. On a particular day, the temperature might rise as high as 118°F and drop down only to a brisk 95°F. Figure 12.18 shows the function $T$, where the output of $T(x)$ is the temperature in Fahrenheit degrees and the input $x$ is the time of day, using a 24-hour clock on a particular summer day.

When we analyze this graph, we notice a specific characteristic. There are no breaks in the graph. We could trace the graph without picking up our pencil. This single observation tells us a great deal about the function. In this section, we will investigate functions with and without breaks.

**Determining Whether a Function Is Continuous at a Number**

Let’s consider a specific example of temperature in terms of date and location, such as June 27, 2013, in Phoenix, AZ. The graph in Figure 12.18 indicates that, at 2 a.m., the temperature was 96°F. By 2 p.m. the temperature had risen to 116°F, and by 4 p.m. it was 118°F. Sometime between 2 a.m. and 4 p.m., the temperature outside must have been exactly 110.5°F. In fact, any temperature between 96°F and 118°F occurred at some point that day. This means all real numbers in the output between 96°F and 118°F are generated at some point by the function according to the intermediate value theorem,

Look again at Figure 12.18. There are no breaks in the function’s graph for this 24-hour period. At no point did the temperature cease to exist, nor was there a point at which the temperature jumped instantaneously by several degrees. A function that has no holes or breaks in its graph is known as a **continuous function**. Temperature as a function of time is an example of a continuous function.

If temperature represents a continuous function, what kind of function would not be continuous? Consider an example of dollars expressed as a function of hours of parking. Let’s create the function $D$, where $D(x)$ is the output representing cost in dollars for parking $x$ number of hours. See Figure 12.19.

Suppose a parking garage charges $4.00 per hour or fraction of an hour, with a $25 per day maximum charge. Park for two hours and five minutes and the charge is $12. Park an additional hour and the charge is $16. We can never be charged $13, $14, or $15. There are real numbers between 12 and 16 that the function never outputs. There are breaks in the function’s graph for this 24-hour period, points at which the price of parking jumps instantaneously by several dollars.
A function that has any hole or break in its graph is known as a **discontinuous function**. A stepwise function, such as parking-garage charges as a function of hours parked, is an example of a discontinuous function.

So how can we decide if a function is continuous at a particular number? We can check three different conditions. Let’s use the function \( y = f(x) \) represented in **Figure 12.20** as an example.

### Condition 1
According to Condition 1, the function \( f(a) \) defined at \( x = a \) must exist. In other words, there is a \( y \)-coordinate at \( x = a \) as in **Figure 12.21**.

### Condition 2
According to Condition 2, at \( x = a \) the limit, written \( \lim_{x \to a} f(x) \), must exist. This means that at \( x = a \) the left-hand limit must equal the right-hand limit. Notice as the graph of \( f \) in **Figure 12.20** approaches \( x = a \) from the left and right, the same \( y \)-coordinate is approached. Therefore, Condition 2 is satisfied. However, there could still be a hole in the graph at \( x = a \).

### Condition 3
According to Condition 3, the corresponding \( y \) coordinate at \( x = a \) fills in the hole in the graph of \( f \). This is written \( \lim_{x \to a} f(x) = f(a) \).
Satisfying all three conditions means that the function is continuous. All three conditions are satisfied for the function represented in Figure 12.22 so the function is continuous as \( x = a \).

**Figure 12.22** All three conditions are satisfied. The function is continuous at \( x = a \).

**Figure 12.23** through **Figure 12.26** provide several examples of graphs of functions that are not continuous at \( x = a \) and the condition or conditions that fail.

**Figure 12.23** Condition 2 is satisfied. Conditions 1 and 3 both fail.

**Figure 12.24** Conditions 1 and 2 are both satisfied. Condition 3 fails.

**Figure 12.25** Condition 1 is satisfied. Conditions 2 and 3 fail.
**Definition of Continuity**

A function \( f(x) \) is **continuous** at \( x = a \) provided all three of the following conditions hold true:

- **Condition 1:** \( f(a) \) exists.
- **Condition 2:** \( \lim_{x \to a} f(x) \) exists at \( x = a \).
- **Condition 3:** \( \lim_{x \to a} f(x) = f(a) \).

If a function \( f(x) \) is not continuous at \( x = a \), the function is **discontinuous** at \( x = a \).

**Identifying a Jump Discontinuity**

Discontinuity can occur in different ways. We saw in the previous section that a function could have a left-hand limit and a right-hand limit even if they are not equal. If the left- and right-hand limits exist but are different, the graph “jumps” at \( x = a \). The function is said to have a jump discontinuity.

As an example, look at the graph of the function \( y = f(x) \) in Figure 12.27. Notice as \( x \) approaches \( a \) how the output approaches different values from the left and from the right.

**Jump Discontinuity**

A function \( f(x) \) has a **jump discontinuity** at \( x = a \) if the left- and right-hand limits both exist but are not equal:

\[
\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) .
\]

**Identifying Removable Discontinuity**

Some functions have a discontinuity, but it is possible to redefine the function at that point to make it continuous. This type of function is said to have a removable discontinuity. Let’s look at the function \( y = f(x) \) represented by the graph in Figure 12.28. The function has a limit. However, there is a hole at \( x = a \). The hole can be filled by extending the domain.
to include the input $x = a$ and defining the corresponding output of the function at that value as the limit of the function at $x = a$.

![Figure 12.28](image)

**Figure 12.28** Graph of function $f$ with a removable discontinuity at $x = a$.

### Removable Discontinuity

A function $f(x)$ has a **removable discontinuity** at $x = a$ if the limit, $\lim_{x \to a} f(x)$, exists, but either

1. $f(a)$ does not exist, or
2. $f(a)$, the value of the function at $x = a$ does not equal the limit, $f(a) \neq \lim_{x \to a} f(x)$.

### Example 12.14

#### Identifying Discontinuities

Identify all discontinuities for the following functions as either a jump or a removable discontinuity.

a. $f(x) = \frac{x^2 - 2x - 15}{x - 5}$

b. $g(x) = \begin{cases} x + 1 & x < 2 \\ -x & x \geq 2 \end{cases}$

#### Solution

**Solution**

12.14 Identify all discontinuities for the following functions as either a jump or a removable discontinuity.

a. $f(x) = \frac{x^2 - 6x}{x - 6}$

b. $g(x) = \begin{cases} \sqrt{x}, & 0 \leq x < 4 \\ 2x, & x \geq 4 \end{cases}$

### Recognizing Continuous and Discontinuous Real-Number Functions

Many of the functions we have encountered in earlier chapters are continuous everywhere. They never have a hole in them, and they never jump from one value to the next. For all of these functions, the limit of $f(x)$ as $x$ approaches $a$ is the same as the value of $f(x)$ when $x = a$. So $\lim_{x \to a} f(x) = f(a)$. There are some functions that are continuous everywhere and some that are only continuous where they are defined on their domain because they are not defined for all real numbers.
Examples of Continuous Functions

The following functions are continuous everywhere:

<table>
<thead>
<tr>
<th>Function Type</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polynomial functions</td>
<td>$f(x) = x^4 - 9x^2$</td>
</tr>
<tr>
<td>Exponential functions</td>
<td>$f(x) = 4^x + 2 - 5$</td>
</tr>
<tr>
<td>Sine functions</td>
<td>$f(x) = \sin(2x) - 4$</td>
</tr>
<tr>
<td>Cosine functions</td>
<td>$f(x) = -\cos\left(x + \frac{\pi}{3}\right)$</td>
</tr>
</tbody>
</table>

Table 12.3

The following functions are continuous everywhere they are defined on their domain:

<table>
<thead>
<tr>
<th>Function Type</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Logarithmic functions</td>
<td>$f(x) = 2\ln(x), \ x &gt; 0$</td>
</tr>
<tr>
<td>Tangent functions</td>
<td>$f(x) = \tan(x) + 2, \ x \neq \frac{\pi}{2} + k\pi, \ k \text{ is an integer}$</td>
</tr>
<tr>
<td>Rational functions</td>
<td>$f(x) = \frac{x^2 - 25}{x - 7}, \ x \neq 7$</td>
</tr>
</tbody>
</table>

Table 12.4

Given a function $f(x)$, determine if the function is continuous at $x = a$.

1. Check Condition 1: $f(a)$ exists.
2. Check Condition 2: $\lim_{x \to a} f(x)$ exists at $x = a$.
3. Check Condition 3: $\lim_{x \to a} f(x) = f(a)$.
4. If all three conditions are satisfied, the function is continuous at $x = a$. If any one of the conditions is not satisfied, the function is not continuous at $x = a$.

Example 12.15

Determining Whether a Piecewise Function is Continuous at a Given Number

Determine whether the function $f(x) = \begin{cases} 4x, & x \leq 3 \\ 8 + x, & x > 3 \end{cases}$ is continuous at

a. $x = 3$
b. \( x = \frac{8}{3} \)

**Solution**

**Example 12.15**

Determine whether the function \( f(x) = \begin{cases} \frac{1}{x}, & x \leq 2 \\ 9x - 11.5, & x > 2 \end{cases} \) is continuous at \( x = 2 \).

**Determining Whether a Rational Function is Continuous at a Given Number**

Determine whether the function \( f(x) = \frac{x^2 - 25}{x - 5} \) is continuous at \( x = 5 \).

**Solution**

**Analysis**

See Figure 12.28. Notice that for Condition 2 we have

\[
\lim_{{x \to 5}} \frac{x^2 - 25}{x - 5} = \lim_{{x \to 5}} \frac{(x - 5)(x + 5)}{x - 5} = \lim_{{x \to 5}} (x + 5) = 5 + 5 = 10 \Rightarrow \text{Condition 2 is satisfied.}
\]

At \( x = 5 \), there exists a removable discontinuity. See Figure 12.28.
Determine whether the function \( f(x) = \frac{9 - x^2}{x^2 - 3x} \) is continuous at \( x = 3 \). If not, state the type of discontinuity.

**Determining the Input Values for Which a Function Is Discontinuous**

Now that we can identify continuous functions, jump discontinuities, and removable discontinuities, we will look at more complex functions to find discontinuities. Here, we will analyze a piecewise function to determine if any real numbers exist where the function is not continuous. A **piecewise function** may have discontinuities at the boundary points of the function as well as within the functions that make it up.

