

An Atlas of Engineering Dynamic Systems Models and Transfer Functions

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This document presents the models and/or transfer functions of some real-world control systems.

Models are the mathematical descriptions of real-world systems, simplified by various assumptions, ignoring some nonlinear and higher effects. Models are collections of ordinary differential equations and algebraic equations. These equations must be linearized, if necessary, to work in classical controls.

Transfer functions are the mathematical vehicle of classical controls. Transfer functions are defined as the Laplace transform of the output variable divided by the Laplace transform of the input variable, with zero initial conditions. Transfer functions represent the system dynamics, as described by the simplified model – they yield the simulated system output given various inputs. Transfer functions can be derived for the open-loop, closed-loop, and/or smaller system components. Block diagrams are used for graphical representation, where the blocks have transfer functions representing the dynamics of certain system components, while the arrows represent system variables.

The models and transfer functions summarized in this document only give the bottom-line results, without derivations or much explanation. The reader is referred to the various references for more details.

Common System Variables

We start with a table of like variables playing the same role in different engineering systems.

System	Rate $r(t)$	Quantity $\int r(t)dt$	Effort $e(t)$	Impulse $\int e(t)dt$
translational mechanical	velocity $v(t)$	displacement $x(t)$	force $f(t)$	impulse
rotational mechanical	angular velocity $\omega(t)$	angular displacement $\theta(t)$	torque $\tau(t)$	angular impulse
electrical	current $i(t)$	charge $q(t)$	voltage $v(t)$	flux $\phi(t)$
incompressible fluid	volume flow rate $q(t)$	volume $V(t)$	pressure $p(t)$	none
compressible fluid	mass flow rate $q_m(t)$	mass $m(t)$	pressure $p(t)$	none
thermal	heat flow rate $q(t)$	heat energy $Q(t)$	temperature $T(t)$	none

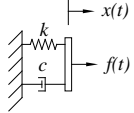
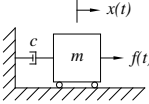
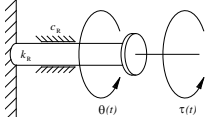
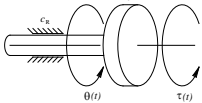
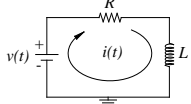
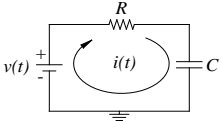
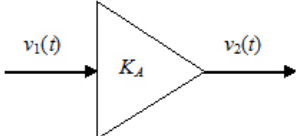
Zeroth-Order System Examples

Name	Model	$G(s)$
gear ratio	$n = \frac{\omega_{IN}(t)}{\omega_{OUT}(t)} = \frac{\theta_{IN}(t)}{\theta_{OUT}(t)} = \frac{\tau_{OUT}(t)}{\tau_{IN}(t)}$	$\frac{\Omega_{OUT}(s)}{\Omega_{IN}(s)} = \frac{\Theta_{OUT}(s)}{\Theta_{IN}(s)} = \frac{1}{n} \quad \frac{T_{OUT}(s)}{T_{IN}(s)} = n$
rack and pinion	$l(t) = r\theta(t) \quad \tau(t) = rf(t)$	$\frac{L(s)}{\Theta(s)} = \frac{V(s)}{\Omega(s)} = r \quad \frac{F(s)}{T(s)} = \frac{1}{r}$
Hooke's Law	$f(t) = kx(t) \quad \tau(t) = k_R\theta(t)$	$\frac{F(s)}{X(s)} = k \quad \frac{T(s)}{\Theta(s)} = k_R$
series / parallel springs	$x(t) = \left[\frac{1}{k_1} + \frac{1}{k_2} \right] f(t) \quad f(t) = (k_1 + k_2)x(t)$	$\frac{F(s)}{X(s)} = \frac{k_1 k_2}{k_1 + k_2} \quad \frac{F(s)}{X(s)} = k_1 + k_2$
viscous damping	$f(t) = cv(t) \quad \tau(t) = c_R\omega(t)$	$\frac{F(s)}{V(s)} = c \quad \frac{T(s)}{\Omega(s)} = c_R$
Newton's Second Law Euler's Rotational Law	$f(t) = ma(t) \quad \tau(t) = J\alpha(t)$	$\frac{A(s)}{F(s)} = \frac{1}{m} \quad \frac{A(s)}{T(s)} = \frac{1}{J}$
accelerometer, low-frequency (Dorf & Bishop)	$(k/m)x(t) = \omega^2 x_{IN}(t)$	$\frac{X(s)}{X_{IN}(s)} = \frac{\omega^2}{k/m}$
motor torque	$\tau(t) = K_T i(t)$	$\frac{T(s)}{I(s)} = K_T$
back emf	$v_B(t) = K_B \omega_M(t)$	$\frac{V_B(s)}{\Omega_M(s)} = K_B$

Zeroth-Order System Examples (continued)

Name	Model	$G(s)$
capacitor ($q(t) \sim$ charge)	$q(t) = Cv(t)$	$\frac{Q(s)}{V(s)} = C \quad \frac{V(s)}{Q(s)} = \frac{1}{C}$
resistor	$v(t) = Ri(t)$	$\frac{V(s)}{I(s)} = R \quad \frac{I(s)}{V(s)} = \frac{1}{R}$
inductor ($\phi(t) \sim$ flux)	$\phi(t) = Li(t)$	$\frac{\Phi(s)}{I(s)} = L \quad \frac{I(s)}{\Phi(s)} = \frac{1}{L}$
potentiometer	$v_1(t)R_2 = v_2(t)(R_1 + R_2)$	$\frac{V_2(s)}{V_1(s)} = \frac{R_2}{R_1 + R_2}$
tachometer	$v(t) = K_t \omega(t)$	$\frac{V(s)}{\Omega(s)} = K_t$
DC amplifier, zero time constant	$v_2(t) = K_A v_1(t)$	$\frac{V_2(s)}{V_1(s)} = K_A$
series / parallel resistors	$v(t) = (R_1 + R_2)i(t) \quad i(t) = \left[\frac{1}{R_1} + \frac{1}{R_2} \right] v(t)$	$\frac{V(s)}{I(s)} = R_1 + R_2 \quad \frac{V(s)}{I(s)} = \frac{R_1 R_2}{R_1 + R_2}$

