LOCATION OF FAULT REGIONS IN ANALOG CIRCUITS

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Abstract

The emphasis in this paper is on locating subnetworks or regions containing all the faults of the network. Nodal equations and voltage measurements are used by the method. A necessary topological condition for fault location is formulated and used to localize the effect of faults in subnetworks. It is shown that effect of faults in one subnetwork can be represented by equivalent faults at the partition nodes if fault analysis in the other subnetwork is to be performed. Coates flow-graph representation of a network is used for topological considerations.

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I. INTRODUCTION

Testing of analog circuits with the aim of fault location is important in network analysis. There are different approaches to the problem depending on the information available from tests conducted on the network. Generally, the network topology is known and we try to identify the faulty elements and evaluate them. If the number of measurements is large enough we can evaluate all elements and single out the faulty ones [1,2]. However, when the number of measurements is limited we can use various methods to predict regions where faults may appear [3,4]. To verify whether a predicted region contains all the faults, the multiple-fault location method based on the multiport description of a network can be used [5].

In this paper, we present a method based on the nodal equations which extends the possibilities of the above-mentioned multiport method. Topological restrictions on multiple-fault location are discussed. These are effectively used to locate faulty regions. Some illustrative examples and practical remarks for effective calculations are given.

II. MULTIPLE-FAULT EVALUATION BY NODAL EQUATIONS

In this section we discuss the method of multiple fault location on the basis of the nodal equations. The principal difference between the nodal and the multiport approach is that in the multiport approach we aim to find changes in element values whereas in the nodal method we design the changes in nodal currents only. Changes in element values can be computed by the nodal method after the network topology is considered.
Nodal Equations for Faulty Network

Let us assume that the network has \( n+1 \) nodes, \( m \) of them accessible, and \( f < m \) is the number of faulty elements. The nodal equations for the nominal values of the elements have the form

\[
\mathbf{Y} \mathbf{V} = \mathbf{I},
\]

(1)

For the faulty network, assuming the same excitations, we obtain

\[
(\mathbf{Y} + \Delta \mathbf{Y})(\mathbf{V} + \Delta \mathbf{V}) = \mathbf{I}.
\]

(2)

Thus

\[
\mathbf{Y} \Delta \mathbf{V} = -\Delta \mathbf{Y} \mathbf{V}',
\]

(3)

where \( \mathbf{V}' = \mathbf{V} + \Delta \mathbf{V} \) is the vector of nodal voltages in the faulty network. We can compute \( \Delta \mathbf{V} \) assuming that \( \mathbf{Y} \) is nonsingular and obtain

\[
\Delta \mathbf{V} = -\mathbf{Y}^{-1} \Delta \mathbf{Y} \mathbf{V}'.
\]

(4)

Let us denote \( \Delta \mathbf{A} = -\Delta \mathbf{Y} \mathbf{V}' \). \( \Delta \mathbf{A} \) represents changes in nodal currents caused by faulty elements. The relation (4) becomes

\[
\Delta \mathbf{V} = \mathbf{Y}^{-1} \Delta \mathbf{A}.
\]

(5)

We can assume that a few elements are faulty, in which case \( \Delta \mathbf{A} \) has the form

\[
\Delta \mathbf{A} = \begin{bmatrix} \mathbf{0} \\ \Delta \mathbf{A}^F \\ \mathbf{0} \end{bmatrix}.
\]

(6)

Assuming that the first \( m \) nodal voltages can be measured we obtain

\[
\begin{bmatrix} \Delta \mathbf{V}^M \\ \Delta \mathbf{V}^{N-M} \end{bmatrix} = \mathbf{Y}^{-1} \begin{bmatrix} \mathbf{0} \\ \Delta \mathbf{A}^F \end{bmatrix},
\]

(7)

where \( N \) indicates the set of all nodes and \( M \) the set of measurement nodes. Hence,

\[
\Delta \mathbf{V}^M = \mathbf{Z}_{MF} \Delta \mathbf{A}^F,
\]

(8)
where
\[
Z^{-1} = \begin{bmatrix}
Z_{MN} & Z_{MF} & Z_{M2} \\
Z_{N-M,1} & Z_{N-M,F} & Z_{N-M,2}
\end{bmatrix}.
\] (9)

Relation (8) has to be satisfied when the set \( F \) of network nodes includes all nodes associated with faulty elements in the network.

According to Theorem 1 of Huang, Lin and Liu [6], if equation (8) is consistent, and \( \text{rank} \ Z_{MX} = f + 1 \) for any set of columns \( X \) of \( Z_{MN} \) such that \( \text{card} \ X = f + 1 \), then (8) has a unique solution \( \Delta J^F \) almost surely.

In practice, however, this requirement is too strong, especially if we are interested in a certain set of faults \( F \). Using the work of Huang, Lin and Liu, we can formulate the following result.

