Math 442/542  
Winter 2011

Homework 3 Solution

1. For the Revised Simplex we first need to fix some representation of the problem (with equality constraints!) which we will use to get the data \((A, b)\) and \(c\) in each step.

\[
\begin{align*}
\text{max } & -2x_1 - x_2 - 3x_3 \\
\text{s.t. } & 5x_1 + 2x_2 + 7x_3 = 420 \\
& 3x_1 + 2x_2 + 5x_3 - x_4 = 280 \\
& x_i \geq 0 \quad \forall i = 1, \ldots, 4
\end{align*}
\]

Normally, we also have to do phase I, but the problem statements says to start with phase II. So we begin with the feasible basis found at the end of phase I: \(B = \{4, 3\}, N = \{1, 2\}\).

**Step 1:** Compute \(B^{-1}, B^{-1}b, c^T_B B^{-1}b\). \(B\) consists of the 4th and the 3rd column of the constraint matrix in the above representation, and so

\[
B = \begin{pmatrix} 0 & 7 \\ -1 & 5 \end{pmatrix}
\]

Computing \(B^{-1}\), we get

\[
B^{-1} = \begin{pmatrix} \frac{5}{7} & -1 \\ 1 & 0 \end{pmatrix}
\]

Now

\[
x_B = \begin{pmatrix} x_4 \\ x_3 \end{pmatrix} = B^{-1}b = \begin{pmatrix} \frac{5}{7} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 420 \\ 280 \end{pmatrix} = \begin{pmatrix} 20 \\ 60 \end{pmatrix}
\]

So the current solution is \(x = (0, 0, 60, 20)\) with objective function value

\[
z = c^T_B B^{-1}b = c^T_B x_B = (c_4, c_3) \begin{pmatrix} 20 \\ 60 \end{pmatrix} = (0, -3) \begin{pmatrix} 20 \\ 60 \end{pmatrix} = -180
\]

**Step 2:** Test optimality, or find entering basic variable. For this we need to compute the row 0 coefficients:

\[
c^T_B B^{-1}N - c^T_N = (c_4, c_3)B^{-1}[A_1, A_2] - (c_1, c_2)
\]

\[
= (0, -3) \begin{pmatrix} \frac{5}{7} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 3 & 2 \end{pmatrix} - (-2, -1)
\]

\[
= \left( -\frac{3}{7}, 0 \right) \begin{pmatrix} 5 & 2 \\ 3 & 2 \end{pmatrix} - (-2, -1)
\]

\[
= \left( -\frac{15}{7}, -\frac{6}{7} \right) - (-2, -1)
\]

\[
= \left( -\frac{1}{7}, \frac{1}{7} \right).
\]

Note that we want to increase \(x_1\) here.
Step 3: To be able to perform the min-ratio test we need to compute the entering column of the tableau:

\[
B^{-1}A_1 = \begin{pmatrix}
\frac{5}{4} & -1 \\
\frac{1}{7} & 0
\end{pmatrix}
\begin{pmatrix}
5 \\
3
\end{pmatrix}
= \begin{pmatrix}
\frac{4}{5} \\
\frac{7}{4}
\end{pmatrix}
\]

Step 4: We do the min-ratio test and determine that \(x_4\) leaves the basis.

Step 5: We get that \(B = \{1, 3\}\) and \(N = \{2, 4\}\) and we can move on to the next iteration.

Step 1: Compute \(B^{-1}, B^{-1}b, c_B^T B^{-1}b\). \(B\) now consists of the 1st and the 3rd columns of the constraint matrix in the above representation, and so

\[
B = \begin{pmatrix}
5 & 7 \\
3 & 5
\end{pmatrix}
\]

We can either compute \(B^{-1}\) from scratch or just perform the usual pivot operation on the previous \(B^{-1}\) to get the new \(B^{-1}\) (in this case, multiply the first row with \(\frac{7}{4}\) and subtract \(\frac{5}{4}\) times the first row from the second row). In any case we get that

\[
B^{-1} = \begin{pmatrix}
\frac{5}{4} & \frac{-7}{4} \\
\frac{-3}{4} & \frac{5}{4}
\end{pmatrix}
\]