To determine the real numbers for which a piecewise function composed of polynomial functions is not continuous, recall that polynomial functions themselves are continuous on the set of real numbers. Any discontinuity would be at the boundary points. So we need to explore the three conditions of continuity at the boundary points of the piecewise function.

**Given a piecewise function, determine whether it is continuous at the boundary points.**

1. For each boundary point \( a \) of the piecewise function, determine the left- and right-hand limits as \( x \) approaches \( a \), as well as the function value at \( a \).

2. Check each condition for each value to determine if all three conditions are satisfied.

3. Determine whether each value satisfies condition 1: \( f(a) \) exists.

4. Determine whether each value satisfies condition 2: \( \lim_{x \to a} f(x) \) exists.

5. Determine whether each value satisfies condition 3: \( \lim_{x \to a} f(x) = f(a) \).

6. If all three conditions are satisfied, the function is continuous at \( x = a \). If any one of the conditions fails, the function is not continuous at \( x = a \).

**Example 12.17**
Determining the Input Values for Which a Piecewise Function Is Discontinuous

Determine whether the function \( f \) is discontinuous for any real numbers.

\[
f(x) = \begin{cases} 
  x + 1, & x < 2 \\
  \frac{x}{4}, & 2 \leq x < 4 \\
  x^2 - 11, & x \geq 4 
\end{cases}
\]

**Solution**

**Analysis**

See Figure 12.28. At \( x = 4 \), there exists a jump discontinuity. Notice that the function is continuous at \( x = 2 \).

Graph is continuous at \( x = 2 \) but shows a jump discontinuity at \( x = 4 \).

**12.17**

Determine where the function \( f(x) = \begin{cases} 
  \frac{\pi x}{4}, & x < 2 \\
  \frac{x}{4}, & 2 \leq x \leq 6 \\
  2\pi x, & x > 6 
\end{cases} \) is discontinuous.

**Determining Whether a Function Is Continuous**

To determine whether a piecewise function is continuous or discontinuous, in addition to checking the boundary points, we must also check whether each of the functions that make up the piecewise function is continuous.

**Given a piecewise function, determine whether it is continuous.**

1. Determine whether each component function of the piecewise function is continuous. If there are discontinuities, do they occur within the domain where that component function is applied?
2. For each boundary point \( x = a \) of the piecewise function, determine if each of the three conditions hold.

**Example 12.18**
Determining Whether a Piecewise Function Is Continuous

Determine whether the function below is continuous. If it is not, state the location and type of each discontinuity.

\[ f(x) = \begin{cases} 
\sin(x), & x < 0 \\
 x^3, & x > 0 
\end{cases} \]

**Solution**

**Analysis**

See Figure 12.28. There exists a removable discontinuity at \( x = 0 \); \( \lim_{x \to 0} f(x) = 0 \), thus the limit exists and is finite, but \( f(a) \) does not exist.

Function has removable discontinuity at 0.

Access these online resources for additional instruction and practice with continuity.

- Continuity at a Point (http://openstaxcollege.org/l/continuitypoint)
- Continuity at a Point: Concept Check (http://openstaxcollege.org/l/contconcept)
12.3 EXERCISES

Verbal

113. State in your own words what it means for a function \( f \) to be continuous at \( x = c \).

114. State in your own words what it means for a function to be continuous on the interval \((a, b)\).

Algebraic

For the following exercises, determine why the function \( f \) is discontinuous at a given point \( a \) on the graph. State which condition fails.

115. \( f(x) = \ln |x + 3|, a = -3 \)

116. \( f(x) = \ln |5x - 2|, a = \frac{2}{3} \)

117. \( f(x) = \frac{x^2 - 16}{x + 4}, a = -4 \)

118. \( f(x) = \frac{x^2 - 16x}{x}, a = 0 \)

119. \( f(x) = \begin{cases} x, & x \neq 3 \\ 2x, & x = 3 \end{cases}, a = 3 \)

120. \( f(x) = \begin{cases} 5, & x \neq 0 \\ 3, & x = 0 \end{cases}, a = 0 \)

121. \( f(x) = \begin{cases} \frac{2 - x}{3}, & x \neq 2 \\ x = 2 \end{cases}, a = 2 \)

122. \( f(x) = \begin{cases} \frac{1}{x + 6}, & x = -6 \\ \frac{1}{x^2}, & x \neq -6 \end{cases}, a = -6 \)

123. \( f(x) = \begin{cases} 3 + x, & x < 1 \\ x, & x = 1 \\ x^2, & x > 1 \end{cases}, a = 1 \)

124. \( f(x) = \begin{cases} 3 - x, & x < 1 \\ x, & x = 1 \\ 2x^2, & x > 1 \end{cases}, a = 1 \)

125. \( f(x) = \begin{cases} 3 + 2x, & x < 1 \\ x, & x = 1 \\ -x^2, & x > 1 \end{cases}, a = 1 \)

126. \( f(x) = \begin{cases} x^2, & x < -2 \\ 2x + 1, & x = -2 \\ x^3, & x > -2 \end{cases}, a = -2 \)

127. \( f(x) = \begin{cases} \frac{x^2 - 9}{x + 3}, & x < -3 \\ \frac{x - 9}{x}, & x = -3 \\ 1, & x > -3 \end{cases}, a = -3 \)
128. \[ f(x) = \begin{cases} 
\frac{x^2 - 9}{x + 3}, & x < -3 \\
9x, & x = -3 \\
-6, & x > -3 
\end{cases} \ a = 3 \]

129. \[ f(x) = \frac{x^2 - 4}{x - 2}, \quad a = 2 \]

130. \[ f(x) = \frac{25 - x^2}{x^2 - 10x + 25}, \quad a = 5 \]

131. \[ f(x) = \frac{x^3 - 9x}{x^2 + 11x + 24}, \quad a = -3 \]

132. \[ f(x) = \frac{x^3 - 27}{x^2 - 3x}, \quad a = 3 \]

133. \[ f(x) = \frac{1}{|x|}, \quad a = 0 \]

134. \[ f(x) = \frac{2|x + 2|}{x + 2}, \quad a = -2 \]

For the following exercises, determine whether or not the given function \( f \) is continuous everywhere. If it is continuous everywhere it is defined, state for what range it is continuous. If it is discontinuous, state where it is discontinuous.

135. \( f(x) = x^3 - 2x - 15 \)

136. \[ f(x) = \frac{x^2 - 2x - 15}{x - 5} \]

137. \[ f(x) = 2 \cdot 3^x + 4 \]

138. \[ f(x) = -\sin(3x) \]

139. \[ f(x) = \frac{|x - 2|}{x^2 - 2x} \]

140. \[ f(x) = \tan(x) + 2 \]

141. \[ f(x) = 2x + \frac{5}{x} \]

142. \[ f(x) = \log_2 (x) \]

143. \[ f(x) = \ln x^2 \]

144. \[ f(x) = e^{2x} \]

145. \[ f(x) = \sqrt{x - 4} \]

146. \[ f(x) = \sec(x) - 3 \]

147. \[ f(x) = x^2 + \sin(x) \]

148. Determine the values of \( b \) and \( c \) such that the following function is continuous on the entire real number line.
Graphical

For the following exercises, refer to Figure 12.29. Each square represents one square unit. For each value of $a$, determine which of the three conditions of continuity are satisfied at $x = a$ and which are not.

149. $x = -3$
150. $x = 2$
151. $x = 4$

For the following exercises, use a graphing utility to graph the function $f(x) = \sin\left(\frac{12\pi}{x}\right)$ as in Figure 12.30. Set the $x$-axis a short distance before and after 0 to illustrate the point of discontinuity.

152. Which conditions for continuity fail at the point of discontinuity?
153. Evaluate $f(0)$.
154.
Solve for $x$ if $f(x) = 0$.

155. What is the domain of $f(x)$?

For the following exercises, consider the function shown in Figure 12.31.

156. At what $x$-coordinates is the function discontinuous?

157. What condition of continuity is violated at these points?

158. Consider the function shown in Figure 12.32. At what $x$-coordinates is the function discontinuous? What condition(s) of continuity were violated?

159. Construct a function that passes through the origin with a constant slope of 1, with removable discontinuities at $x = -7$ and $x = 1$.

160. The function $f(x) = \frac{x^3 - 1}{x - 1}$ is graphed in Figure 12.33. It appears to be continuous on the interval $[-3, 3]$, but there is an $x$-value on that interval at which the function is discontinuous. Determine the value of $x$ at which the function is discontinuous, and explain the pitfall of utilizing technology when considering continuity of a function by examining its graph.
161. Find the limit \( \lim_{x \to 1} f(x) \) and determine if the following function is continuous at \( x = 1 \):

\[
f(x) = \begin{cases} 
  x^2 + 4 & x \neq 1 \\
  2 & x = 1 
\end{cases}
\]

(12.25)

162. The graph of \( f(x) = \frac{\sin(2x)}{x} \) is shown in Figure 12.34. Is the function \( f(x) \) continuous at \( x = 0 \)? Why or why not?
12.4 | Derivatives

Learning Objectives

In this section, you will:

12.4.1 Find the derivative of a function.
12.4.2 Find instantaneous rates of change.
12.4.3 Find an equation of the tangent line to the graph of a function at a point.
12.4.4 Find the instantaneous velocity of a particle.

The average teen in the United States opens a refrigerator door an estimated 25 times per day. Supposedly, this average is up from 10 years ago when the average teenager opened a refrigerator door 20 times per day \(^2\).

It is estimated that a television is on in a home 6.75 hours per day, whereas parents spend an estimated 5.5 minutes per day having a meaningful conversation with their children. These averages, too, are not the same as they were 10 years ago, when the television was on an estimated 6 hours per day in the typical household, and parents spent 12 minutes per day in meaningful conversation with their kids.

What do these scenarios have in common? The functions representing them have changed over time. In this section, we will consider methods of computing such changes over time.

Finding the Average Rate of Change of a Function

The functions describing the examples above involve a change over time. Change divided by time is one example of a rate. The rates of change in the previous examples are each different. In other words, some changed faster than others. If we were to graph the functions, we could compare the rates by determining the slopes of the graphs.

A tangent line to a curve is a line that intersects the curve at only a single point but does not cross it there. (The tangent line may intersect the curve at another point away from the point of interest.) If we zoom in on a curve at that point, the curve appears linear, and the slope of the curve at that point is close to the slope of the tangent line at that point.