**Result 1**

If equation (8) is consistent and \( \text{rank} \ Z_{MFX} = f + 1 \), where \( F_x \) is the set of columns of the matrix \( Z_{MN} \),

\[ F_x = F \cup x, \quad \forall x \in N-F \]

then (8) has a unique solution almost surely.

**Remark 1**

The condition on rank stated in Result 1 is equivalent to the existence of a square, nonsingular \((f+1) \times (f+1)\) submatrix of \( Z_{MFX} \).

**Remark 2**

In applying Result 1 to nodal fault location we consider specific candidates for faulty nodes \( F \). It is possible to evaluate \( f \) faulty
nodes even when the network may be not f-node-fault testable as defined in [6]. Moreover, even if f faults can not be uniquely identified we can still isolate a fault region containing these faults (see Section IV).

Reduction of the Number of Equations

It is clear from Result 1 that in order to design \( \Delta J_F^L \) we must have at least \( (1 + \text{card } F) \) measurement nodes. This may cause some redundancies in the case of isolated faults. If there is an isolated fault in the network it causes changes in two elements of the \( \Delta J_F^L \) vector. In the example shown in Fig. 1 we have \( \Delta J_K^F = -\Delta Y_e J^T = -\Delta J_j^F \). In such a case vector \( \Delta J_F^L \) will contain variables which are not independent. We can transform the equation (8) to reduce the column rank of the coefficient matrix \( Z_{MF} \). The reduction realized depends on the location of different faults. Let us discuss the following two cases.

1) The case of isolated faults

If an isolated fault appears between nodes k and j (see Fig. 1) then equation (8) can be written in the form

\[
\Delta J^M = \left[ \begin{array}{c}
\Delta J_1^F \\
\vdots \\
\Delta J_k^F \\
\Lambda \\
\vdots \\
\Delta J_j^F \\
\Delta J_r^F
\end{array} \right]
\]

or, after summing columns \( \Lambda_k \) and \( \Lambda_j \) and deleting column \( \Lambda_j \),
2) The case of connected faults

If connected faults form an ungrounded subtree in the network then the number of variables in $\Delta_j^F$ can be reduced by one in similar way to Case 1. The reduction holds for every connected subgraph formed by faulty elements. If the subgraph contains a circuit then the number of variables can not be reduced.

The nodal approach is restricted to two-terminal elements and voltage controlled current sources only, but it can be extended to any linear active network using the modified nodal description [7].

\[
\Delta^M = [a_1 \cdots a_k - a_j \cdots a_{j-1} a_{j+1} \cdots a_r]^T
\]

\[
\begin{bmatrix}
\Delta_j^F \\
\vdots \\
\Delta_k^F \\
\Delta_{j-1}^F \\
\Delta_{j+1}^F \\
\vdots \\
\Delta_r^F
\end{bmatrix}
\]

III. TOPOLOGICAL RESTRICTIONS

In this section we will discuss the problem of the placement of measurements in the network to make possible the identification of a certain set of faults on the basis of network topology.

A necessary condition for nodal fault evaluation will be formulated. This will be used later to isolate faulty regions. Namely, it will be shown that faults in the incident subnetwork can be represented by equivalent faulty currents at the common nodes.

Let $Z_{EFX}$ denote a square submatrix of $Z_{MFX}$ and $Y (F, 1E)$ denote the
submatrix of \( Y \) obtained by removing \( F_x \) rows and \( E \) columns. Using the equivalence [9]
\[
\det Z_{EF}^X \neq 0 \iff \det Y (F_x;E) \neq 0
\]
(12)
we can find topological restrictions for the fault location problem. We can use the approach presented in [10]. Let us assume that the topological equations for the nodal admittance matrix and the Coates graph representation of the network are
\[
Y = \lambda_+ Y_e \lambda_+^T
\]
(13)
where the element \( ij \) of \( \lambda_+ \) is equal to 1 if the \( j \)th edge is directed towards the \( i \)th vertex, otherwise zero, and the element \( ij \) of \( \lambda_- \) is equal to 1 if the \( j \)th edge is directed away from the \( i \)th vertex, otherwise zero and \( Y_e \) is a diagonal matrix of element admittances.

The submatrix \( Y (F_x;E) \) can be presented in the form [8]
\[
Y (F_x;E) = \lambda_- F_x Y_e \lambda_+^T
\]
(14)
where \( \lambda_- F_x \) (\( \lambda_+ E \)) is obtained from \( \lambda_+ \) (\( \lambda_- \)) by removing rows \( F_x \) (\( E \)), respectively.

Let \( G \) denote a directed Coates graph [11] and let \( S \) denote a set of node pairs of \( G \), namely, \( S = \{(v_{s1}, v_{e1}), \ldots, (v_{sk}, v_{ek})\} \), where \( v_{pi} \neq v_{nm} \) for \( i \neq m \) (\( p,n = s,e \)).