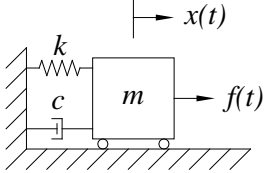
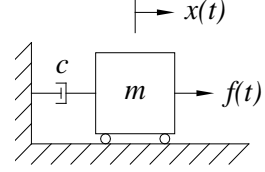
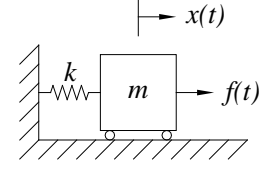
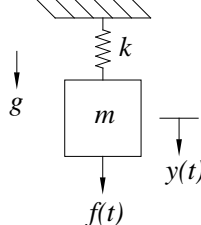
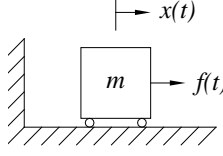
First-Order System Examples

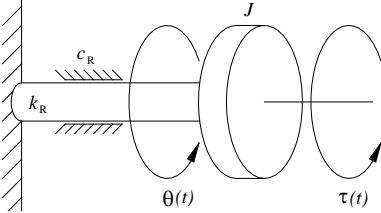
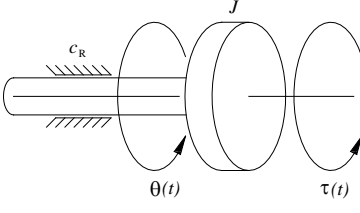
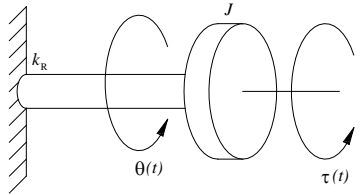
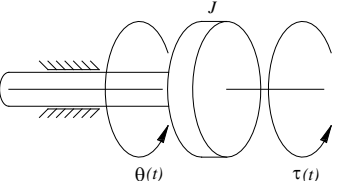
Name	Diagram	Model	$G(s)$	τ
massless translational mechanical system		$c\dot{x}(t) + kx(t) = f(t)$	$\frac{X(s)}{F(s)} = \frac{1}{cs + k}$	$\frac{c}{k}$
springless translational mechanical system		$m\dot{v}(t) + cv(t) = f(t)$	$\frac{V(s)}{F(s)} = \frac{1}{ms + c}$	$\frac{m}{c}$
inertialess rotational mechanical system		$c_R\dot{\theta}(t) + k_R\theta(t) = \tau(t)$	$\frac{\Theta(s)}{T(s)} = \frac{1}{c_Rs + k_R}$	$\frac{c_R}{k_R}$
springless rotational mechanical system		$J\dot{\omega}(t) + c_R\omega(t) = \tau(t)$	$\frac{\Omega(s)}{T(s)} = \frac{1}{Js + c_R}$	$\frac{J}{c_R}$
LR series electrical circuit		$L\frac{di(t)}{dt} + Ri(t) = v(t)$	$\frac{I(s)}{V(s)} = \frac{1}{Ls + R}$	$\frac{L}{R}$
RC series electrical circuit		$R\dot{q}(t) + \frac{1}{C}q(t) = v(t)$ $Ri(t) + \frac{1}{C}\int i(t)dt = v(t)$	$\frac{Q(s)}{V(s)} = \frac{C}{RCs + 1}$ $\frac{I(s)}{V(s)} = \frac{Cs}{RCs + 1}$	RC RC
DC amplifier		$v_2(t) = K_A v_1(t)$	$\frac{V_2(s)}{V_1(s)} = \frac{K_A}{\tau s + 1}$	τ

First-Order System Examples (continued)

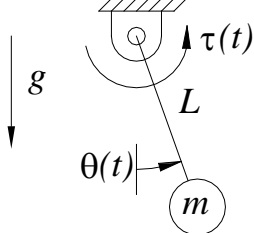
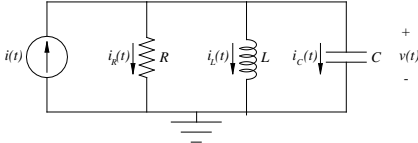
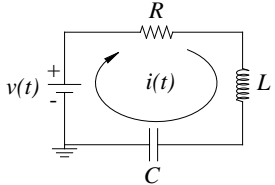
Name	Model	$G(s)$
differentiator	$v(t) = \frac{dx(t)}{dt} \quad a(t) = \frac{dv(t)}{dt}$ $i(t) = \frac{dq(t)}{dt} \quad v(t) = \frac{d\phi(t)}{dt}$	$\frac{V(s)}{X(s)} = s \quad \frac{A(s)}{V(s)} = s$ $\frac{I(s)}{Q(s)} = s \quad \frac{V(s)}{\Phi(s)} = s$
integrator	$x(t) = \int v(t)dt \quad v(t) = \int a(t)dt$ $q(t) = \int i(t)dt \quad \phi(t) = \int v(t)dt$	$\frac{X(s)}{V(s)} = \frac{1}{s} \quad \frac{V(s)}{A(s)} = \frac{1}{s}$ $\frac{Q(s)}{I(s)} = \frac{1}{s} \quad \frac{\Phi(s)}{V(s)} = \frac{1}{s}$
capacitor	$i(t) = C \frac{dv(t)}{dt} \quad v(t) = \frac{1}{C} \int i(t)dt$	$\frac{I(s)}{V(s)} = Cs \quad \frac{V(s)}{I(s)} = \frac{1}{Cs}$
resistor	$q(t) = \frac{1}{R} \int v(t)dt \quad v(t) = R \frac{dq(t)}{dt}$	$\frac{Q(s)}{V(s)} = \frac{1}{Rs} \quad \frac{V(s)}{Q(s)} = Rs$
inductor	$i(t) = \frac{1}{L} \int v(t)dt \quad v(t) = L \frac{di(t)}{dt}$	$\frac{I(s)}{V(s)} = \frac{1}{Ls} \quad \frac{V(s)}{I(s)} = Ls$