Figure 12.35 represents the function \( f(x) = x^3 - 4x \). We can see the slope at various points along the curve.

- slope at \( x = -2 \) is 8
- slope at \( x = -1 \) is -1
- slope at \( x = 2 \) is 8

\[ \text{Figure 12.35} \quad \text{Graph showing tangents to curve at } -2, -1, \text{ and } 2. \]

\(^2\) http://www.csun.edu/science/health/docs/tv&health.html Source provided.
Let’s imagine a point on the curve of function \( f \) at \( x = a \) as shown in Figure 12.36. The coordinates of the point are \((a, f(a))\). Connect this point with a second point on the curve a little to the right of \( x = a \), with an \( x\)-value increased by some small real number \( h \). The coordinates of this second point are \((a + h, f(a + h))\) for some positive-value \( h \).

We can calculate the slope of the line connecting the two points \((a, f(a))\) and \((a + h, f(a + h))\), called a secant line, by applying the slope formula,

\[
slope = \frac{\text{change in } y}{\text{change in } x}
\]

We use the notation \( m_{\text{sec}} \) to represent the slope of the secant line connecting two points.

\[
 m_{\text{sec}} = \frac{f(a + h) - f(a)}{(a + h) - a} \tag{12.27}
\]

The slope \( m_{\text{sec}} \) equals the average rate of change between two points \((a, f(a))\) and \((a + h, f(a + h))\).

\[
 m_{\text{sec}} = \frac{f(a + h) - f(a)}{h} \tag{12.28}
\]

**The Average Rate of Change between Two Points on a Curve**

The average rate of change (AROC) between two points \((a, f(a))\) and \((a + h, f(a + h))\) on the curve of \( f \) is the slope of the line connecting the two points and is given by

\[
 \text{AROC} = \frac{f(a + h) - f(a)}{h} \tag{12.29}
\]

**Example 12.19**

**Finding the Average Rate of Change**

Find the average rate of change connecting the points \((2, -6)\) and \((-1, 5)\).
12.18 Find the average rate of change connecting the points (−5, 1.5) and (−2.5, 9).

Understanding the Instantaneous Rate of Change

Now that we can find the average rate of change, suppose we make \( h \) in Figure 12.36 smaller and smaller. Then \( a + h \) will approach \( a \) as \( h \) gets smaller, getting closer and closer to 0. Likewise, the second point \( (a + h, f(a + h)) \) will approach the first point, \( (a, f(a)) \). As a consequence, the connecting line between the two points, called the secant line, will get closer and closer to being a tangent to the function at \( x = a \), and the slope of the secant line will get closer and closer to the slope of the tangent at \( x = a \). See Figure 12.37.

![Figure 12.37](image)

The connecting line between two points moves closer to being a tangent line at \( x = a \).

Because we are looking for the slope of the tangent at \( x = a \), we can think of the measure of the slope of the curve of a function \( f \) at a given point as the rate of change at a particular instant. We call this slope the instantaneous rate of change, or the derivative of the function at \( x = a \). Both can be found by finding the limit of the slope of a line connecting the point at \( x = a \) with a second point infinitesimally close along the curve. For a function \( f \) both the instantaneous rate of change of the function and the derivative of the function at \( x = a \) are written as \( f'(a) \), and we can define them as a two-sided limit that has the same value whether approached from the left or the right.

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{12.30}
\]

The expression by which the limit is found is known as the difference quotient.

**Definition of Instantaneous Rate of Change and Derivative**

The derivative, or instantaneous rate of change, of a function \( f \) at \( x = a \), is given by

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{12.31}
\]

The expression \( \frac{f(a + h) - f(a)}{h} \) is called the difference quotient.

We use the difference quotient to evaluate the limit of the rate of change of the function as \( h \) approaches 0.

**Derivatives: Interpretations and Notation**

The derivative of a function can be interpreted in different ways. It can be observed as the behavior of a graph of the function or calculated as a numerical rate of change of the function.
The derivative of a function $f(x)$ at a point $x = a$ is the slope of the tangent line to the curve $f(x)$ at $x = a$. The derivative of $f(x)$ at $x = a$ is written $f'(a)$.

The derivative $f'(a)$ measures how the curve changes at the point $(a, f(a))$.

The derivative $f'(a)$ may be thought of as the instantaneous rate of change of the function $f(x)$ at $x = a$.

If a function measures distance as a function of time, then the derivative measures the instantaneous velocity at time $t = a$.

### Notations for the Derivative

The equation of the derivative of a function $f(x)$ is written as $y' = f'(x)$, where $y = f(x)$. The notation $f'(x)$ is read as “$f$ prime of $x$.” Alternate notations for the derivative include the following:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x)$$

(12.32)

The expression $f'(x)$ is now a function of $x$; this function gives the slope of the curve $y = f(x)$ at any value of $x$.

The derivative of a function $f(x)$ at a point $x = a$ is denoted $f'(a)$.

---

**Given a function $f$, find the derivative by applying the definition of the derivative.**

1. Calculate $f(a + h)$.
2. Calculate $f(a)$.
3. Substitute and simplify $\frac{f(a + h) - f(a)}{h}$.
4. Evaluate the limit if it exists: $f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$.

---

**Example 12.20**

### Finding the Derivative of a Polynomial Function

Find the derivative of the function $f(x) = x^2 - 3x + 5$ at $x = a$.

**Solution**

---

**Try it** Find the derivative of the function $f(x) = 3x^2 + 7x$ at $x = a$.

---

**Finding Derivatives of Rational Functions**

To find the derivative of a rational function, we will sometimes simplify the expression using algebraic techniques we have already learned.

---

**Example 12.21**
Finding the Derivative of a Rational Function

Find the derivative of the function \( f(x) = \frac{3 + x}{2 - x} \) at \( x = a \).

Solution

\[ f(x) = \frac{3 + x}{2 - x} \]

Finding Derivatives of Functions with Roots

To find derivatives of functions with roots, we use the methods we have learned to find limits of functions with roots, including multiplying by a conjugate.

Example 12.22

Finding the Derivative of a Function with a Root

Find the derivative of the function \( f(x) = 4\sqrt{x} \) at \( x = 36 \).

Solution

\[ f(x) = 4\sqrt{x} \]

Finding Instantaneous Rates of Change

Many applications of the derivative involve determining the rate of change at a given instant of a function with the independent variable time—which is why the term instantaneous is used. Consider the height of a ball tossed upward with an initial velocity of 64 feet per second, given by \( s(t) = -16t^2 + 64t + 6 \), where \( t \) is measured in seconds and \( s(t) \) is measured in feet. We know the path is that of a parabola. The derivative will tell us how the height is changing at any given point in time. The height of the ball is shown in Figure 12.38 as a function of time. In physics, we call this the “s-t graph.”
Example 12.23

Finding the Instantaneous Rate of Change

Using the function above, \( s(t) = -16t^2 + 64t + 6 \), what is the instantaneous velocity of the ball at 1 second and 3 seconds into its flight?

Solution

The position of the ball is given by \( s(t) = -16t^2 + 64t + 6 \). What is its velocity 2 seconds into flight?

Using Graphs to Find Instantaneous Rates of Change

We can estimate an instantaneous rate of change at \( x = a \) by observing the slope of the curve of the function \( f(x) \) at \( x = a \). We do this by drawing a line tangent to the function at \( x = a \) and finding its slope.
Given a graph of a function $f(x)$, find the instantaneous rate of change of the function at $x = a$.

1. Locate $x = a$ on the graph of the function $f(x)$.
2. Draw a tangent line, a line that goes through $x = a$ at $a$ and at no other point in that section of the curve. Extend the line far enough to calculate its slope as

$$\frac{\text{change in } y}{\text{change in } x} \quad (12.33)$$

Example 12.24

Estimating the Derivative at a Point on the Graph of a Function

From the graph of the function $y = f(x)$ presented in Figure 12.39, estimate each of the following:

$f(0); \ f(2); \ f'(0); \ f'(2)$
12.23 Using the graph of the function \( f(x) = x^3 - 3x \) shown in Figure 12.40, estimate: \( f(1) \), \( f'(1) \), \( f(0) \), and \( f'(0) \).

![Figure 12.40](image)

**Using Instantaneous Rates of Change to Solve Real-World Problems**

Another way to interpret an instantaneous rate of change at \( x = a \) is to observe the function in a real-world context. The unit for the derivative of a function \( f(x) \) is

\[
\text{output units} \quad \frac{\text{input unit}}{\text{input unit}}
\]

Such a unit shows by how many units the output changes for each one-unit change of input. The instantaneous rate of change at a given instant shows the same thing: the units of change of output per one-unit change of input.

One example of an instantaneous rate of change is a marginal cost. For example, suppose the production cost for a company to produce \( x \) items is given by \( C(x) \), in thousands of dollars. The derivative function tells us how the cost is changing for any value of \( x \) in the domain of the function. In other words, \( C'(x) \) is interpreted as a marginal cost, the additional cost in thousands of dollars of producing one more item when \( x \) items have been produced. For example, \( C'(11) \) is the approximate additional cost in thousands of dollars of producing the 12th item after 11 items have been produced. \( C'(11) = 2.50 \) means that when 11 items have been produced, producing the 12th item would increase the total cost by approximately $2,500.00.

**Example 12.25**

**Finding a Marginal Cost**

The cost in dollars of producing \( x \) laptop computers in dollars is \( f(x) = x^2 - 100x \). At the point where 200 computers have been produced, what is the approximate cost of producing the 201st unit?
Example 12.26
Interpreting a Derivative in Context

A car leaves an intersection. The distance it travels in miles is given by the function \( f(t) \), where \( t \) represents hours. Explain the following notations:
\[
f(0) = 0; \quad f'(1) = 60; \quad f(1) = 70; \quad f(2.5) = 150
\]

Solution

A runner runs along a straight east-west road. The function \( f(t) \) gives how many feet eastward of her starting point she is after \( t \) seconds. Interpret each of the following as it relates to the runner.
\[
f(0) = 0; \quad f(10) = 150; \quad f'(10) = 15; \quad f'(20) = -10; \quad f(40) = -100
\]

Finding Points Where a Function’s Derivative Does Not Exist

To understand where a function’s derivative does not exist, we need to recall what normally happens when a function \( f(x) \) has a derivative at \( x = a \). Suppose we use a graphing utility to zoom in on \( x = a \). If the function \( f(x) \) is differentiable, that is, if it is a function that can be differentiated, then the closer one zooms in, the more closely the graph approaches a straight line. This characteristic is called linearity.