Now:

\[
x_B = \begin{pmatrix}
x_1 \\
x_3
\end{pmatrix} = B^{-1}b = \begin{pmatrix}
\frac{5}{4} & \frac{-7}{4} \\
\frac{-3}{4} & \frac{5}{4}
\end{pmatrix}
\begin{pmatrix}
420 \\
280
\end{pmatrix}
= \begin{pmatrix}
35 \\
35
\end{pmatrix}
\]

We get that \(x = (35, 0, 35, 0)\) is our current basic feasible solution. The current objective function value is

\[
z = c_B^T B^{-1}b = \begin{pmatrix} c_1, c_3 \end{pmatrix} \begin{pmatrix} 35 \\
35 \end{pmatrix} = (-2, -3) \begin{pmatrix} 35 \\
35 \end{pmatrix} = -175
\]

Step 2: Test optimality, or find entering basic variable. For this we need to compute the row 0 coefficients:

\[
c_B^T B^{-1}N - c_N^T B^{-1}a = (c_1, c_3) B^{-1} [A_2, A_4] - (c_2, c_4)
\]

\[
= (-2, -3) \begin{pmatrix}
\frac{5}{4} & \frac{-7}{4} \\
\frac{-3}{4} & \frac{5}{4}
\end{pmatrix}
\begin{pmatrix}
2 \\
0
\end{pmatrix} - (-1, 0)
\]

\[
= (-1, \frac{1}{4}) - (-1, 0)
\]

\[
= (0, \frac{1}{4}).
\]

Here we see that our current solution is in fact an optimal one. But since there is a zero non-basic objective function coefficient we can do one more iteration to get an alternative optimal basic feasible solution. To do this, we let \(x_2\) enter the basis.

Step 3: To be able to perform the min-ratio test we need to compute the entering column of the tableau:

\[
B^{-1}A_2 = \begin{pmatrix}
\frac{5}{4} & \frac{-7}{4} \\
\frac{-3}{4} & \frac{5}{4}
\end{pmatrix}
\begin{pmatrix}
2 \\
0
\end{pmatrix}
= \begin{pmatrix}
4 \\
6
\end{pmatrix}
\]

Step 4: We do the min-ratio test and determine that \(x_3\) leaves the basis.

Step 5: We get that \(B = \{1, 2\}\) and \(N = \{3, 4\}\) and we can move on to the next iteration.
**Step 1:** Compute $B^{-1}, B^{-1}b, c_B^T B^{-1}b$. $B$ now consists of the 1st and the 2nd columns of the constraint matrix in the above representation, and so

$$B = \begin{pmatrix} 5 & 2 \\ 3 & 2 \end{pmatrix}$$

We compute

$$B^{-1} = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

Now:

$$x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = B^{-1}b = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 420 \\ 280 \end{pmatrix} = \begin{pmatrix} 70 \\ 35 \end{pmatrix}$$

We get that $x = (70, 35, 0, 0)$ is our current basic feasible solution. The current objective function value is

$$z = c_B^T B^{-1}b = c_B^T x_B = \begin{pmatrix} c_1 & c_2 \end{pmatrix} \begin{pmatrix} 70 \\ 35 \end{pmatrix} = (-2, -1) \begin{pmatrix} 70 \\ 35 \end{pmatrix} = -175$$

**Step 2:** Test optimality, or find entering basic variable. For this we need to compute the row 0 coefficients:

$$c_B^T B^{-1}N - c_N^T = (c_1, c_2) B^{-1} [A_3, A_4] - (c_3, c_4)$$

$$= (-2, -1) \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 5 & -1 \end{pmatrix} - (-3, 0)$$

$$= (-\frac{1}{4}, -\frac{1}{4}) \begin{pmatrix} 7 & 0 \\ 5 & -1 \end{pmatrix} - (-3, 0)$$

$$= (-3, 1) - (-3, 0)$$

$$= (0, 1)$$.