**Definition [12]**

A \( k \)-connection of a graph \( G \) is a subgraph \( c_S \) of the graph, such that elements of \( c_S \) form a set of \( k \) node-disjoint directed paths and node-disjoint directed circuits incident with all graph nodes. The starting point and the endpoint of the paths are indicated by the pairs of \( S \).
Following Starzyk et al. [10] we can formulate the following theorem.

**Theorem 1**

If \( \det Y (F_x;E) \neq 0 \) then there exists at least one \( k \)-connection \( c_S \) in the graph \( G(F_x;E) \) obtained from the Coates graph of the network after deleting all the edges incoming to nodes \( F_x \) and all the edges outgoing from nodes \( E \), where

\[
S = \{(v_s, v_e); v_s \in F_x, v_e \in E\},
\]

\[ k = \text{card } S = \text{card } E = \text{card } F_x, \]

where \( (v_s, v_e) \) represents a path directed from the node \( v_s \) to the node \( v_e \), or isolated node in the case \( v_s = v_e \).

**Proof**

According to the Cauchy-Binet theorem [9] and relation (14), we have

\[
\det Y (F_x;E) = \sum \det C^- \det C^+, \tag{17}
\]

where \( C^- \) is a major submatrix of \( \lambda_{F_x} Y \) with order equal to \( (n - \text{card } F_x) \), and \( C^+ \) is the corresponding major submatrix of \( \lambda_{E}^T \). If \( \det Y (F_x;E) \neq 0 \), then there exists at least one pair of corresponding determinants, both different from zero. A major determinant of \( \lambda_{F_x} Y \) is different from zero if and only if there exists one nonzero element in every row of the chosen submatrix \( C^- \). This corresponds to the set of \( (n - \text{card } F_x) \) edges, such that every edge has a different endpoint, belonging to the set of nodes \( (N - F_x) \). The corresponding submatrix is different from zero if the same edges have different origins, belonging to the set of nodes \( (N - E) \). Now it is easy to check that these edges form a \( k \)-connection, as stated in Theorem 1 and as illustrated by Fig. 2.
Corollary

When \( f \) faults appear in a subgraph connected to the rest of the graph through \( c \) common nodes, we must have at least \( f-c+1 \) measurements inside this subgraph to identify all these faults uniquely (see Fig. 3).

IV. LOCATION OF FAULT REGIONS

In this section we will use Theorem 1 to prove that any number of faults in one subnetwork can be represented by faults at the common nodes with the other subnetwork. This kind of representation allows us to solve fault evaluation problem locally at the subnetwork level. Usually, smaller number of variables are involved while solving fault evaluation problem at the subnetwork level, therefore a solution can be obtained much easier.

Let us investigate the problem of two subnetworks, the graphs of which \( G_1 \) and \( G_2 \) have \( c \) common nodes \( C \). Assume for simplicity that all measurement nodes \( M \) are placed in \( G_1 \). If all the faulty nodes are in \( G_2 \) and \( f > c \) then in general we cannot identify them uniquely because the \( k \)-connection required by Theorem 1 does not exist. However, in this case all the faults in \( G_2 \) as seen by the measurements can be represented as faults at the set of nodes \( C \). To this end let us prove the following theorem.

Theorem 2

All the columns of \( Z_{MX} \) are a linear combination of the columns of \( Z_{MC} \), where \( X \) is any set of nodes from \( G_2 \) and \( Z_{MX} \) and \( Z_{MC} \) are submatrices of the matrix defined in (9).
Proof

Assume for simplicity that $Z_{MC}$ is of full column rank (the proof can be easily extended to the case of any rank of $Z_{MC}$). To prove the theorem it is enough to show that

$$\text{rank}(Z_{MC}^t Z_{MX}) = \text{rank} Z_{MC}^t \quad (18)$$

Suppose that $\text{rank} (Z_{MC}^t Z_{MX}) > c$. Let us denote $Z_{E,U}$ a square, maximum rank submatrix of $[Z_{MC}^t Z_{MX}]$. Then $\det Y(U;E) \neq 0$ and according to the Theorem 1 there exists a $k$-connection $c_S$ with starting points in $U$ and endpoints in $E$. But all the nodes $U$ belong to $G_2$ while $E$ belong to $G_1$, which means that the set of isolated nodes in the $k$-connection equals $E \cup U \cup C$. After removing the isolated nodes $E \cup C$ the $k$-connection will link the nodes $E \cup U \cup C$ with the nodes $U \cup E \cup C$. As a result of our assumption, $\text{card} E > c$, consequently $\text{card} (E \cup U \cup C) > (C - E \cup U \cup C)$, which would imply the impossible situation that the number of node-disjoint paths is greater than the number of nodes that they must pass through. Equation (18) has been proved by contradiction.

As a result of Theorem 2 we are able to locate the fault region $F_{w,f}$ even if $m << f$, where the fault region is defined as any set of $w$ nodes which contains $f$ faulty nodes.