Look at the graph in Figure 12.41. The closer we zoom in on the point, the more linear the curve appears.

We might presume the same thing would happen with any continuous function, but that is not so. The function \( f(x) = |x| \), for example, is continuous at \( x = 0 \), but not differentiable at \( x = 0 \). As we zoom in close to 0 in Figure 12.42, the graph does not approach a straight line. No matter how close we zoom in, the graph maintains its sharp corner.
Figure 12.42  Graph of the function \( f(x) = |x| \), with \( x \)-axis from \(-0.1\) to \(0.1\) and \( y \)-axis from \(-0.1\) to \(0.1\).

We zoom in closer by narrowing the range to produce Figure 12.43 and continue to observe the same shape. This graph does not appear linear at \( x = 0 \).

Figure 12.43  Graph of the function \( f(x) = |x| \), with \( x \)-axis from \(-0.001\) to \(0.001\) and \( y \)-axis from \(-0.001\) to \(0.001\).

What are the characteristics of a graph that is not differentiable at a point? Here are some examples in which function \( f(x) \) is not differentiable at \( x = a \).

In Figure 12.44, we see the graph of

\[
f(x) = \begin{cases} 
  x^2, & x \leq 2 \\
  8 - x, & x > 2 
\end{cases}
\]  

(12.35)

Notice that, as \( x \) approaches 2 from the left, the left-hand limit may be observed to be 4, while as \( x \) approaches 2 from the right, the right-hand limit may be observed to be 6. We see that it has a discontinuity at \( x = 2 \).
Figure 12.44  The graph of \( f(x) \) has a discontinuity at \( x = 2 \).

In Figure 12.45, we see the graph of \( f(x) = |x| \). We see that the graph has a corner point at \( x = 0 \).

Figure 12.45  The graph of \( f(x) = |x| \) has a corner point at \( x = 0 \).

In Figure 12.46, we see that the graph of \( f(x) = x^{\frac{2}{3}} \) has a cusp at \( x = 0 \). A cusp has a unique feature. Moving away from the cusp, both the left-hand and right-hand limits approach either infinity or negative infinity. Notice the tangent lines as \( x \) approaches 0 from both the left and the right appear to get increasingly steeper, but one has a negative slope, the other has a positive slope.

Figure 12.46  The graph of \( f(x) = x^{\frac{2}{3}} \) has a cusp at \( x = 0 \).

In Figure 12.47, we see that the graph of \( f(x) = x^{\frac{1}{3}} \) has a vertical tangent at \( x = 0 \). Recall that vertical tangents are vertical lines, so where a vertical tangent exists, the slope of the line is undefined. This is why the derivative, which measures the slope, does not exist there.
The graph of \( f(x) = x^{\frac{1}{3}} \) has a vertical tangent at \( x = 0 \).

**Differentiability**

A function \( f(x) \) is differentiable at \( x = a \) if the derivative exists at \( x = a \), which means that \( f'(a) \) exists.

There are four cases for which a function \( f(x) \) is not differentiable at a point \( x = a \).

1. When there is a discontinuity at \( x = a \).
2. When there is a corner point at \( x = a \).
3. When there is a cusp at \( x = a \).
4. Any other time when there is a vertical tangent at \( x = a \).

**Example 12.27**

**Determining Where a Function Is Continuous and Differentiable from a Graph**

Using Figure 12.48, determine where the function is

a. continuous
b. discontinuous
c. differentiable
d. not differentiable

At the points where the graph is discontinuous or not differentiable, state why.
12.25 Determine where the function \( y = f(x) \) shown in Figure 12.49 is continuous and differentiable from the graph.

**Finding an Equation of a Line Tangent to the Graph of a Function**

The equation of a tangent line to a curve of the function \( f(x) \) at \( x = a \) is derived from the point-slope form of a line, \( y = m(x - x_1) + y_1 \). The slope of the line is the slope of the curve at \( x = a \) and is therefore equal to \( f'(a) \), the derivative of \( f(x) \) at \( x = a \). The coordinate pair of the point on the line at \( x = a \) is \( (a, f(a)) \).

If we substitute into the point-slope form, we have
The equation of the tangent line is

\[ y = f'(a)(x - a) + f(a). \]  

(12.36)

**The Equation of a Line Tangent to a Curve of the Function**

The equation of a line tangent to the curve of a function \( f \) at a point \( x = a \) is

\[ y = f'(a)(x - a) + f(a). \]  

(12.37)

**Example 12.28**

**Finding the Equation of a Line Tangent to a Function at a Point**

Find the equation of a line tangent to the curve \( f(x) = x^2 - 4x \) at \( x = 3 \).

**Solution**

1. Find the derivative of \( f(x) \) at \( x = 3 \) using \( f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \).
2. Evaluate the function at \( x = 3 \). This is \( f(a) \).
3. Substitute \( (a, f(a)) \) and \( f'(a) \) into \( y = f'(a)(x - a) + f(a) \).
4. Write the equation of the tangent line in the form \( y = mx + b \).

**Analysis**

We can use a graphing utility to graph the function and the tangent line. In so doing, we can observe the point of tangency at \( x = 3 \) as shown in Figure 12.49.
Graph confirms the point of tangency at \( x = 3 \).

**12.26** Find the equation of a tangent line to the curve of the function \( f(x) = 5x^2 - x + 4 \) at \( x = 2 \).

### Finding the Instantaneous Speed of a Particle

If a function measures position versus time, the derivative measures displacement versus time, or the speed of the object. A change in speed or direction relative to a change in time is known as velocity. The velocity at a given instant is known as instantaneous velocity.

In trying to find the speed or velocity of an object at a given instant, we seem to encounter a contradiction. We normally define speed as the distance traveled divided by the elapsed time. But in an instant, no distance is traveled, and no time elapses. How will we divide zero by zero? The use of a derivative solves this problem. A derivative allows us to say that even while the object’s velocity is constantly changing, it has a certain velocity at a given instant. That means that if the object traveled at that exact velocity for a unit of time, it would travel the specified distance.

#### Instantaneous Velocity

Let the function \( s(t) \) represent the position of an object at time \( t \). The **instantaneous velocity** or velocity of the object at time \( t = a \) is given by

\[
s'(a) = \lim_{h \to 0} \frac{s(a + h) - s(a)}{h}.
\]  

**(12.38)**

#### Example 12.29

**Finding the Instantaneous Velocity**

A ball is tossed upward from a height of 200 feet with an initial velocity of 36 ft/sec. If the height of the ball in feet after \( t \) seconds is given by \( s(t) = -16t^2 + 36t + 200 \), find the instantaneous velocity of the ball at \( t = 2 \).
This result means that at time $t = 2$ seconds, the ball is dropping at a rate of 28 ft/sec.

**Try it**

12.27 A fireworks rocket is shot upward out of a pit 12 ft below the ground at a velocity of 60 ft/sec. Its height in feet after $t$ seconds is given by $s = -16t^2 + 60t - 12$. What is its instantaneous velocity after 4 seconds?

Access these online resources for additional instruction and practice with derivatives.
- [Estimate the Derivative](http://openstaxcollege.org/l/estimatederiv)
- [Estimate the Derivative Ex. 4](http://openstaxcollege.org/l/estimatederiv4)
12.4 EXERCISES

Verbal

163. How is the slope of a linear function similar to the derivative?

164. What is the difference between the average rate of change of a function on the interval \([x, x + h]\) and the derivative of the function at \(x\)?

165. A car traveled 110 miles during the time period from 2:00 P.M. to 4:00 P.M. What was the car's average velocity? At exactly 2:30 P.M., the speed of the car registered exactly 62 miles per hour. What is another name for the speed of the car at 2:30 P.M.? Why does this speed differ from the average velocity?

166. Explain the concept of the slope of a curve at point \(x\).

167. Suppose water is flowing into a tank at an average rate of 45 gallons per minute. Translate this statement into the language of mathematics.

Algebraic

For the following exercises, use the definition of derivative \(\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}\) to calculate the derivative of each function.

168. \(f(x) = 3x - 4\)

169. \(f(x) = -2x + 1\)

170. \(f(x) = x^2 - 2x + 1\)

171. \(f(x) = 2x^2 + x - 3\)

172. \(f(x) = 2x^2 + 5\)

173. \(f(x) = \frac{-1}{x - 2}\)

174. \(f(x) = \frac{2 + x}{1 - x}\)

175. \(f(x) = \frac{5 - 2x}{3 + 2x}\)

176. \(f(x) = \sqrt{1 + 3x}\)

177. \(f(x) = 3x^3 - x^2 + 2x + 5\)

178. \(f(x) = 5\)

179. \(f(x) = 5\pi\)

For the following exercises, find the average rate of change between the two points.

180. \((-2, 0)\) and \((-4, 5)\)

181. \((4, -3)\) and \((-2, -1)\)

182. \((0, 5)\) and \((6, 5)\)

183. \((7, -2)\) and \((7, 10)\)
For the following polynomial functions, find the derivatives.

184.  \( f(x) = x^3 + 1 \)

185.  \( f(x) = -3x^2 - 7x = 6 \)

186.  \( f(x) = 7x^2 \)

187.  \( f(x) = 3x^3 + 2x^2 + x - 26 \)

For the following functions, find the equation of the tangent line to the curve at the given point \( x \) on the curve.

188.  \( f(x) = 2x^2 - 3x \quad x = 3 \)

189.  \( f(x) = x^3 + 1 \quad x = 2 \)

190.  \( f(x) = \sqrt{x} \quad x = 9 \)

For the following exercise, find \( k \) such that the given line is tangent to the graph of the function.

191.  \( f(x) = x^2 - kx, \quad y = 4x - 9 \)

**Graphical**

For the following exercises, consider the graph of the function \( f \) and determine where the function is continuous/discontinuous and differentiable/not differentiable.

192.
For the following exercises, use Figure 12.50 to estimate either the function at a given value of $x$ or the derivative at a given value of $x$, as indicated.