Second-Order System Examples

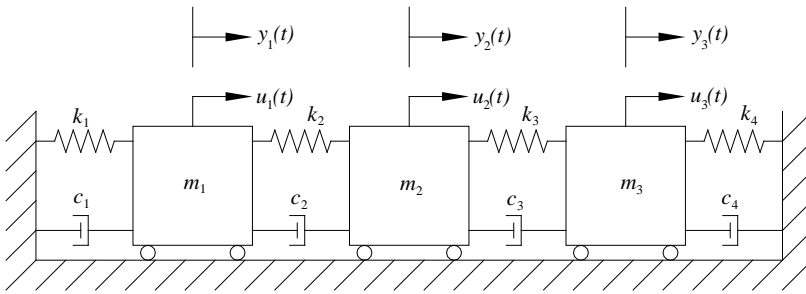
Name	Diagram	Model	$G(s)$
translational mechanical system		$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)$ $m \frac{dv(t)}{dt} + cv(t) + k \int v(t) dt = f(t)$	$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$ $\frac{V(s)}{F(s)} = \frac{s}{ms^2 + cs + k}$
springless translational mechanical system		$m\ddot{x}(t) + c\dot{x}(t) = f(t)$	$\frac{X(s)}{F(s)} = \frac{1}{s(ms + c)}$
damperless translational mechanical system		$m\ddot{x}(t) + kx(t) = f(t)$	$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + k}$
damperless translational mechanical system, vertical		$m\ddot{y}(t) + ky(t) = f(t)$	$\frac{Y(s)}{F(s)} = \frac{1}{ms^2 + k}$
mass-only translational mechanical system		$m\ddot{x}(t) = f(t)$	$\frac{X(s)}{F(s)} = \frac{1}{ms^2}$

Name	Diagram	Model	$G(s)$
rotational mechanical system	 <p>The diagram shows a cylindrical rotor with moment of inertia J mounted on a shaft. The shaft is fixed to a wall on the left. A damper with coefficient c_R and a spring with constant k_R are connected between the shaft and the wall. The angular displacement is $\theta(t)$ and the applied torque is $\tau(t)$.</p>	$J\ddot{\theta}(t) + c_R\dot{\theta}(t) + k_R\theta(t) = \tau(t)$	$\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + c_Rs + k_R}$
springless rotational mechanical system	 <p>The diagram shows a cylindrical rotor with moment of inertia J mounted on a shaft. The shaft is fixed to a wall on the left. A damper with coefficient c_R is connected between the shaft and the wall. The angular displacement is $\theta(t)$ and the applied torque is $\tau(t)$.</p>	$J\ddot{\theta}(t) + c_R\dot{\theta}(t) = \tau(t)$	$\frac{\Theta(s)}{T(s)} = \frac{1}{s(Js + c_R)}$
damperless rotational mechanical system	 <p>The diagram shows a cylindrical rotor with moment of inertia J mounted on a shaft. The shaft is fixed to a wall on the left. A spring with constant k_R is connected between the shaft and the wall. The angular displacement is $\theta(t)$ and the applied torque is $\tau(t)$.</p>	$J\ddot{\theta}(t) + k_R\theta(t) = \tau(t)$	$\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + k_R}$
inertia-only rotational mechanical system	 <p>The diagram shows a cylindrical rotor with moment of inertia J mounted on a shaft. The shaft is fixed to a wall on the left. The angular displacement is $\theta(t)$ and the applied torque is $\tau(t)$.</p>	$J\ddot{\theta}(t) = \tau(t)$	$\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2}$

Second-Order System Examples (continued)

Name	Diagram	Model	$G(s)$
torqued pendulum (linearized)		$\ddot{\theta}(t) + \frac{g}{L}\theta(t) = \frac{1}{mL^2}\tau(t)$	$\frac{\Theta(s)}{T(s)} = \frac{1}{mL^2 \left[s^2 + \frac{g}{L} \right]}$
parallel RLC circuit		$C \frac{dv(t)}{dt} + \frac{v(t)}{R} + \frac{1}{L} \int v(t) dt = i(t)$ $C \ddot{\phi}(t) + \frac{1}{R} \dot{\phi}(t) + \frac{1}{L} \phi(t) = i(t)$	$\frac{V(s)}{I(s)} = \frac{RLs}{CRLs^2 + Ls + R}$ $\frac{\Phi(s)}{I(s)} = \frac{RL}{CRLs^2 + Ls + R}$
series RLC circuit		$L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int i(t) dt = v(t)$ $L \ddot{q}(t) + R\dot{q}(t) + \frac{1}{C} q(t) = v(t)$	$\frac{I(s)}{V(s)} = \frac{Cs}{LCs^2 + RCs + 1}$ $\frac{Q(s)}{V(s)} = \frac{C}{LCs^2 + RCs + 1}$
accelerometer (Dorf & Bishop)		$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = -m\ddot{x}_{IN}(t)$	$\frac{X(s)}{X_{IN}(s)} = \frac{-s^2}{s^2 + (c/m)s + (k/m)}$
double differentiator		$a(t) = \frac{d^2 x(t)}{dt^2}$	$\frac{A(s)}{X(s)} = s^2$
double integrator		$x(t) = \iint a(t) dt$	$\frac{X(s)}{A(s)} = \frac{1}{s^2}$