Let us decompose the graph $G$ into subgraphs $G_1$, ..., $G_k$. Let $c_i$ denote the cardinality of the set $C_i$ of those nodes of $G_i$ which are incident to the other graphs, so

$$C_i \triangleq \bigcup_{j=1}^k (N_{j} \cap N_{i} \cap N_{j} \cap N_{i} \cap N_{j} \cap N_{i})$$

where $N_j$, $N_i$ represent sets of nodes of subgraphs $G_j$ and $G_i$, respectively. Let $M_i$ denote the set of measurement nodes belonging to $G_i$ and $m_i = \text{card} M_i$. We can formulate an important theorem.
Theorem 3

If subgraph $G_i$ contains $f_i$ faults with

$$m_i > c_i + f_i,$$

(20)

and if the conditions stated in Result 1 are satisfied for $M = M_i$, $F = C_i \cup F_i$, $N = N_i$ and $f = \text{card } F$, then $F_i$ consists of the unique set of faulty nodes in $G_i$ almost surely and faulty currents can be found by solving (8).

Proof

Let us assume that $F_o$ is the set of faults outside $G_i$. The effect of faulty currents at these nodes can be seen through eq. (8), namely

$$\Delta V^M = Z_{MF_o} \Delta J^o + Z_{MF_i} \Delta J^f_i$$

But according to Theorem 2 $Z_{MF_o} \Delta J^o$, as a linear combination of $Z_{MF_o}$ columns, can be replaced by a linear combination of $Z_{MC_i}$ columns, so

$$\Delta V^M = Z_{MC_i} \Delta J^i + Z_{MF_i} \Delta J^f_i$$

where $\Delta J^i$ is a vector of equivalent faulty currents at the interconnection nodes. If (8) can be solved uniquely (with the help of Result 1) and $\Delta J^f$ calculated, then $\Delta J^i$ is a unique set of faulty nodes in $G_i$ as stated.
Remark

If any subnetwork has been found fault free we can remove the corresponding set of nodes from \( C_i \) (19). If a set of faulty nodes \( F_j \) in any subnetwork has been previously established we can use the set directly rather than its effect at common nodes.

Corollary

\( C_i \) in (19) can be replaced by \( C_i' \),

\[
C_i' = \bigcup_{j=1}^{k_1} N_j \cap N_i \cup \bigcup_{j=k_1}^{k} F_j
\]

(21)

where \( k_1 \) denotes those subgraphs where faulty nodes have not been evaluated.

Note that when \( F_j \) has been uniquely evaluated, then on the basis of (20)

\[ m_j > f_j \]

and these measurements can be used together with \( m_i \) to evaluate faulty nodes \( F_i \). Let

\[
M_i' = M_i \cup \bigcup_{j=k_1}^{k} M_j
\]

(22)

Theorem 4

If the conditions stated in Result 1 are satisfied for \( M = M_i' \), \( F = C_i' \cup F_i \), \( N = N_i \) and \( f = \text{card } F \), then \( F_i \) consists of the unique set of faulty nodes in \( G_i \) almost surely and faulty currents can be found by solving (8).

Proof is very similar to the proof of Theorem 3.
Reduction of the Number of Equations

According to Theorem 2 we have

\[ Z_{MF_0} = Z_{MC_1} \]

where \( Z \) is a transformation matrix with card \( C_1 \) rows and card \( F_0 \) columns.

Therefore the rank \( Z_{MF_0} \leq \min \{ \text{card } C_1, \text{card } F_0 \} \). With this observation we can restrict the number of unknown faulty currents to the rank of \( Z_{MF_0} \), using reduction of the number of equations as discussed in Section II.

On the other hand, in case the incident subnetwork does not contain the reference node then equivalent faulty currents at the common nodes are not independent, as they must satisfy Kirchhoff current law. For each such subnetwork, we can reduce number of equivalent faulty currents by 1, replacing set of \( Z_{MC_1} \) columns by their linear combinations with the resulting matrix having smaller number of columns. Therefore, the number of equations to be solved and the number of measurements required for element evaluation in the subnetwork will be appropriately reduced.

V. NETWORK PARTITIONING INTO FAULT REGIONS

An important problem in multiple-fault location is to guess the set \( F_{w,f} \) that contains all faulty nodes but has a number of elements \( w < m \). We discuss how to choose this proper set of nodes.

The fault region can be predicted or designed initially by the approximate fault isolation method described in [13]. If we have no
initial information about the system we can try to guess the proper set \( F \) but then the probability of being correct is low because the number of different combinations is equal to \( \binom{n}{f'} \), (cf. [5]). Below we describe an algorithm which can be used to detect the fault region.

**Topological Algorithm**

**Step 1**

Decompose the Coates signal-flow graph \( G \) of the network into subgraphs \( G_{1}, \ldots, G_{k} \) separated by a small number of nodes.