Figure 12.50

196. $f(-1)$
197. $f(0)$
198. $f(1)$
199. $f(2)$
200. \( f(3) \)

201. \( f'( -1) \)

202. \( f'(0) \)

203. \( f'(1) \)

204. \( f'(2) \)

205. \( f'(3) \)

206. Sketch the function based on the information below:
\[ f''(x) = 2x, \quad f(2) = 4 \]

Technology

207. Numerically evaluate the derivative. Explore the behavior of the graph of \( f(x) = x^2 \) around \( x = 1 \) by graphing the function on the following domains: \([0.9, 1.1], [0.99, 1.01], [0.999, 1.001], \) and \([0.9999, 1.0001]\). We can use the feature on our calculator that automatically sets Ymin and Ymax to the Xmin and Xmax values we preset. (On some of the commonly used graphing calculators, this feature may be called ZOOM FIT or ZOOM AUTO). By examining the corresponding range values for this viewing window, approximate how the curve changes at \( x = 1 \), that is, approximate the derivative at \( x = 1 \).

Real-World Applications

For the following exercises, explain the notation in words. The volume \( f(t) \) of a tank of gasoline, in gallons, \( t \) minutes after noon.

208. \( f(0) = 600 \)

209. \( f'(30) = -20 \)

210. \( f(30) = 0 \)

211. \( f'(200) = 30 \)

212. \( f'(240) = 500 \)

For the following exercises, explain the functions in words. The height, \( s \), of a projectile after \( t \) seconds is given by \( s(t) = -16t^2 + 80t \).

213. \( s(2) = 96 \)

214. \( s'(2) = 16 \)

215. \( s(3) = 96 \)

216. \( s'(3) = -16 \)

217. \( s(0) = 0, \quad s(5) = 0. \)

For the following exercises, the volume \( V \) of a sphere with respect to its radius \( r \) is given by \( V = \frac{4}{3}\pi r^3 \).

218. Find the average rate of change of \( V \) as \( r \) changes from 1 cm to 2 cm.

219. Find the instantaneous rate of change of \( V \) when \( r = 3 \) cm.
For the following exercises, the revenue generated by selling \( x \) items is given by \( R(x) = 2x^2 + 10x \).

220. Find the average change of the revenue function as \( x \) changes from \( x = 10 \) to \( x = 20 \).

221. Find \( R'(10) \) and interpret.

222. Find \( R'(15) \) and interpret. Compare \( R'(15) \) to \( R'(10) \), and explain the difference.

For the following exercises, the cost of producing \( x \) cellphones is described by the function \( C(x) = x^2 - 4x + 1000 \).

223. Find the average rate of change in the total cost as \( x \) changes from \( x = 10 \) to \( x = 15 \).

224. Find the approximate marginal cost, when 15 cellphones have been produced, of producing the 16\({}^{\text{th}}\) cellphone.

225. Find the approximate marginal cost, when 20 cellphones have been produced, of producing the 21\({}^{\text{st}}\) cellphone.

**Extension**

For the following exercises, use the definition for the derivative at a point \( x = a \), \( \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(a)}{\Delta x} \), to find the derivative of the functions.

226. \( f(x) = \frac{1}{x^2} \)

227. \( f(x) = 5x^2 - x + 4 \)

228. \( f(x) = -x^2 + 4x + 7 \)

229. \( f(x) = \frac{-4}{3 - x^2} \)
CHAPTER 12 REVIEW

KEY TERMS

average rate of change: the slope of the line connecting the two points \((a, f(a))\) and \((a + h, f(a + h))\) on the curve of \(f(x)\); it is given by \(\text{AROC} = \frac{f(a + h) - f(a)}{h}\).

continuous function: a function that has no holes or breaks in its graph

derivative: the slope of a function at a given point; denoted \(f'(a)\), at a point \(x = a\) it is \(f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}\), providing the limit exists.

differentiable: a function \(f(x)\) for which the derivative exists at \(x = a\). In other words, if \(f'(a)\) exists.

discontinuous function: a function that is not continuous at \(x = a\)

instantaneous rate of change: the slope of a function at a given point; at \(x = a\) it is given by \(f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}\).

instantaneous velocity: the change in speed or direction at a given instant; a function \(s(t)\) represents the position of an object at time \(t\), and the instantaneous velocity or velocity of the object at time \(t = a\) is given by \(s'(a) = \lim_{h \to 0} \frac{s(a + h) - s(a)}{h}\).

jump discontinuity: a point of discontinuity in a function \(f(x)\) at \(x = a\) where both the left and right-hand limits exist, but \(\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)\)

left-hand limit: the limit of values of \(f(x)\) as \(x\) approaches \(a\) from the left, denoted \(\lim_{x \to a^-} f(x) = L\). The values of \(f(x)\) can get as close to the limit \(L\) as we like by taking values of \(x\) sufficiently close to \(a\) such that \(x < a\) and \(x \neq a\). Both \(a\) and \(L\) are real numbers.

limit: when it exists, the value, \(L\), that the output of a function \(f(x)\) approaches as the input \(x\) gets closer and closer to \(a\) but does not equal \(a\). The value of the output, \(f(x)\), can get as close to \(L\) as we choose to make it by using input values of \(x\) sufficiently near to \(x = a\), but not necessarily at \(x = a\). Both \(a\) and \(L\) are real numbers, and \(L\) is denoted \(\lim_{x \to a} f(x) = L\).

properties of limits: a collection of theorems for finding limits of functions by performing mathematical operations on the limits

removable discontinuity: a point of discontinuity in a function \(f(x)\) where the function is discontinuous, but can be redefined to make it continuous

right-hand limit: the limit of values of \(f(x)\) as \(x\) approaches \(a\) from the right, denoted \(\lim_{x \to a^+} f(x) = L\). The values of \(f(x)\) can get as close to the limit \(L\) as we like by taking values of \(x\) sufficiently close to \(a\) where \(x > a\), and \(x \neq a\). Both \(a\) and \(L\) are real numbers.

secant line: a line that intersects two points on a curve

tangent line: a line that intersects a curve at a single point
two-sided limit: the limit of a function \( f(x) \), as \( x \) approaches \( a \), is equal to \( L \), that is, \( \lim_{x \to a} f(x) = L \) if and only if

\[
\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).
\]

**KEY EQUATIONS**

<table>
<thead>
<tr>
<th>average rate of change</th>
<th>AROC = ( \frac{f(a+h) - f(a)}{h} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>derivative of a function</td>
<td>( f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} )</td>
</tr>
</tbody>
</table>

Table 12.5

**KEY CONCEPTS**

12.1 Finding Limits: Numerical and Graphical Approaches

- A function has a limit if the output values approach some value \( L \) as the input values approach some quantity \( a \). See Example 12.1.
- A shorthand notation is used to describe the limit of a function according to the form \( \lim_{x \to a} f(x) = L \), which indicates that as \( x \) approaches \( a \), both from the left of \( x = a \) and the right of \( x = a \), the output value gets close to \( L \).
- A function has a left-hand limit if \( f(x) \) approaches \( L \) as \( x \) approaches \( a \) where \( x < a \). A function has a right-hand limit if \( f(x) \) approaches \( L \) as \( x \) approaches \( a \) where \( x > a \).
- A two-sided limit exists if the left-hand limit and the right-hand limit of a function are the same. A function is said to have a limit if it has a two-sided limit.
- A graph provides a visual method of determining the limit of a function.
- If the function has a limit as \( x \) approaches \( a \), the branches of the graph will approach the same \( y \)-coordinate near \( x = a \) from the left and the right. See Example 12.2.
- A table can be used to determine if a function has a limit. The table should show input values that approach \( a \) from both directions so that the resulting output values can be evaluated. If the output values approach some number, the function has a limit. See Example 12.3.
- A graphing utility can also be used to find a limit. See Example 12.4.

12.2 Finding Limits: Properties of Limits

- The properties of limits can be used to perform operations on the limits of functions rather than the functions themselves. See Example 12.5.
- The limit of a polynomial function can be found by finding the sum of the limits of the individual terms. See Example 12.6 and Example 12.7.
- The limit of a function that has been raised to a power equals the same power of the limit of the function. Another method is direct substitution. See Example 12.8.
- The limit of the root of a function equals the corresponding root of the limit of the function.
- One way to find the limit of a function expressed as a quotient is to write the quotient in factored form and simplify. See Example 12.9.
- Another method of finding the limit of a complex fraction is to find the LCD. See Example 12.10.
- A limit containing a function containing a root may be evaluated using a conjugate. See Example 12.11.
- The limits of some functions expressed as quotients can be found by factoring. See Example 12.12.
One way to evaluate the limit of a quotient containing absolute values is by using numeric evidence. Setting it up piecewise can also be useful. See Example 12.13.

### 12.3 Continuity

- A continuous function can be represented by a graph without holes or breaks.
- A function whose graph has holes is a discontinuous function.
- A function is continuous at a particular number if three conditions are met:
  - Condition 1: \( f(a) \) exists.
  - Condition 2: \( \lim_{x \to a} f(x) \) exists at \( x = a \).
  - Condition 3: \( \lim_{x \to a} f(x) = f(a) \).
- A function has a jump discontinuity if the left- and right-hand limits are different, causing the graph to “jump.”
- A function has a removable discontinuity if it can be redefined at its discontinuous point to make it continuous. See Example 12.14.
- Some functions, such as polynomial functions, are continuous everywhere. Other functions, such as logarithmic functions, are continuous on their domain. See Example 12.15 and Example 12.16.
- For a piecewise function to be continuous each piece must be continuous on its part of the domain and the function as a whole must be continuous at the boundaries. See Example 12.17 and Example 12.18.

### 12.4 Derivatives

- The slope of the secant line connecting two points is the average rate of change of the function between those points. See Example 12.19.
- The derivative, or instantaneous rate of change, is a measure of the slope of the curve of a function at a given point, or the slope of the line tangent to the curve at that point. See Example 12.20, Example 12.21, and Example 12.22.
- The difference quotient is the quotient in the formula for the instantaneous rate of change: \( \frac{f(a + h) - f(a)}{h} \).
- Instantaneous rates of change can be used to find solutions to many real-world problems. See Example 12.23.
- The instantaneous rate of change can be found by observing the slope of a function at a point on a graph by drawing a line tangent to the function at that point. See Example 12.24.
- Instantaneous rates of change can be interpreted to describe real-world situations. See Example 12.25 and Example 12.26.
- Some functions are not differentiable at a point or points. See Example 12.27.
- The point-slope form of a line can be used to find the equation of a line tangent to the curve of a function. See Example 12.28.
- Velocity is a change in position relative to time. Instantaneous velocity describes the velocity of an object at a given instant. Average velocity describes the velocity maintained over an interval of time.
- Using the derivative makes it possible to calculate instantaneous velocity even though there is no elapsed time. See Example 12.29.