Three-dof translational mechanical system¹



$$\begin{aligned} m_1 \ddot{y}_1(t) + (c_1 + c_2) \dot{y}_1(t) + (k_1 + k_2) y_1(t) - c_2 \dot{y}_2(t) - k_2 y_2(t) &= u_1(t) \\ m_2 \ddot{y}_2(t) + (c_2 + c_3) \dot{y}_2(t) + (k_2 + k_3) y_2(t) - c_2 \dot{y}_1(t) - k_2 y_1(t) - c_3 \dot{y}_3(t) - k_3 y_3(t) &= u_2(t) \\ m_3 \ddot{y}_3(t) + (c_3 + c_4) \dot{y}_3(t) + (k_3 + k_4) y_3(t) - c_3 \dot{y}_2(t) - k_3 y_2(t) &= u_3(t) \end{aligned}$$

Inverted pendulum¹

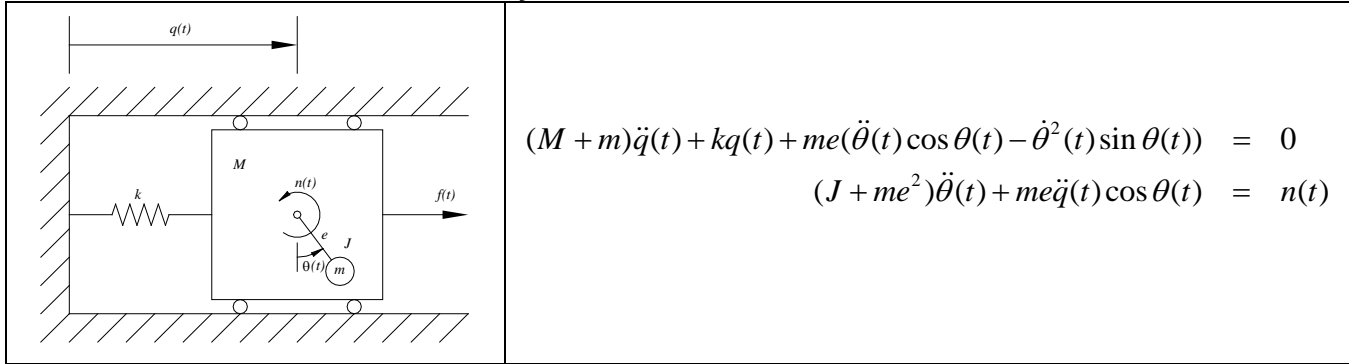
	$\begin{aligned} (m_1 + m_2) \ddot{w}(t) - m_2 L \cos \theta(t) \ddot{\theta}(t) + m_2 L \sin \theta(t) \dot{\theta}(t)^2 &= f(t) \\ m_2 L^2 \ddot{\theta}(t) - m_2 L \cos \theta(t) \ddot{w}(t) - m_2 g L \sin \theta(t) &= 0 \\ (m_1 + m_2) \ddot{w}(t) - m_2 L \ddot{\theta}(t) &= f(t) \\ -m_2 \ddot{w}(t) + m_2 L \ddot{\theta}(t) - m_2 g \theta(t) &= 0 \end{aligned}$
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Ball and beam system¹

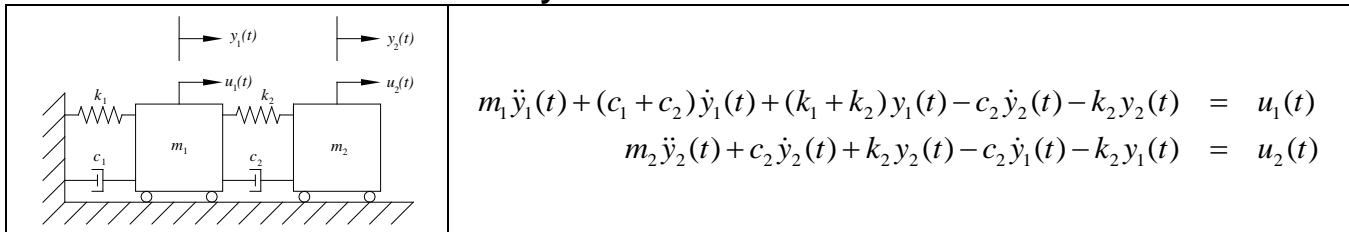
	$\begin{aligned} \left[\frac{J_b}{r^2} + m \right] \ddot{p}(t) + m g \sin \theta(t) - m p(t) \dot{\theta}(t)^2 &= 0 \\ \left[m p(t)^2 + J + J_b \right] \ddot{\theta}(t) + 2 m p(t) \dot{p}(t) \dot{\theta}(t) + m g p(t) \cos \theta(t) &= \tau(t) \end{aligned}$
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¹ R.L. Williams II and D.A. Lawrence, 2007, Linear State-Space Control Systems, John Wiley & Sons, Inc.

