The Volume of a Ball in 4-Space

Let $B(a)$ be the (closed) ball centered at the origin of radius $a$ in $\mathbb{R}^4$. So,

$$B(a) = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 \leq a^2 \}.$$ 

Let $S(a)$ be the sphere centered at the origin of radius $a$ in $\mathbb{R}^4$. So,

$$S(a) = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = a^2 \}.$$ 

Let $V(a)$ be the volume of $B(a)$ and $A(a)$ be the (surface) area of $S(a)$. Now,

$$V(a) = \iiint_{B(a)} dV = \int_a^{-a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} \int_{-\sqrt{a^2-x^2-y^2-z^2}}^{\sqrt{a^2-x^2-y^2-z^2}} dw \, dz \, dy \, dx.$$ 

Integrating once and using the symmetry we see the iterated integral is equal to

$$16 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \sqrt{a^2-x^2-y^2-z^2} \, dz \, dy \, dx.$$ 

We can use MATLAB to compute this iterated integral.

1. Enter the following commands:

   ```matlab
   syms x y z
   syms a positive .................................................................. Makes a > 0.
   16*int(int(int(sqrt(a^2-x^2-y^2-z^2),z,0,sqrt(a^2-x^2-y^2)), ... 
     y,0,sqrt(a^2-x^2-y^2)),0,a)
   ```

2. Given that

   $$V(a) = \int_0^a A(r) \, dr$$

   find $A(a)$. Hint: Use the Fundamental Theorem of Calculus.

3. Modify the commands above to find the volume of the ball of radius $a$ in $\mathbb{R}^5$. Then find the (surface) area of the sphere of radius $a$ in $\mathbb{R}^5$.

4. Using the formulas that you already know in $\mathbb{R}^2$ and in $\mathbb{R}^3$, try to come up with a general formula for the volume of the ball of radius $a$ and the (surface) area of the sphere of radius $a$ in $\mathbb{R}^n$. (Hint: You will need different formulas for $n$ even and $n$ odd.)