### CHAPTER 12 REVIEW EXERCISES

m10358 (http://legacy.cnx.org/content/m10358/latest/)

For the following exercises, use Figure 12.51.
230. \( \lim_{x \to -1^+} f(x) \)

231. \( \lim_{x \to -1^-} f(x) \)

232. \( \lim_{x \to -1} f(x) \)

233. \( \lim_{x \to 3} f(x) \)

234. At what values of \( x \) is the function discontinuous? What condition of continuity is violated?

235. Using Table 12.6, estimate \( \lim_{x \to 0} f(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>-0.1</th>
<th>-0.01</th>
<th>-0.001</th>
<th>0</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
<th>0.15</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F(x) )</td>
<td>2.875</td>
<td>2.92</td>
<td>2.998</td>
<td>Undefined</td>
<td>2.9987</td>
<td>2.865</td>
<td>2.78145</td>
<td>2.678</td>
</tr>
</tbody>
</table>

Table 12.6

For the following exercises, with the use of a graphing utility, use numerical or graphical evidence to determine the left- and right-hand limits of the function given as \( x \) approaches \( a \). If the function has limit as \( x \) approaches \( a \), state it. If not, discuss why there is no limit.

236. \( f(x) = \begin{cases} |x| - 1, & \text{if } x \neq 1 \\ x^3, & \text{if } x = 1 \end{cases} \quad a = 1 \)

237. \( f(x) = \begin{cases} \frac{1}{x + 1}, & \text{if } x = -2 \\ (x + 1)^2, & \text{if } x \neq -2 \end{cases} \quad a = -2 \)
238. \( f(x) = \begin{cases} \sqrt{x} + 3, & \text{if } x < 1 \\ -\sqrt{x}, & \text{if } x > 1 \end{cases} a = 1 \)

m10359 (http://legacy.cnx.org/content/m10359/latest/)
For the following exercises, find the limits if \( \lim_{x \to c} f(x) = -3 \) and \( \lim_{x \to c} g(x) = 5 \).

239. \( \lim_{x \to c} (f(x) + g(x)) \)

240. \( \lim_{x \to c} \frac{f(x)}{g(x)} \)

241. \( \lim_{x \to c} (f(x) \cdot g(x)) \)

242. \( \lim_{x \to 0^+} f(x), \quad f(x) = \begin{cases} 3x^2 + 2x + 1, & x > 0 \\ 5x + 3, & x < 0 \end{cases} \)

243. \( \lim_{x \to 0^-} f(x), \quad f(x) = \begin{cases} 3x^2 + 2x + 1, & x > 0 \\ 5x + 3, & x < 0 \end{cases} \)

244. \( \lim_{x \to 3^+} (3x - \lfloor x \rfloor) \)

For the following exercises, use algebraic techniques.

245. \( \lim_{h \to 0} \left( \frac{(h + 6)^2 - 36}{h} \right) \)

246. \( \lim_{x \to 25} \left( \frac{x^2 - 625}{\sqrt{x} - 5} \right) \)

247. \( \lim_{x \to 1} \left( \frac{-x^2 - 9x}{x} \right) \)

248. \( \lim_{x \to 4} \frac{7 - \sqrt{12x + 1}}{x - 4} \)

249. \( \lim_{x \to -3} \left( \frac{\frac{1}{3} + \frac{1}{x}}{3 + x} \right) \)

m10356 (http://legacy.cnx.org/content/m10356/latest/)
For the following exercises, use numerical evidence to determine whether the limit exists at \( x = a \). If not, describe the behavior of the graph of the function at \( x = a \).

250. \( f(x) = \frac{-2}{x - 4}; \quad a = 4 \)

251. \( f(x) = \frac{-2}{(x - 4)^2}; \quad a = 4 \)

252. \( f(x) = \frac{-x}{x^2 - x - 6}; \quad a = 3 \)
253. \[ f(x) = \frac{6x^2 + 23x + 20}{4x^2 - 25}; \quad a = \frac{-5}{2} \]

254. \[ f(x) = \frac{\sqrt{x} - 3}{9 - x}; \quad a = 9 \]

For the following exercises, determine where the given function \( f(x) \) is continuous. Where it is not continuous, state which conditions fail, and classify any discontinuities.

255. \[ f(x) = x^2 - 2x - 15 \]

256. \[ f(x) = \frac{x^2 - 2x - 15}{x - 5} \]

257. \[ f(x) = \frac{x^2 - 2x}{x^2 - 4x + 4} \]

258. \[ f(x) = \frac{x^3 - 125}{2x^2 - 12x + 10} \]

259. \[ f(x) = \frac{x^2 - 1}{2 - x} \]

260. \[ f(x) = \frac{x + 2}{x^2 - 3x - 10} \]

261. \[ f(x) = \frac{x + 2}{x^3 + 8} \]

m10360 (http://legacy.cnx.org/content/m10360/latest/)

For the following exercises, find the average rate of change \( \frac{f(x + h) - f(x)}{h} \).

262. \[ f(x) = 3x + 2 \]

263. \[ f(x) = 5 \]

264. \[ f(x) = \frac{1}{x + 1} \]

265. \[ f(x) = \ln(x) \]

266. \[ f(x) = e^{2x} \]

For the following exercises, find the derivative of the function.

267. \[ f(x) = 4x - 6 \]

268. \[ f(x) = 5x^2 - 3x \]
269. Find the equation of the tangent line to the graph of \( f(x) \) at the indicated \( x \) value.

\[ f(x) = -x^3 + 4x; \ x = 2. \]

For the following exercises, with the aid of a graphing utility, explain why the function is not differentiable everywhere on its domain. Specify the points where the function is not differentiable.

270. \( f(x) = \frac{x}{|x|} \)

271. Given that the volume of a right circular cone is \( V = \frac{1}{3}\pi r^2 h \) and that a given cone has a fixed height of 9 cm and variable radius length, find the instantaneous rate of change of volume with respect to radius length when the radius is 2 cm. Give an exact answer in terms of \( \pi \)

CHAPTER 12 PRACTICE TEST

For the following exercises, use the graph of \( f \) in Figure 12.52.

![Figure 12.52](Image)

272. \( f(1) \)

273. \( \lim_{x \to -1^+} f(x) \)

274. \( \lim_{x \to -1^-} f(x) \)

275. \( \lim_{x \to -1} f(x) \)

276. \( \lim_{x \to -2} f(x) \)

277. At what values of \( x \) is \( f \) discontinuous? What property of continuity is violated?

For the following exercises, with the use of a graphing utility, use numerical or graphical evidence to determine the left- and right-hand limits of the function given as \( x \) approaches \( a \). If the function has a limit as \( x \) approaches \( a \), state it. If not, discuss why there is no limit.

278. \( f(x) = \begin{cases} \frac{1}{x} - 3, & \text{if } x \leq 2 \\ x^3 + 1, & \text{if } x > 2 \end{cases} \ a = 2 \)
For the following exercises, evaluate each limit using algebraic techniques.

279. \[ \lim_{x \to -5} \left( \frac{\frac{1}{5} + \frac{1}{x}}{10 + 2x} \right) \]

280. \[ \lim_{h \to 0} \left( \frac{\sqrt{h^2 + 25} - 5}{h^2} \right) \]

281. \[ \lim_{h \to 0} \left( \frac{1}{h} - \frac{1}{h^2 + h} \right) \]

For the following exercises, determine whether or not the given function \( f \) is continuous. If it is continuous, show why. If it is not continuous, state which conditions fail.

282. \( f(x) = \sqrt{x^2 - 4} \)

283. \( f(x) = \frac{x^3 - 4x^2 - 9x + 36}{x^3 - 3x^2 + 2x - 6} \)

For the following exercises, use the definition of a derivative to find the derivative of the given function at \( x = a \).

284. \( f(x) = \frac{3}{5 + 2x} \)

285. \( f(x) = \frac{3}{\sqrt{x}} \)

286. \( f(x) = 2x^2 + 9x \)

287. For the graph in Figure 12.53, determine where the function is continuous/discontinuous and differentiable/not differentiable.

![Figure 12.53](image-url)
For the following exercises, with the aid of a graphing utility, explain why the function is not differentiable everywhere on its domain. Specify the points where the function is not differentiable.

288. \( f(x) = |x - 2| - |x + 2| \)

289. \( f(x) = \frac{2}{1 + e^{\frac{x}{2}}} \)

For the following exercises, explain the notation in words when the height of a projectile in feet, \( s \), is a function of time \( t \) in seconds after launch and is given by the function \( s(t) \).

290. \( s(0) \)

291. \( s(2) \)

292. \( s'(2) \)

293. \( \frac{s(2) - s(1)}{2 - 1} \)

294. \( s(t) = 0 \)

For the following exercises, use technology to evaluate the limit.

295. \( \lim_{x \to 0} \frac{\sin(x)}{3x} \)

296. \( \lim_{x \to 0} \frac{\tan^2(x)}{2x} \)

297. \( \lim_{x \to 0} \frac{\sin(x)(1 - \cos(x))}{2x^2} \)

298. Evaluate the limit by hand.

\[ \lim_{x \to 1} f(x), \text{ where } f(x) = \begin{cases} 4x - 7 & x \neq 1 \\ x^2 - 4 & x = 1 \end{cases} \]

At what value(s) of \( x \) is the function below discontinuous?

\[ f(x) = \begin{cases} 4x - 7 & x \neq 1 \\ x^2 - 4 & x = 1 \end{cases} \]

For the following exercises, consider the function whose graph appears in Figure 12.54.

![Figure 12.54](http://legacy.cnx.org/content/col11667/1.2)
299. Find the average rate of change of the function from \( x = 1 \) to \( x = 3 \).

300. Find all values of \( x \) at which \( f'(x) = 0 \).

301. Find all values of \( x \) at which \( f'(x) \) does not exist.

302. Find an equation of the tangent line to the graph of \( f \) at the indicated point: \( f(x) = 3x^2 - 2x - 6 \), \( x = -2 \)

For the following exercises, use the function \( f(x) = x(1 - x)^\frac{2}{5} \).