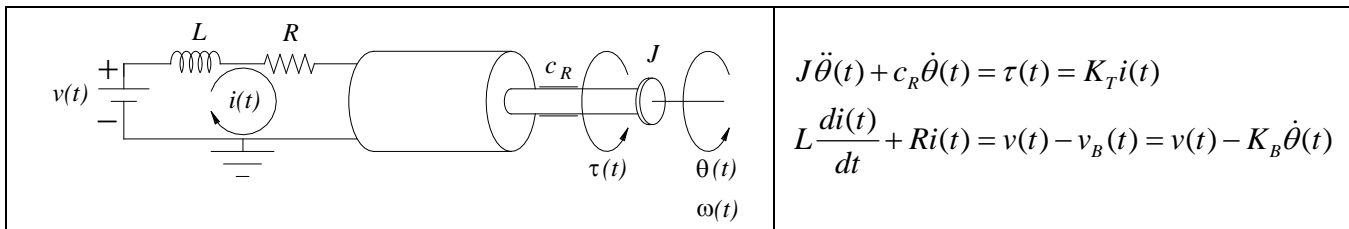
Non-linear Proof-Mass Actuator System¹



Two-dof translational mechanical system¹



DC Servomotor¹



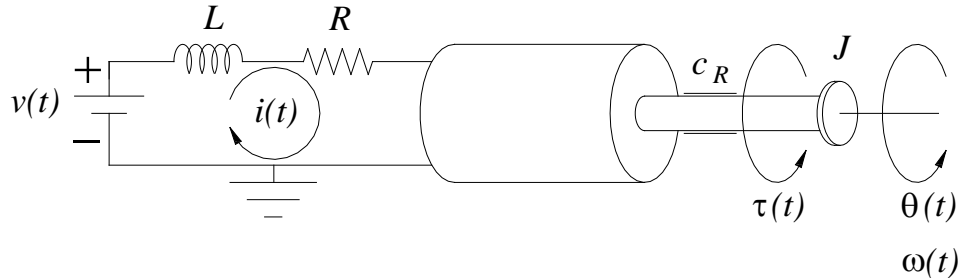
$$K_T = K_B = K$$

$$G_\omega(s) = \frac{\Omega(s)}{V(s)} = \frac{K}{(Ls + R)(Js + c_R) + K^2}$$

$$G_\theta(s) = \frac{\Theta(s)}{V(s)} = \frac{K}{s((Ls + R)(Js + c_R) + K^2)}$$

Simplified DC servomotor (ignoring back emf and inductor)

The figure below shows a simple diagram for deriving the model of a DC servomotor, which is a rotational electromechanical system. On the circuit side, $v(t)$ is the input armature voltage, L is the inductance constant, R is the resistance constant, and $i(t)$ is the armature circuit current. On the rotational mechanical side, J is the lumped rotational inertia of the motor shaft and load, c_R is the rotational viscous damping coefficient, and the output is angular displacement $\theta(t)$ (whose time derivative is angular velocity $\omega(t)$).



From an earlier derivation, the RL series circuit model is:

$$L \frac{di(t)}{dt} + Ri(t) = v(t)$$

where we have ignored the motor back emf voltage. Usually the time constant for the electrical system is much smaller than the time constant for the rotational mechanical system, which means that the electrical system current $i(t)$ rises much faster than the mechanical displacement $\theta(t)$. Therefore, we can ignore the circuit dynamics ($L \approx 0$), so the electrical circuit model simplifies to $Ri(t) = v(t)$, which is simply Ohm's Law.

In a DC servomotor, the generated motor torque is proportional to the circuit current, a linear proportional relationship that holds good for nearly the entire range of operation of the motor:

$$\tau(t) = K_T i(t)$$

K_T is the motor torque constant, which is stamped on the motor housing, available from the motor manufacturer, or determinable by experiment.

The rotational mechanical system dynamic model is derived from a free-body diagram of the rotating motor shaft, using Euler's rotational dynamics law $\sum M = J\ddot{\theta}(t)$:

$$J\ddot{\theta}(t) + c_R \dot{\theta}(t) = \tau(t)$$

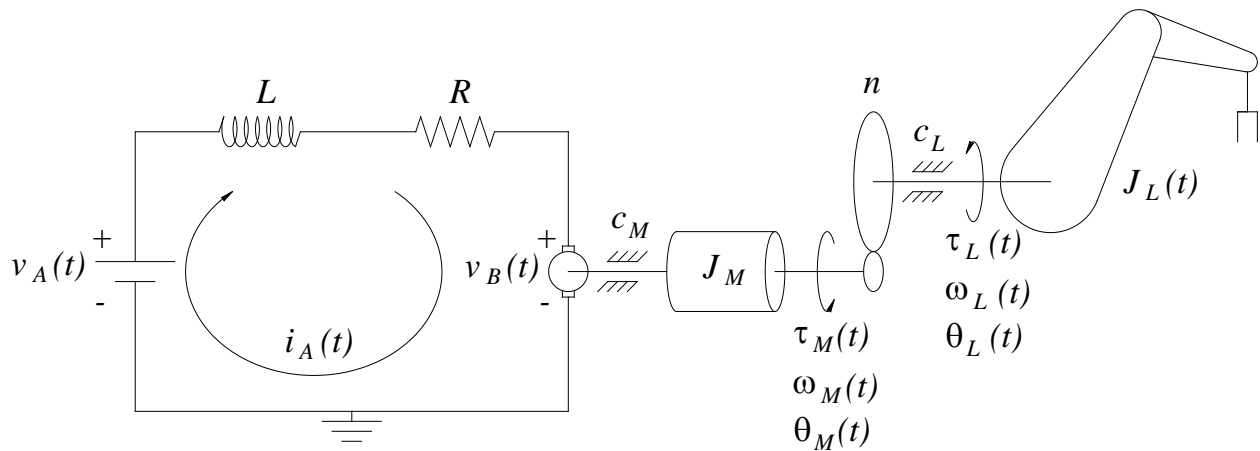
Substituting the electrical models into the rotational mechanical system dynamic model yields:

$$J\ddot{\theta}(t) + c_R \dot{\theta}(t) = \tau(t) = K_T i(t) = \frac{K_T}{R} v(t)$$

This is a linear, lumped-parameter, constant-coefficient, second-order ODE. The same model written for angular velocity $\omega(t)$ output is a first-order model:

$$J\dot{\omega}(t) + c_R \omega(t) = \frac{K_T}{R} v(t)$$

Armature circuit / DC servomotor / gear box / robot joint



$$LJ\ddot{\theta}_L(t) + (Lc + RJ)\dot{\theta}_L(t) + (Rc + K_T K_B)\theta_L(t) = \frac{K_T}{n} v_A(t)$$

$$LJ\dot{\omega}_L(t) + (Lc + RJ)\omega_L(t) + (Rc + K_T K_B)\omega_L(t) = \frac{K_T}{n} v_A(t)$$

where $J = J_M + \frac{J_L}{n^2}$ and $c = c_M + \frac{c_L}{n^2}$ are the effective polar inertia and viscous damping coefficient reflected to the motor shaft.