**Comment**

It is better if the common nodes contain measurement points.

**Step 2**

Choose the subgraph \( G_{i} \) (or combination of subgraphs) containing \( m_{i} \) measurement nodes \( M_{i} \), such that

\[
m_{i} > c_{i}.
\]

If there is no such subgraph stop.

**Step 3**

If the chosen subgraph \( G_{i} \) contains \( f_{i} \) faults such that

\[
m_{i} > c_{i} + f_{i}
\]
evaluate them using Theorem 3.

**Comment**

Realization of this step allows us to find all the faults in a certain region and to calculate the actual values of the faulty elements. Then all the currents and voltages in subgraph \( G_{i} \) can be calculated and \( G_{i} \) may be represented as the set of external excitations with known voltages for incident subnetworks.

**Step 4**

If no subgraph chosen in Step 2 satisfies the condition stated in
Step 3 then try different combinations of two or more subgraphs and check if Step 3 may be realized for the combination considered. If not, stop.

Step 5

For the remaining subgraphs use Theorem 4. If there is no success stop.

Example 2

Assume two subgraphs $G_1$, $G_2$ separated by $c_1 = c_2 = 2$ common nodes. Let subgraph $G_1$ have $m_1 = 5$ measurements and no faults. The conditions stated in Steps 2 and 3 are obviously satisfied because $c_1 + f_1 = 2 < m_1$. So no matter how many faults are in subgraph $G_2$ we find $G_1$ as nonfaulty and can use two measurements from $G_1$ together with measurements placed in $G_2$ to locate the faults in $G_2$.

Graph Partition

A question that arises while dealing with the topological algorithm is whether a desired decomposition, which satisfies the condition stated in Step 3 of the Topological Algorithm can be found efficiently. This question is particularly important in location of fault regions in large circuits where there is a large number of possible partitions. Since an optimal partition is $np$-complete problem [14], an heuristic technique is proposed.

We employ the approach presented in [14] to develop a partition strategy iterated with the topological algorithm. In each iteration we want to identify one subgraph $G_i$ in which all faults included in $G_i$ can be evaluated. In the worst case all measurement nodes must be tried as starting points for the graph partition. If no success the combinatorial
algorithm must be used. After each iteration the remaining graph to be partitioned \((G_r)\) is obtained and another subgraph \(G_i\) is to be extracted from \(G_r\). The greedy strategy presented in [14] is realized in extracting subgraphs \(G_i\). Each such subgraph is built up starting from a measurement node to which we add adjacent nodes, check conditions for fault evaluation and move to the next cut-set. See [14] for details of the greedy strategy. Each time a new measurement points are added to \(G_i\) we update a set of measurement points in this subgraph \(M_i\). Initially \(M_i\) contains the starting point only. The proposed partition algorithm is as follows:

**Partition Algorithm:**

(* initialize variables *)

\[ i := 0; M_r := M; G_r := G; \]

repeat (* iterations with different starting points *)

select a new starting point \(s\) from \(M_r\);

\[ i := i + 1; f_1 := 0; M_1 := s; \]

repeat (* search through the remaining graph \(G_r\) *)

select next cut-set in \(G_r\) using greedy strategy and update \(G_r, G_i\) and \(M_i\);

if \(m_i > c_i + f_i\) then

begin (* \(c_i\) represents the cardinality of the cut-set, \(f_i\) is the number of expected faults in \(G_i\) *)

for \(j := f_i\) to \(m_i - c_i - 1\) do

begin

if \(j\) faults can be evaluated then

begin

print\((G_i)\);

\[ M_r := M_r - M_i; \]

\[ M_i := M_i - M_i; \]

end

end

end
go to (1);
end;
end;
f_i := m_i - c_i; (* larger number of faults is assumed *)
end;
until G_r = 0;
i := i - 1;
(1) G_r := G - U_j G_j; (* G_r is reevaluated for next iteration *)
until ( M_r ,= 0 or all M_r nodes have been tried as new starting
points without a change in M_r );

The output of the partition algorithm contains subgraphs G_i for which
conditions stated in Theorem 3 or Theorem 4 are satisfied.

VI. PRACTICAL REMARKS

It is known that the solution of the equation

\[ A \tilde{x} = b, \]  (24)

where \( A \) is an \( m \times f \) full column rank matrix \( f < m \), exists if and only if
it can be transformed to the form

\[
\begin{bmatrix}
  x & x & x \\
  0 & x & x \\
  \vdots & \vdots & \vdots \\
  0 & x \\
\end{bmatrix}
\begin{bmatrix}
  \tilde{x} \\
\end{bmatrix}
= \begin{bmatrix}
  b_1 \\
\end{bmatrix}
\]  (25)

after row manipulation, where \( b_1 \) is a column vector having \( f \) elements.
The form (24) is also convenient to obtain the solution of the set of
For ill-conditioned systems the method of Householder orthogonal transformations can be used to reduce to zero the subdiagonal elements of $A$ [15].