303. Graph the function \( f(x) = x(1 - x)^\frac{2}{5} \) by entering \( f(x) = x\left((1 - x)^{\frac{1}{5}}\right) \) and then by entering \( f(x) = x\left((1 - x)^{\frac{1}{5}}\right)^2 \).

304. Explore the behavior of the graph of \( f(x) \) around \( x = 1 \) by graphing the function on the following domains, [0.9, 1.1], [0.99, 1.01], [0.999, 1.001], and [0.9999, 1.0001]. Use this information to determine whether the function appears to be differentiable at \( x = 1 \).

For the following exercises, find the derivative of each of the functions using the definition: \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \)

305. \( f(x) = 2x - 8 \)

306. \( f(x) = 4x^2 - 7 \)

307. \( f(x) = x - \frac{1}{2}x^2 \)

308. \( f(x) = \frac{1}{x + 2} \)

309. \( f(x) = \frac{3}{x - 1} \)

310. \( f(x) = -x^3 + 1 \)

311. \( f(x) = x^2 + x^3 \)

312. \( f(x) = \sqrt{x - 1} \)
ANSWER KEY

Chapter 2

253. Yes
255. Increasing.
257. \( y = -3x + 26 \)
259. 3
261. \( y = 2x - 2 \)
263. Not linear.
265. parallel
267. \((-9, 0); (0, -7)\)
269. Line 1: \( m = -2 \); Line 2: \( m = -2 \); Parallel
271. \( y = -0.2x + 21 \)

273.
275. 250.
277. 118,000.
279. \( y = -300x + 11,500 \)
281. a) 800; b) 100 students per year; c) \( P(t) = 100t + 1700 \)
283. 18,500
285. $91,625
287. Extrapolation.

289.
291. Midway through 2024.
293. \( y = -1.294x + 49.412; \quad r = -0.974 \)
295. Early in 2022
297. 7,660
298. Yes.
300. Increasing
302. \( y = -1.5x - 6 \)
304. \( y = -2x - 1 \)
306. No.
308. Perpendicular
310. \((-7, 0); (0, -2)\)
312. \( y = -0.25x + 12 \)
314.
316. 150
318. 165,000
320. \( y = 875x + 10,675 \)
322. a) 375; b) dropped an average of 46.875, or about 47 people per year; c) \( y = -46.875t + 1250 \)

324.
326. Early in 2018
328. \( y = 0.00455x + 1979.5 \)
330. \( r = 0.999 \)

**Chapter 3**

669. \( 2 - 2i \)
671. \( 24 + 3i \)
673. \( \{2 + i, 2 - i\} \)
675. \( f(x) = (x - 2)^2 - 9 \) vertex (2, -9), intercepts (5, 0); (-1, 0); (0, -5)

677. \( f(x) = \frac{3}{25}(x + 2)^2 + 3 \)

679. 300 meters by 150 meters, the longer side parallel to river.

681. Yes, degree = 5, leading coefficient = 4

683. Yes, degree = 4, leading coefficient = 1

685. As \( x \to -\infty \), \( f(x) \to -\infty \), as \( x \to \infty \), \( f(x) \to \infty \)

687. -3 with multiplicity 2, \(-\frac{1}{2}\) with multiplicity 1, -1 with multiplicity 3

689. 4 with multiplicity 1

691. \(-\frac{1}{2}\) with multiplicity 1, 3 with multiplicity 3

693. \( x^2 + 4 \) with remainder 12

695. \( x^2 - 5x + 20 - \frac{61}{x + 3} \)

697. \( 2x^2 - 2x - 3 \), so factored form is \( (x + 4)(2x^2 - 2x - 3) \)

699. \(-2, \quad 4, \quad -\frac{1}{2}\)

701. \(1, \quad 3, \quad 4, \quad \frac{1}{2}\)

703. 0 or 2 positive, 1 negative

705. Intercepts (-2, 0) and \((0, -\frac{2}{3})\), Asymptotes \( x = 5 \) and \( y = 1 \).
707. Intercepts (3, 0), (-3, 0), and \(0, \frac{27}{2}\). Asymptotes \(x = 1, \quad x = -2, \quad y = 3\).

709. \(y = x - 2\)

711. \(f^{-1}(x) = \sqrt{x} + 2\)

713. \(f^{-1}(x) = \sqrt{x + 11} - 3\)

715. \(f^{-1}(x) = \frac{(x + 3)^2 - 5}{4}, \quad x \geq -3\)

717. \(y = 64\)

719. \(y = 72\)

721. 148.5 pounds

723. \(20 - 10i\)

725. \(\{2 + 3i, \quad 2 - 3i\}\)

727. As \(x \to -\infty, \quad f(x) \to -\infty, \quad as \ x \to \infty, \quad f(x) \to \infty\)

729. \(f(x) = (x + 1)^2 - 9, \quad \text{vertex } (-1, -9), \quad \text{intercepts } (2, 0); (-4, 0); (0, -8)\)

731. 60,000 square feet

733. 0 with multiplicity 4, 3 with multiplicity 2

735. \(2x^2 - 4x + 11 - \frac{26}{x + 2}\)

737. \(2x^2 - x - 4. \quad \text{So factored form is } (x + 3)(2x^2 - x - 4)\)

739. \(-\frac{1}{2} \quad (\text{has multiplicity } 2), \quad -\frac{1 \pm \sqrt{15}}{2}\)

741. \(-2 \quad (\text{has multiplicity } 3), \quad \pm i\)

743. \(f(x) = 2(2x - 1)^3(x + 3)\)
745. Intercepts \((-4, 0), (0, -\frac{4}{3})\). Asymptotes \(x = 3, x = -1, y = 0\).

\[y = x + 4\]

\[f^{-1}(x) = \frac{\sqrt[3]{x + 4}}{3}\]

751. \(y = 18\)

753. 4 seconds

**Chapter 6**

168. Amplitude: 3; period: \(2\pi\); midline: \(y = 3\); no asymptotes

170. Amplitude: 3; period: \(2\pi\); midline: \(y = 0\); no asymptotes
172. amplitude: 3; period: $2\pi$; midline: $y = -4$; no asymptotes

174. amplitude: 6; period: $\frac{2\pi}{3}$; midline: $y = -1$; no asymptotes

176. stretching factor: none; period: $\pi$; midline: $y = -4$; asymptotes: $x = \frac{\pi}{2} + \pi k$, where $k$ is an integer
178. stretching factor: 3; period: \( \frac{\pi}{4} \); midline: \( y = -2 \); asymptotes: \( x = \frac{\pi}{8} + \frac{\pi}{4}k \), where \( k \) is an integer

180. amplitude: none; period: \( 2\pi \); no phase shift; asymptotes: \( x = \frac{\pi}{2}k \), where \( k \) is an odd integer
182. amplitude: none; period: \(\frac{2\pi}{3}\); no phase shift; asymptotes: \(x = \frac{\pi}{3}k\), where \(k\) is an integer

184. amplitude: none; period: \(4\pi\); no phase shift; asymptotes: \(x = 2\pi k\), where \(k\) is an integer

186. largest: 20,000; smallest: 4,000
188. amplitude: 8,000; period: 10; phase shift: 0
190. In 2007, the predicted population is 4,413. In 2010, the population will be 11,924.
192. 5 in.
194. 10 seconds
196. \(\frac{\pi}{6}\)
198. \(\frac{\pi}{4}\)
200. \(\frac{\pi}{3}\)
202. No solution
204. \(\frac{12}{5}\)
206. The graphs are not symmetrical with respect to the line $y = x$. They are symmetrical with respect to the $y$-axis.

![Graphs](image)

208. The graphs appear to be identical.

![Graphs](image)

209. Amplitude: 0.5; period: $2\pi$; midline $y = 0$
211. amplitude: 5; period: $2\pi$; midline: $y = 0$

213. amplitude: 1; period: $2\pi$; midline: $y = 1$

215. amplitude: 3; period: $6\pi$; midline: $y = 0$
217. amplitude: none; period: \( \pi \); midline: \( y = 0 \); asymptotes: \( x = \frac{2\pi}{3} + \pi k \), where \( k \) is an integer

![Graph of a function](image1)

219. amplitude: none; period: \( \frac{2\pi}{3} \); midline: \( y = 0 \); asymptotes: \( x = \frac{2\pi}{3} k \), where \( k \) is an integer

![Graph of a function](image2)

221. amplitude: none; period: \( 2\pi \); midline: \( y = -3 \)

![Graph of a function](image3)

223. amplitude: 2; period: 2; midline: \( y = 0 \); \( f(x) = 2\sin(\pi(x - 1)) \)
225. amplitude: 1; period: 12; phase shift: −6; midline $y = −3$

227. $D(t) = 68 − 12\sin\left(\frac{\pi}{12}x\right)$

229. period: $\frac{\pi}{6}$; horizontal shift: −7

231. $f(x) = \sec(\pi x)$; period: 2; phase shift: 0

233. 4

235. The views are different because the period of the wave is $\frac{1}{25}$. Over a bigger domain, there will be more cycles of the graph.

237. $\frac{3}{5}$

239. On the approximate intervals $(0.5, 1), (1.6, 2.1), (2.6, 3.1), (3.7, 4.2), (4.7, 5.2), (5.6, 6.28)$

241. $f(x) = 2\cos\left(12\left(x + \frac{\pi}{4}\right)\right) + 3$

243. This graph is periodic with a period of $2\pi$.

245. $\frac{\pi}{3}$

247. $\frac{\pi}{2}$
249. $\sqrt{1 - (1 - 2x)^2}$
251. $\frac{1}{\sqrt{1 + x^4}}$
253. $\frac{x}{x + 1}$
255. False
257. approximately 0.07 radians

Chapter 8

574. Not possible
576. $C = 120^\circ$, $a = 23.1$, $c = 34.1$
578. distance of the plane from point $A$: 2.2 km, elevation of the plane: 1.6 km
580. $B = 71.0^\circ$, $C = 55.0^\circ$, $a = 12.8$
582. 40.6 km
584.