Numerical Parameters

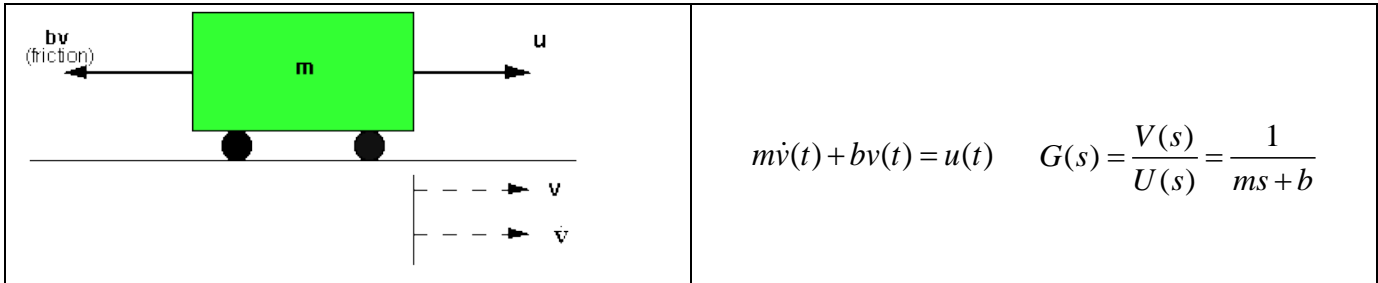
Parameter	Value	Units	Name
L	0.0006	H	armature inductance
R	1.40	Ω	armature resistance
k_B	0.00867	V/deg/s	motor back emf constant
J_M	0.00844	lb _f -in-s ²	motor shaft polar inertia
b_M	0.00013	lb _f -in/deg/s	motor shaft damping constant
k_T	4.375	lb _f -in/A	torque constant
n	200	unitless	gear ratio
J_L	1	lb _f -in-s ²	load shaft polar inertia
b_L	0.5	lb _f -in/deg/s	load shaft damping constant

$$G_\omega(s) = \frac{\Omega_L(s)}{V_A(s)} = \frac{K_T / n}{LJs^2 + (Lc + RJ)s + (Rc + K_T K_B)} = \frac{5}{s^2 + 11s + 1010}$$

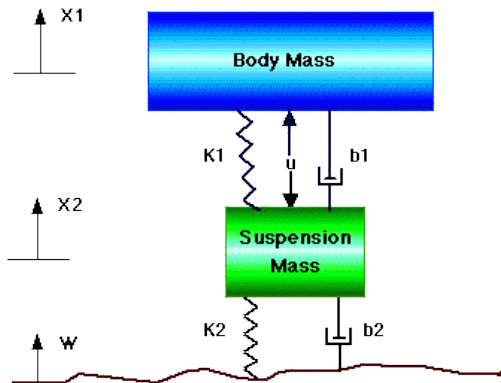
$$G_\theta(s) = \frac{\Theta_L(s)}{V_A(s)} = \frac{K_T / n}{s(LJs^2 + (Lc + RJ)s + (Rc + K_T K_B))} = \frac{5}{s(s^2 + 11s + 1010)}$$

The models of the last two pages (DC servomotor and robot joint) cannot vibrate (since there is no torsional spring) unless driven by a sinusoidal voltage input.

Automobile Cruise Control²



Automobile Suspension System²



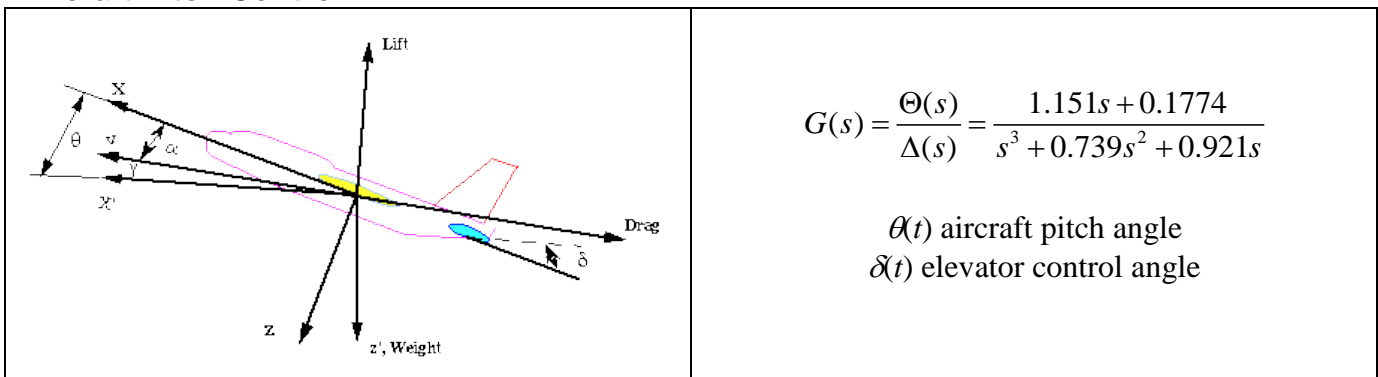
$$m_1 \ddot{x}_1(t) + b_1(\dot{x}_1(t) - \dot{x}_2(t)) + k_1(x_1(t) - x_2(t)) = u(t)$$

$$m_2 \ddot{x}_2(t) + b_2(\dot{x}_2(t) - \dot{w}(t)) + k_2(x_2(t) - w(t)) + c_2 \dot{y}_2(t) - b_1(\dot{x}_1(t) - \dot{x}_2(t)) - k_1(x_1(t) - x_2(t)) = -u(t)$$

$$G_u(s) = \frac{X_1(s) - X_2(s)}{U(s)} = \frac{(m_1 + m_2)s^2 + b_2s + K_2}{(m_1s^2 + b_1s + K_1)(m_2s^2 + (b_1 + b_2)s + (K_1 + K_2)) - (b_1s + K_1)^2}$$

$$G_w(s) = \frac{X_1(s) - X_2(s)}{W(s)} = \frac{-m_1(b_2s + K_2)s^2}{(m_1s^2 + b_1s + K_1)(m_2s^2 + (b_1 + b_2)s + (K_1 + K_2)) - (b_1s + K_1)^2}$$