For practical situations when both measurement errors and effects of tolerances appear, the technique proposed by Bandler, Biernacki and Salama [13] can be used. In the first stage of computation we solve optimization problem that can be stated as

$$\text{minimize} \sum_{i=1}^{n} (|\text{Re}(\Delta J_i^F)| + |\text{Im}(\Delta J_i^F)|)$$

subject to linear equality constraints (8). Solution of this problem gives us the most likely faulty elements. Then the verification technique in the presence of tolerances can be used to check (8) in the way described in [13].

The nodal approach discussed in this paper has the following advantages as compared with the multiport methods [5].

1. Fault regions can be located even if fault elements form a circuit or cutset.
2. We do not face the situation of block dependent systems when only one element in a circuit or cutset is not faulty.

It should be noted, however, that the evaluation of faulty elements on the basis of identified changes in current excitations representing the faults is not always possible. For example, when only one element in a circuit is not faulty, then the problem of identification is not solvable, which is a simple consequence of the transformation of current excitations (cf. [16]). In such case evaluation of faulty currents should be repeated with independent current excitations.
VII. EXAMPLES

To see how the method works in practical networks let us discuss two examples where the number of measurements was less than the number of faults.

Example 3

The resistive network shown in Fig. 4 has 20 nodes and 39 resistors with nominal values \( R_i = 1 \) for \( i \neq 12 \) and \( R_{12} = 2 \Omega \). We take measurements at nodes number 9, 11, 13 and 19. These measurements are simulated for the faulty network where we increase the value of resistors \( R_k \) (\( k = 5, 6, \ldots, 12 \)) and \( R_{37} \) from the nominal by 1 \( \Omega \) and \( J = 1 \)A. They are equal to

\[
\begin{align*}
V_9 &= 0.1316 \text{V}, \quad V_{11} = 0.06177 \text{V}, \quad V_{13} = 0.03797 \text{V}, \quad V_{19} = 0.05351 \text{V}.
\end{align*}
\]

The graph of the network can be decomposed into three subgraphs as shown in Fig. 5.

All the measurements are placed in subgraph \( G_2 \). Following the topological algorithm we find that subgraph \( G_2 \) has a number of measurements \( m_2 \) larger than \( c_2 \). In this case \( m_2 > c_2 + f_2 \) only for \( f_2 = 0 \), so we can only check if subgraph \( G_2 \) represents a nonfaulty subgraph. We take \( F_x \) as in Theorem 3 having \( F_2 = 0 \). So, in this case \( F = N_1 \cap N_2 \cup N_3 \cap N_2 = \{9, 13, 18\} \). We check that the rank of \( Z_{MFx} \) is 4 for all \( x \in N_2 \cap F = \{7, 11, 12, 19, 20\} \), so on the basis of Result 1, if equation (8) is consistent, then the solution \( \Delta J^F \) represents all the faulty elements.

The equation (8) now has the form

\[
\begin{bmatrix}
\Delta V^M_9 \\
\Delta V^M_{11} \\
\Delta V^M_{13} \\
\Delta V^M_{19}
\end{bmatrix} =
\begin{bmatrix}
0.0676 \\
0.0319 \\
0.0197 \\
0.0277
\end{bmatrix}
= 
\begin{bmatrix}
0.506 & 0.144 & 0.170 \\
0.237 & 0.384 & 0.572 \\
0.144 & 0.536 & 0.381 \\
0.205 & 0.396 & 0.698
\end{bmatrix}
\begin{bmatrix}
\Delta J^F_9 \\
\Delta J^F_{13} \\
\Delta J^F_{18}
\end{bmatrix}
\]

(27)
We check consistency using the transformation to the form of (25) and obtain the transformed equation

\[
\begin{bmatrix}
0.506 & 0.144 & 0.170 \\
0 & 0.317 & 0.493 \\
0 & 3 \cdot 10^{-29} & -0.435 \\
0 & 4 \cdot 10^{-29} & 3 \cdot 10^{-30}
\end{bmatrix}
\begin{bmatrix}
\Delta J^F_9 \\
\Delta J^F_{13} \\
\Delta J^F_{18}
\end{bmatrix}
= 
\begin{bmatrix}
0.0676 \\
0.0003 \\
10^{-15} \\
8.6 \cdot 10^{-16}
\end{bmatrix}
\] (28)

The last element of the right-hand side is almost zero so we recognize (8) as consistent and solving for \(\Delta J^F_2\), we obtain

\(\Delta J^F_9 = 0.1333\text{A}, \Delta J^F_{13} = 0.000949\text{A}, \Delta J^F_{18} = -3 \cdot 10^{-15}\text{A.}\)