586. (0, 2)
588. (9.8489, 203.96°)
590. $r = 8$
592. $x^2 + y^2 = 7x$
594. \( y = -x \)

596. symmetric with respect to the line \( \theta = \frac{\pi}{2} \)

598.

600.

602. 5
604. $\text{cis}\left(-\frac{\pi}{3}\right)$
606. $2.3 + 1.9i$
608. $60\text{cis}\left(\frac{\pi}{2}\right)$
610. $3\text{cis}\left(\frac{4\pi}{3}\right)$
612. $25\text{cis}\left(\frac{3\pi}{2}\right)$
614. $5\text{cis}\left(\frac{3\pi}{4}\right), 5\text{cis}\left(\frac{7\pi}{4}\right)$

618. $x^2 + \frac{1}{2}y = 1$
620. \[
x(t) = -2 + 6t \\
y(t) = 3 + 4t
\]
622. $y = -2x^5$

624. \[
x(t) = (80\cos(40^\circ))t \\
y(t) = -16t^2 + (80\sin(40^\circ))y + 4
\]
The ball is 14 feet high and 184 feet from where it was launched. 3.3 seconds

626. not equal
628. 4$i$
630. $-\frac{3\sqrt{10}}{10}i - \frac{\sqrt{10}}{10}j$
632. Magnitude: $3\sqrt{2}$, Direction: $225^\circ$
634. 16
636.

\[ u - v \]
\[ u + v \]
\[ 3v \]

638. \( \alpha = 67.1^\circ, \gamma = 44.9^\circ, a = 20.9 \)

640. 1712 miles

642. \((1, \sqrt{3})\)

644. \(y = -3\)

646.

648. \(\sqrt{106}\)
650. \(-\frac{5}{2} + i\frac{\sqrt{3}}{2}\)
652. \(4 \text{cis}(21^\circ)\)
654. \(2\sqrt{3} \text{cis}(18^\circ), 2\sqrt{3} \text{cis}(198^\circ)\)
656. \(y = 2(x - 1)^2\)
658.

\[
\begin{align*}
\text{Chapter 9} \\
514. \text{No} \\
516. (-2, 3) \\
518. (4, -1) \\
520. \text{No solutions exist.} \\
522. (300, 60, 000) \\
523. (400, 30, 000) \\
524. (10, -10, 10) \\
526. \text{No solutions exist.} \\
528. (-1, -2, 3) \\
530. \left( x, \frac{8x}{5}, \frac{14x}{5} \right) \\
532. 11, 17, 33 \\
534. (2, -3), (3, 2) \\
536. \text{No solution} \\
538. \text{No solution}
\end{align*}
\]
540.

542.

544. \( \frac{2}{x+2} \cdot \frac{-4}{x+1} \)

546. \( \frac{7}{x+5} \cdot \frac{-15}{(x+5)^2} \)

548. \( \frac{3}{x-5} \cdot \frac{-4x+1}{x^2+5x+25} \)

550. \( \frac{x-4}{(x^2-2)} \cdot \frac{5x+3}{(x^2-2)^2} \)

552. \[
\begin{bmatrix}
-16 & 8 \\
-4 & -12
\end{bmatrix}
\]

554. undefined; dimensions do not match

556. undefined; inner dimensions do not match

558. \[
\begin{bmatrix}
113 & 28 & 10 \\
44 & 81 & -41 \\
84 & 98 & -42
\end{bmatrix}
\]

560. \[
\begin{bmatrix}
-127 & -74 & 176 \\
-2 & 11 & 40 \\
28 & 77 & 38
\end{bmatrix}
\]

562. undefined; inner dimensions do not match

564. \( x - 3z = 7 \)

566. \[
\begin{bmatrix}
-2 & 2 & 1 & 7 \\
2 & -8 & 5 & 0 \\
19 & -10 & 22 & 3
\end{bmatrix}
\]

568. \[
\begin{bmatrix}
1 & 0 & 3 & 12 \\
-1 & 4 & 0 & 0 \\
0 & 1 & -2 & -7
\end{bmatrix}
\]

570. No solutions exist.

572. No solutions exist.
574. \[
\begin{bmatrix}
\frac{7}{2} & 1 \\
\frac{1}{6} & 1 \\
\end{bmatrix}
\]

576. No inverse exists.

578. \((-20, 40)\)

580. \((-1, 0.2, 0.3)\)

582. 17% oranges, 34% bananas, 39% apples

584. 0

586. 6

588. \((6, \frac{1}{2})\)

590. \((x, 5x + 3)\)

592. \((0, 0, -\frac{1}{2})\)

594. Yes

596. No solutions exist.

598. \(\frac{1}{20}(10, 5, 4)\)

600. \((x, \frac{16x}{5} - \frac{13x}{5})\)

602. \((-2\sqrt{2}, -\sqrt{17}), (-2\sqrt{2}, \sqrt{17}), (2\sqrt{2}, -\sqrt{17}), (2\sqrt{2}, \sqrt{17})\)

604.

606. \(\frac{5}{3x+1} - \frac{2x+3}{(3x+1)^2}\)

608. \[
\begin{bmatrix}
17 & 51 \\
-8 & 11 \\
\end{bmatrix}
\]

610. \[
\begin{bmatrix}
12 & -20 \\
-15 & 30 \\
\end{bmatrix}
\]

612. \(-\frac{1}{8}\)

614. \[
\begin{bmatrix}
14 & -2 & 13 & 140 \\
-2 & 3 & -6 & -1 \\
1 & -5 & 12 & 11 \\
\end{bmatrix}
\]

616. No solutions exist.

618. \((100, 90)\)

620. \((\frac{1}{100}, 0)\)

622. 32 or more cell phones per day

Chapter 11

431. 2, 4, 7, 11

433. 13, 103, 1003, 10003
435. The sequence is arithmetic. The common difference is $d = \frac{5}{3}$.

437. 18, 10, 2, -6, -14

439. $a_1 = -20$, $a_n = a_{n-1} + 10$

441. $a_n = \frac{1}{3}n + \frac{13}{24}$

443. $r = 2$

445. 4, 16, 64, 256, 1024

447. 3, 12, 48, 192, 768

449. $a_n = -\frac{1}{3} \cdot \left(\frac{1}{3}\right)^{n-1}$

451. $\sum_{m=0}^{5} \left(\frac{1}{2}m + 5\right)$

453. $S_{11} = 110$

455. $S_9 \approx 23.95$

457. $S = \frac{135}{4}$

459. $S, 5617.61$

461. 6

463. $10^4 = 10,000$

465. $P(18, 4) = 73,440$

467. $C(15, 6) = 5005$

469. $2^{50} = 1.13 \times 10^{15}$

471. $\frac{8!}{3!2!} = 3360$

473. 490,314

475. $131,072a^{17} + 1,114,112a^{16}b + 4,456,448a^{15}b^2$

477. Table 11.17.

479. $\frac{1}{6}$

481. $\frac{5}{9}$

483. $\frac{4}{9}$

485. $1 - \frac{C(350, 8)}{C(500, 8)} \approx 94.4\%$

487. $\frac{C(150, 3)C(350, 5)}{C(500, 8)} \approx 25.6\%$

488. -14, -6, -2, 0

490. The sequence is arithmetic. The common difference is $d = 0.9$.

492. $a_1 = -2$, $a_n = a_{n-1} - \frac{3}{2}$, $a_{22} = -\frac{67}{2}$

494. The sequence is geometric. The common ratio is $r = \frac{1}{2}$.

496. $a_1 = 1$, $a_n = -\frac{1}{2} \cdot a_{n-1}$

498. $\sum_{k=-3}^{15} \left(3k^2 - \frac{5}{6}k\right)$

500. $S_7 = -2604.2$

502. Total in account: $140,355.75$; Interest earned: $14,355.75$

504. $5 \times 3 \times 2 \times 3 \times 2 = 180$
506. \( C(15, 3) = 455 \)

508. \( \frac{10!}{2!13!12!} = 151,200 \)

510. \( \frac{429}{14 \cdot 16} \)

512. \( \frac{4}{7} \)

514. \( \frac{5}{7} \)

516. \( \frac{C(14, 3)C(26, 4)}{C(40, 7)} = 29.2\% \)

Chapter 12

230. 2

232. does not exist

Discontinuous at \( x = -1 \) (\( \lim_{x \to a} f(x) \) does not exist), \( x = 3 \) (jump discontinuity), and \( x = 7 \) (\( \lim_{x \to a} f(x) \) does not exist).

234. 3

237. \( \lim_{x \to -2} f(x) = 1 \)

239. 2

241. -15

243. 3

245. 12

247. -10

249. \( -\frac{1}{9} \)

251. At \( x = 4 \), the function has a vertical asymptote.

253. removable discontinuity at \( a = -\frac{5}{2} \)

255. continuous on \((-\infty, \infty)\)

257. removable discontinuity at \( x = 2 \). \( f(2) \) is not defined, but limits exist.

259. discontinuity at \( x = 0 \) and \( x = 2 \). Both \( f(0) \) and \( f(2) \) are not defined.

261. removable discontinuity at \( x = -2 \). \( f(-2) \) is not defined.

263. 0

265. \( \frac{\ln(x + h) - \ln(x)}{h} \)

267. = 4

269. \( y = -8x + 16 \)

271. \( 12\pi \)

272. 3

274. 0

276. -1

278. \( \lim_{x \to 2^-} f(x) = -\frac{5}{2}a \) and \( \lim_{x \to 2^+} f(x) = 9 \) Thus, the limit of the function as \( x \) approaches 2 does not exist.

279. \( -\frac{1}{50} \)

281. 1

283. removable discontinuity at \( x = 3 \)

285. \( f'(x) = -\frac{3}{2a^2} \)

287. discontinuous at \(-2,0\), not differentiable at \(-2,0, 2\).

289. not differentiable at \( x = 0 \) (no limit)

291. the height of the projectile at \( t = 2 \) seconds

293. the average velocity from \( t = 1 \) to \( t = 2 \)
295. \( \frac{1}{3} \)
297. 0
299. 2
300. \( x = 1 \)
302. \( y = -14x - 18 \)
304. The graph is not differentiable at \( x = 1 \) (cusp).
306. \( f'(x) = 8x \)
308. \( f'(x) = -\frac{1}{(2 + x)^2} \)
310. \( f'(x) = -3x^2 \)
312. \( f'(x) = -\frac{1}{2\sqrt{x} - 1} \)
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