Aircraft Pitch Control²



Fluid Heating System³

$\frac{T(s)}{H(s)} = \frac{1}{Cs + \left[QW + \frac{1}{R} \right]}$ $\tau = \frac{RC}{RQW + 1}$	<ul style="list-style-type: none"> • $T(s)$ temperature difference • C thermal capacitance • Q constant flow rate • W water specific heat • R insulation thermal resistance • $H(s)$ heating element heat flow • τ time constant
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Fluid Flow Tank System³

$\frac{\Delta Q_2(s)}{\Delta Q_1(s)} = \frac{1}{\tau s + 1}$ $\frac{\Delta H(s)}{\Delta Q_1(s)} = \frac{R}{RCs + 1}$	<ul style="list-style-type: none"> • Q_1 input flow rate • Q_2 output flow rate • H head • τ RC, time constant • R orifice flow resistance • C cross-sectional tank area
--	--

Generic Sensor³

$$H(s) = \frac{Y_{SENS}(s)}{Y(s)} = \frac{k}{\tau s + 1} \quad k \text{ gain} \quad \tau \text{ time constant}$$

Hydraulic Actuator³

$\frac{Y(s)}{X(s)} = \frac{K}{s(ms + B)}$	$K = \frac{Ak_x}{k_p} \quad B = b + K = \frac{A^2}{k_p}$ $k_x = \left. \frac{\partial g}{\partial x} \right _{x_0} \quad k_p = \left. \frac{\partial g}{\partial P} \right _{P_0}$	<ul style="list-style-type: none"> • $g = g(x, P)$ flow • A piston area
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Robot Elbow Control³

$$G(s) = \frac{\Theta(s)}{T(s)} = \frac{2}{s(s+4)} \quad \tau(t) \text{ torque input} \quad \theta(t) \text{ elbow angle}$$

Space Telescope Pointing Control³

$$G(s) = \frac{\Phi(s)}{T(s)} = \frac{1}{s(s+12)} \quad \tau(t) \text{ torque wheel input} \quad \phi(t) \text{ telescope angle}$$

SkyCam Control³

$$G(s) = \frac{Y(s)}{T(s)} = \frac{1}{s(0.002s + 1)} \quad \tau(t) \text{ torque input} \quad y(t) \text{ displacement}$$

³ Dorf and Bishop, Modern Control Systems, 11th edition, Pearson Prentice Hall, 2007.

VTOL Aircraft Control³

$$G(s) = \frac{Y(s)}{T(s)} = \frac{1}{s(s-1)}$$

 $\tau(t)$ thruster input $y(t)$ vertical displacement**Diabetes Control³**

$$G(s) = \frac{B(s)}{I(s)} = \frac{s+2}{s(s+1)}$$

 $i(t)$ insulin input $b(t)$ blood-sugar level**Racecar Speed Control³**

$$G(s) = \frac{V(s)}{C(s)} = \frac{100}{(s+2)(s+5)}$$

 $c(t)$ throttle input $v(t)$ racecar speed**Weld Bead Depth Control³**

$$G(s) = \frac{Y(s)}{I(s)} = \frac{K}{(0.01s+1)(1.5s+1)}$$

 $i(t)$ current input $y(t)$ weld bead depth**Ferris Wheel Control³**

$$G(s) = \frac{\Omega(s)}{T(s)} = \frac{s+6}{(s+2)(s+4)}$$

 $\tau(t)$ torque input $\omega(t)$ angular velocity**Human "Paper-Pilot" Model³**

$$G(s) = \frac{\Phi(s)}{\Theta_E(s)} = \frac{-(2s+1)(\tau s-2)}{(0.5s+1)(\tau s+2)}$$

 $\theta_E(t)$ angle error $\phi(t)$ elevator angle $0.25 \leq \tau \leq 0.50$ human pilot time constant**Ship Stabilization³**

$$G(s) = \frac{\Theta(s)}{T_f(s)} = \frac{9}{s^2+1.2s+9}$$

 $\tau_f(t)$ fin control torque $\theta(t)$ ship angle**Laser Printer Positioning³**

$$G(s) = \frac{Y(s)}{T(s)} = \frac{4(s+50)}{s^2+30s+200}$$

 $\tau(t)$ control torque $y(t)$ printer head displacement**Vehicle Steering Control³**

$$G(s) = \frac{Y(s)}{\Theta(s)} = \frac{1}{s^2(s+12)}$$

 $\theta(t)$ steering wheel angle $y(t)$ centerline displacement**Welding Robot Control³**

$$G(s) = \frac{Y(s)}{T(s)} = \frac{75(s+1)}{s(s+5)(s+20)}$$

 $\tau(t)$ torque input $y(t)$ displacement

Aircraft Roll Control³

$$G(s) = \frac{\Phi(s)}{Q(s)} = \frac{K}{s(s^2 + 4s + 9)}$$

$q(t)$ hyd. fluid flow $\phi(t)$ roll angle

Elevator Control³

$$G(s) = \frac{Y(s)}{T(s)} = \frac{1}{s(s^2 + 3s + 3)}$$

$\tau(t)$ torque input $y(t)$ displacement

Robot Position Control³

$$G(s) = \frac{Y(s)}{T(s)} = \frac{100}{s \left[\frac{s^2}{6400} + \frac{s}{50} + 1 \right]}$$

$\tau(t)$ torque input $y(t)$ displacement

Space Station Orientation Control³

$$G(s) = \frac{\Phi(s)}{T(s)} = \frac{20K}{s(s^2 + 20s + 100)}$$

$\tau(t)$ torque input $\phi(t)$ station angle

Missile Yaw Control³

$$G(s) = \frac{A(s)}{T(s)} = \frac{-0.5(s^2 - 2500)}{(s - 3)(s^2 + 50s + 1000)}$$

$\tau(t)$ torque input $\alpha(t)$ yaw acceleration

Helicopter Pitch Control³

$$G(s) = \frac{\Phi(s)}{T(s)} = \frac{25(s + 0.03)}{(s + 0.4)(s^2 - 0.36s + 0.16)}$$