From the solution obtained the requirements of Theorem 3 for \(F_2 = 0\) are satisfied so we can be sure that subgraph \(G_2\) is nonfaulty, while \(G_1\) and \(G_3\) represent faulty subnetworks. If \(G_3\) contains only one faulty element, as in this example, we can locate them exactly, using the reduction technique described in Section II. We check the combination of every two nodes in \(G_3\) which are connected by a resistor. After subtracting column 16 from column 14 in matrix \(Z_{MN}\), we may write equation (8) as

\[
\begin{bmatrix}
0.506 & -0.0289 \\
0.237 & -0.0769 \\
0.144 & -0.107 \\
0.205 & -0.0792
\end{bmatrix}
\begin{bmatrix}
\Delta J^F_9 \\
\Delta J^F_{14-16}
\end{bmatrix}
= 
\begin{bmatrix}
0.0676 \\
0.0319 \\
0.0197 \\
0.0277
\end{bmatrix}
\] (29)

Next, we transform it to the form (25) and obtain

\[
\begin{bmatrix}
0.506 & -0.0289 \\
0 & -0.0634 \\
0 & -6 \cdot 10^{-30} \\
0 & -6 \cdot 10^{-30}
\end{bmatrix}
\begin{bmatrix}
\Delta J^F_9 \\
\Delta J^F_{14-16}
\end{bmatrix}
= 
\begin{bmatrix}
0.0676 \\
0.0003 \\
10^{-15} \\
6 \cdot 10^{-16}
\end{bmatrix}
\] (30)
We can solve this obtaining

$$\Delta j^F_9 = 0.1333 \text{A}, \quad \Delta j^F_{14-16} = -0.00475 \text{A}.$$  

This proves that the faulty element is between nodes 14 and 16. We can find its value by exciting the nominal network with the original excitation plus current sources $\Delta j^F_{\text{fa}}$ as excitations at faulty nodes. The solution gives us the values of voltages $V'$ at all the nodes of $G_2$ and $G_3$ (not $G_1$). Now, $\Delta V_{37}$ can be calculated as:

$$\Delta V_{37} = \frac{\Delta j^F_{14-16}}{V_{16} - V_{14}} = \frac{-0.00475}{0.0095} = -\frac{1}{2}, \quad (31)$$

which is the exact change from the nominal value.

Example 4

The active lowpass filter as shown in Fig. 6 has nominal values of elements equal (cf. [17]) to

- $R_1 = 0.182$, $C_2 = 0.01$, $R_3 = 1.57$, $R_5 = 2.64$, $R_6 = 10$, $R_7 = 10$, $R_9 = 100$, $R_{10} = 11.1$, $R_{11} = 2.64$, $C_{12} = 0.01$, $R_{14} = 5.41$, $R_{15} = 1$, $R_{17} = 1$,
- $C_{18} = 0.01$, $R_{19} = 4.84$, $R_{21} = 2.32$, $R_{22} = 10$, $R_{23} = 10$, $R_{25} = 500$,
- $R_{26} = 111.1$, $R_{27} = 1.14$, $R_{28} = 2.32$, $R_{29} = 0.01$, $R_{31} = 72.4$, $R_{32} = 10$,
- $R_{34} = 10$ (all resistors in k\(\Omega\) and capacitors in \(\mu\)F).

Operational amplifiers are modelled by the circuit shown in Fig. 7.

The input current is equal to $j(t) = 10^{-2} \cos(2000t) \text{A}$. Measurements taken at nodes 10, 12, 15 and 17 are equal to

- $V_{10} = -9.866 + j0.6264 \text{ V}$,  $V_{12} = 0.0822 + j0.932 \text{ V}$,  $V_{15} = 20.08 - j1.726 \text{ V}$,  $V_{17} = 2.437 - j0.2094 \text{ V}$.

These measurements were simulated in the faulty network with the faults
\[ \begin{align*}
R_1 &= 0.1, \quad C_2 = 0.02, \quad R_6 = 20, \quad R_7 = 20, \quad R_{11} = 2, \quad R_{14} = 4, \quad R_{15} = 2, \\
R_{17} &= 2, \quad R_{32} = 40
\end{align*} \]

and the gain of the amplifier \( A_g \) was reduced to 50. Again, we decompose the graph of the network into three subgraphs, as shown in Fig. 8. We repeat the steps from Example 4 testing the middle sub-network. We obtain equation (8) for the given set of measurements and \( F = \{10, 15, 17\} \) as

\[
\begin{bmatrix}
\Delta V_{10}^M \\
\Delta V_{17}^M \\
\Delta V_{12}^M \\
\Delta V_{15}^M
\end{bmatrix} =
\begin{bmatrix}
5.43 + j0.246 \\
1.34 - j0.0908 \\
0.0359 + j0.513 \\
11.1 - j0.749
\end{bmatrix}
\quad \text{(32)}
\]