Hence, we have another optimal solution. (Note that the last Step 2 may be skipped since we know that row 0 will not change.)

2. From what is left in the final tableau, we have that the basic variables are $\{x_5, x_3, x_1\}$ in that order. We can also get $B^{-1}$ from the final tableau.

$$B^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & 4 \\ 1 & 0 & -1 \end{pmatrix}$$

We can now get the constraint coefficients of the final tableau by calculating

$$B^{-1}A = \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & 4 \\ 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ -4 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 1 & 1 & 2 \\ 0 & 4 & 1 & -2 & 0 & 4 \\ 1 & 0 & 1 & 0 & -1 \end{pmatrix}$$

The right hand side is

$$B^{-1}b = \begin{pmatrix} 1 & 1 & 2 \\ -2 & 0 & 4 \\ 1 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 1 \end{pmatrix}$$

In the original problem (and using the optimal basis),

$$c_B = (c_5, c_3, c_1) = (0, 2, 6)$$
So the final objective function coefficient for $x_2$ will be

$$c_B \cdot B^{-1} \cdot A_2 - c_2 = (0, 2, 6) \cdot \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} - 1 = 7.$$ 

Finally, the optimal objective function value is

$$c_b \cdot B^{-1} \cdot b = (0, 2, 6) \cdot \begin{pmatrix} 7 \\ 0 \\ 1 \end{pmatrix} = 6$$

Summarizing, the optimal and final tableau is as follows:

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
<td>0</td>
<td>7</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>$x_5$</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>4</td>
<td>1</td>
<td>-2</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

3. (a) Any change in the r.h.s will change the values of the basic variables and hence change the objective function value, so both $b_1$ and $b_2$ are sensitive parameters. If $c_1$ changes by a small amount, $\bar{c}_1 = c_B^T B^{-1} A_1 - c_1$ will still be positive so $x_1$ will still be nonbasic. So a change in $c_1$ will not affect the optimal solution or its value. On the other hand any change in $c_2$ or $c_3$ will have an effect on the objective function value.

(b) We saw earlier that a change in $c_1$ does not affect the current basis, so we only need to make sure that the change does not make $\bar{c}_1$ negative. So we need

$$\bar{c}_1 = c_B^T B^{-1} A_1 - c_1$$

$$\bar{c}_1 = (2, 1) \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - c_1$$

$$\bar{c}_1 = (2, 1) \begin{pmatrix} 3 \\ 5 \end{pmatrix} - c_1 = 11 - c_1 \geq 0$$

So as long as $c_1 \leq 11$, the current tableau is optimal.

The analysis for the other $c_j$’s is a bit more complicated. A change in $c_2$ or $c_3$ affects all coefficients of nonbasic variables, so we have to make sure that they all are nonnegative:

$$\bar{c}_N = c_B^T B^{-1} N - c_N$$

$$\bar{c}_N = (c_3, c_2) \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} - (3, 0, 0)$$

$$\bar{c}_N = (3c_3 + 5c_2 - 3, c_3 + c_2, c_3 + 2c_2) \geq 0$$

So we need to make sure the following inequalities are fulfilled:

$$3c_3 + 5c_2 \geq 3$$
$$c_3 + c_2 \geq 0$$
$$c_3 + 2c_2 \geq 0$$

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Holding $c_2$ at 1, we get that $c_3 \geq \max\{(3 - 5)/3, -1, -2\} = -2/3$. Similarly, keeping $c_3$ at 2 we get that $c_2 \geq \max\{(3 - 6)/5, -2, -1\} = -3/5$.

(c) Finally, changes in the right hand side only affect the final right hand side and the objective function value. We require that $b \geq 0$ (for feasibility). So we need

$$B^{-1}b \geq 0$$

which is the same as saying

$$b_1 + b_2 \geq 0$$
$$b_1 + 2b_2 \geq 0$$

Keeping $b_1$ at its current value 20, we see that $b_2$ must be greater than or equal to $\max\{-20, -10\} = -10$ and similarly, with $b_2$ fixed at 10, $b_1$ must be greater than or equal to $\max\{-10, -20\} = -10$. 