$\tau(t)$ torque input $\phi(t)$ pitch angle

Hydraulic Servovalve Control³

$$G(s) = \frac{Y(s)}{I(s)} = \frac{\frac{s^2}{400\pi^2} + \frac{0.1s}{20\pi} + 1}{(0.02s + 1) \left[\frac{s^2}{196\pi^2} + \frac{0.1s}{14\pi} + 1 \right]}$$

$i(t)$ current input $y(t)$ displacement

Flexible Robot Control³

$$G(s) = \frac{\Theta(s)}{T(s)} = \frac{s + 500}{s(s + 0.0325)(s^2 + 2.57s + 6667)}$$

$\tau(t)$ torque input $\theta(t)$ angle output

Robot Force Control³

$$G(s) = \frac{F(s)}{T(s)} = \frac{K(s + 2.5)}{(s^2 + 2s + 2)(s^2 + 4s + 5)}$$

$\tau(t)$ torque input $f(t)$ force output

Robotic Rubbertuator and Load⁴

$$G(s) = \frac{X(s)}{P(s)} = \frac{10}{s^2 + 10s + 29}$$

$p(t)$ air pressure $x(t)$ displacement

Antenna Azimuth Control⁴

$$G(s) = \frac{\Theta(s)}{V(s)} = \frac{20.83}{s(s+100)(s+1.71)}$$

$v(t)$ input voltage $\theta(t)$ azimuth angle

Robotic Swivel⁴

$$G(s) = \frac{\Omega(s)}{V(s)} = \frac{K}{(s+10)(s^2 + 4s + 10)}$$

$v(t)$ input voltage $\omega(t)$ swivel velocity

Autonomous Submersible Control⁴

$$G(s) = \frac{\Theta(s)}{\Delta_e(s)} = \frac{-0.125(s+0.435)}{(s+1.23)(s^2 + 0.226s + 0.0169)}$$

$\delta_e(t)$ elevator angle $\theta(t)$ pitch angle

Heat Transfer System⁵

$$\frac{T_o(s)}{T_i(s)} = \frac{1}{RCs + 1}$$

$t_i(t)$ input temperature $t_o(t)$ output temperature
 RC thermal time constant

Pneumatic System⁵

$$\frac{P_o(s)}{P_i(s)} = \frac{1}{RCs + 1}$$

$p_i(t)$ input pressure $p_o(t)$ output pressure
 RC pneumatic time constant

Hydraulic System⁵

$$\frac{Q_o(s)}{Q_i(s)} = \frac{1}{RCs + 1}$$

$q_i(t)$ input flow rate $q_o(t)$ output flow rate
 RC hydraulic time constant

Magnetically-Levitated Ball⁶

$$\frac{Y(s)}{V(s)} = \frac{8062}{(s+7)(s+11)(s-11)}$$

$v(t)$ input voltage $y(t)$ ball height

⁴ N.S. Nise, *Control Systems Engineering*, 2nd edition, Cummings, 1995.

⁵ Golnaraghi and Kuo, *Automatic Control Systems*, 9th edition, Wiley, 2010

⁶ researchcentre.apsc.ubc.ca/MECH523

Temperature Control⁷

$$\frac{T(s)}{Q(s)} = \frac{1}{\tau s + 1}$$

$q(t)$ input heat flow rate

$T(t)$ output temperature

τ thermal time constant

Steel Rolling Thickness Control⁷

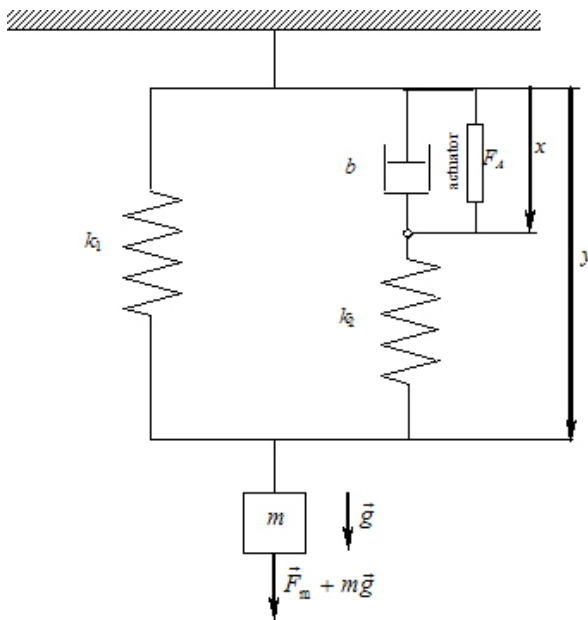
$$\frac{Y(s)}{F(s)} = \frac{K}{s}$$

$f(t)$ roller input force

$y(t)$ output steel thickness

K motor force constant

Human Skeletal Muscle Model⁸



$$\ddot{y}(t) + \frac{k_2}{b} \dot{y}(t) + \frac{(k_1 + k_2)}{m} y(t) + \frac{k_1 k_2}{mb} y(t) = \frac{\dot{F}_m(t)}{m} + \frac{k_2}{mb} (F_A(t) + F_m(t) + mg + k_1 L_{1R})$$

Where:

- $y(t)$ absolute displacement of the muscle end
- m lumped muscle mass
- k_1 linear spring stiffness representing the muscle fascia, parallel elastic component
- k_2 linear spring stiffness representing the connecting tendons, series elastic component
- b viscous damping coefficient representing the muscle energy loss
- $F_A(t)$ muscle actuation force (the integrated effects of all contracting sarcomeres)
- $F_m(t)$ external load applied to the muscle
- g acceleration due to gravity
- L_{1R} neutral length of the muscle (the unstretched length of spring k_1).

⁷ W.J. Palm, Control Systems Engineering, Wiley, 1986.

⁸ Dr. Bob's ME 467/BME567 Biomechanics NotesBook Supplement, derived by Elvedin Kljuno