\[
\begin{bmatrix}
1.06 \times 10^{-2} + j3.43 \times 10^{-5} \\
-2.62 \times 10^{-3} + j5.10^{-5} \\
-2.14 \times 10^{-3} - j1.0^{-3} \\
-2.16 \times 10^{-2} + j4.12 \times 10^{-4}
\end{bmatrix}
\begin{bmatrix}
6.52 \times 10^{-11} - j7.19 \times 10^{-12} \\
5.82 \times 10^{-4} + j7.05 \times 10^{-5} \\
-10^{-2} - j2.41 \times 10^{-4} \\
4.8 \times 10^{-3} + j5.81 \times 10^{-4}
\end{bmatrix}
= \begin{bmatrix}
9.64 \times 10^{-8} + j3.11 \times 10^{-10} \\
1.01 - j3.76 \times 10^{-10} \\
-1.57 \times 10^{-10} - j5.82 \times 10^{-9} \\
-1.25 \times 10^{-7} + j3.1 \times 10^{-9}
\end{bmatrix}
\]

We transform (34) to the form (27) and obtain

\[
\begin{bmatrix}
1.06 \times 10^{-2} + j3.43 \times 10^{-5} \\
-8 \times 10^{-32} + j10^{-33} \\
-3 \times 10^{-34} \\
-2 \times 10^{-30} - j10^{-32}
\end{bmatrix}
\begin{bmatrix}
6.52 \times 10^{-11} + j7.19 \times 10^{-12} \\
5.82 \times 10^{-4} + j7.05 \times 10^{-5} \\
-2 \times 10^{-31} - j2.10^{-32} \\
4 \times 10^{-31} + j6.10^{-32}
\end{bmatrix}
= \begin{bmatrix}
9.64 \times 10^{-8} + j3.11 \times 10^{-10} \\
1.01 - j7.86 \times 10^{-11} \\
17.2 - j1.66 \\
-8 \times 10^{-28} + j8 \times 10^{-29}
\end{bmatrix}
\]

Again, we can recognize equation (8) for these faults as consistent
and solving for \( \Delta J \) we obtain

\[
\Delta J_{10} = -512 + j24.9A, \quad \Delta J_{15} = -1.65 \times 10^{-14} + j7.33 \times 10^{-14}A \\
\Delta J_{17} = -3.43 \times 10^{-4} + j2.91 \times 10^{-5}A.
\]

Once again, we may try to locate the fault in the right-hand side subgraph. After subtracting column 19 from 17 and solving the resulting set of equations, we obtain

\[
\Delta J_{10} = -512 + j24.9A, \\
\Delta J_{17-19} = -3.46 \times 10^{-4} + j2.94 \times 10^{-5}A.
\]

We can see that the change in faulty currents is very small, which indicates weak sensitivity of the given measurements w.r.t. the faulty current at node 19. This result is easy to predict because the faulty current source of value \( 3.46 \times 10^{-4} - j2.94 \times 10^{-5}A \) is connected in parallel with a controlled current source having a current equal to \( J_{19} \approx 0.07 - j0.006A \). Again, we may excite the nominal network adding faulty currents and calculate voltages \( V' \). We calculate

\[
\Delta Y_{32} = \frac{\Delta J_{17-19}}{V'_{19} - V'_{17}} = \frac{-3.46 \times 10^{-4} + j2.94 \times 10^{-5}}{4.62 - j0.392} \approx -0.749 \times 10^{-4}.
\]

This is quite accurate, since the actual change was \(-0.75 \times 10^{-4}\).

VIII. CONCLUSIONS

We have extended the possibilities of multiport methods for multiple-fault location in analog networks. Necessary and sufficient conditions for uniquely evaluating faults have been discussed. Together with topological constraints inherent in a particular network these conditions indicate whether or not the measurements which have been made can be used to evaluate all the faults. Even in cases where the faults can not be evaluated, our analysis can be applied to identify and
isolate faulty and nonfaulty subnetworks. Our recommended strategy then would be to subject the subnetworks containing the faults to further analysis. Our ability to evaluate the faults within a subnetwork depends upon the actual number of faults, the number of measurements within the subnetwork and the information which can be used from outside the subnetwork as seen through the nodes common with the rest of the network. Thus, our approach permits us to use effectively all methods which have been proposed for fault evaluation of networks at the subnetwork level. This partitioning into subnetworks not only increases the efficiency of existing algorithms, especially when we have a large network to analyze, but also permits detailed investigation of specific subnetworks.

REFERENCES


Fig. 1 Changes in nodal current caused by a single fault.
Fig. 2 Example of required 6-connection.
Fig. 3 Illustration of necessary measurements.
Fig. 4 Resistive network example.
Fig. 5 Decomposition of the graph of the resistive network into subgraphs.
Fig. 6 Active lowpass filter example.
Fig. 7 Operational amplifier model used for the active filter.
Fig. 8 Decomposition of the graph of the active filter into subgraphs.