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ABSTRACT. In this note, we give a modern presentation of the material in [5], more in line with the presentation of [1].

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1. INTRODUCTION

The pcf theory as presented in [4] has proven to be a powerful tool for analyzing the combinatorial structure at singular cardinals as well as there successors. Perhaps the most well-known consequence of the pcf-theoretic machinery is the following theorem due to Shelah:

**Theorem 1.1** (Shelah).

\[ \kappa^{\aleph_0} < \max\{\kappa^{\omega_4}, (2^{\aleph_0})^+\}. \]

This contrasts greatly with the situation for regular cardinals, and tells us that we can get meaningful results about the power of singular cardinals in ZFC. On the other hand, we know that some of this machinery can only work for singular cardinals which are not fixed points of the \( \kappa \)-function. Given suitable large cardinal hypotheses, one can use Prikrý-type forcings to blow up the power of some \( \kappa \)-fixed points to be arbitrarily large (see [2] for an overview). So if the pcf machinery can be generalized to \( \kappa \)-fixed points, it can only be done so in a restricted manner.
In [5], Shelah does precisely this. The pcf machinery is relativized to particular ideals over a singular cardinal $\mu$, and provided these ideals satisfy certain hypotheses, many of the theorems of pcf theory generalize to this new setting. Our goal in this note is to present the ideas from [5] in a modern manner.

2. pcf Theory

All results in this section are due to Shelah unless otherwise stated. We begin with some definitions and notation.

**Definition 2.1.** If $I$ is an ideal over a set $A$, we will use $I^*$ to denote the dual filter of $I$.

**Definition 2.2.** Suppose $I$ is an ideal on a set $A$ with $B \subseteq A$. If $B /\notin I$, then we define the ideal:

$$I_B = \{ X \subseteq B : X \in A \}.$$

**Definition 2.3.** Suppose $\mu$ is a singular cardinal, $A$ is a collection of regular cardinals unbounded in $\mu$, and $I$ is a fixed ideal over $A$. Define

$$\text{pcf}_I(A) = \{ \text{cf}(\prod A/D) : D \text{ is an ultrafilter over } A \text{ disjoint from } I \}.$$ 

**Definition 2.4.** Suppose $\mu$ is a singular cardinal and $A$ is a collection of regular cardinals unbounded in $\mu$, and $I$ is a fixed ideal over $\mu$. The ideal $J^I_{<\lambda}[A]$ is defined to be the collection of sets $B \subseteq A$ satisfying:

1. $B \in I$ or;
2. for every ultrafilter $D$ over $A$ disjoint from $I$, if $B \in D$, then $\text{cf}(\prod A/D) < \lambda$.

We will denote these ideals by $J^I_{<\lambda}$ if the set $A$ is clear from context.

One thing to note is that $J^I_{<\lambda}[A]$ can be equivalently defined by asking that $B \in J^I_{<\lambda}[A]$ if and only if there is a representative of $[B]_I$ satisfying property (2) above. We now highlight a number of simple properties of $\text{pcf}_I(A)$ and $J^I_{<\lambda}[A].$

**Lemma 2.1.** Suppose $\mu$ is a singular cardinal, $A$ is a sequence of regular cardinals unbounded in $\mu$, and $I$ is a fixed ideal over $A$.

1. If $\lambda \in \text{pcf}_I(A)$, then $J^I_{<\lambda}$ is proper.
2. If $\lambda \notin \text{pcf}_I(A)$, then $J^I_{<\lambda} = J^I_{<\lambda^+}$.
3. If $F$ is a filter disjoint from $I$ with $\lambda = tcf(\prod A/F)$, then $\lambda \in \text{pcf}_I(A)$.
4. If $B \subseteq A$ is $I$-positive, then $\text{pcf}_{I_B}(B) \subseteq \text{pcf}_I(A)$.
(5) If $I$ and $J$ are ideals over $A$ such that $I \subseteq J$, then $\text{pcf}_J(A) \subseteq \text{pcf}_I(A)$.

(6) If $B \subseteq A$ is such that $B =_I A$, then $\text{pcf}_{t_B}(B) = \text{pcf}_I(A)$.

Proof. (1): If $\lambda \in \text{pcf}_I(A)$, then there is an ultrafilter $D$ disjoint from $I$ such that $\text{cf}(\prod A/D) = \lambda$, and so $D \cap J_{<\lambda} = \emptyset$ by definition. Thus, $J_{<\lambda}$ is proper as $|\prod A/D| > 1$.

(2): If $\lambda \notin \text{pcf}_I(A)$, then there is no ultrafilter $D$ disjoint from $I$ such that $\text{cf}(\prod A/D) = \lambda$. Hence, any $B$ that forces the cofinality of an ultraproduct below $\lambda^+$ will force the cofinality below $\lambda$.

(3): We let $D$ be any ultrafilter extending $F$, then a sequence witnessing that $tcf(\prod A/F) = \lambda$ will witness the same modulo $D$. Thus, $\lambda \in \text{pcf}_I(A)$.

(4): Let $\lambda \in \text{pcf}_{t_B}(B)$, and let $D$ be an ultrafilter over $B$ disjoint from $I_B$ witnessing this. Then we can extend $D$ to an ultrafilter $E$ over $A$ disjoint from $I$ by definition of $I_B$. Note then that $\text{cf}(\prod A/E) = \lambda$ and so $\lambda \in \text{pcf}_I(A)$.

(5): Let $\lambda \in \text{pcf}_J(A)$, then if $D$ is an ultrafilter over $A$ disjoint from $J$ witnessing this, it is also disjoint from $I$ and so $\lambda \in \text{pcf}_I(A)$.

(6): We have already shown that $\text{pcf}_{t_B}(B) \subseteq \text{pcf}_I(A)$ in (4) above, so it suffices to show the other direction. With that in mind, let $\lambda \in \text{pcf}_I(A)$ and let $D$ be an ultrafilter over $A$ such that $\text{cf}(\prod A/D) = \lambda$. Note that $B \in D$ so we can define $D_B = \{X \subseteq B : X \in D\}$ which is an ultrafilter over $B$ extending $I_B$. Now if we let $\vec{f} = \langle f_\xi : \xi < \lambda \rangle$ be cofinal in $\prod A/D$, then we see that $\vec{f} \upharpoonright B = \langle f_\xi \upharpoonright B : \xi < \lambda \rangle$ is cofinal in $\prod B/D_B$. Thus, $\lambda \in \text{pcf}_{t_B}(B)$. \qed

2.1. The Weak pcf Theorem. We would like to show that we can focus our attention on the ideals defined above when working with $\text{pcf}_I(A)$. In particular, we will work towards what Shelah refers to as the weak pcf theorem and derive several corollaries along the way. We begin with more definitions.

Definition 2.5. Let $A$ be a set, $I$ an ideal over a set $X$, and $\lambda$ a cardinal. We say that $I$ is $\lambda$-weakly saturated if any collection of sets partitioning $A$ into disjoint $I$-positive sets has size $< \lambda$. We let $\text{wsat}(I)$ denote the least cardinal $\theta$ such that $I$ is $\theta$-weakly saturated.

We now look at the structure of these pcf ideals that we just defined.

Lemma 2.2. Let $\mu$ be a singular cardinal, and suppose that $A$ is a collection of regular cardinals cofinal in $\mu$. Suppose that $I$ is an ideal over $A$ such that $\text{min}(A) >$
wsat(I), then if \( \lambda \geq \text{wsat}(I) \) is a cardinal with \( J^I_{\leq \lambda}[A] \) proper, then \( \prod A/J^I_{\leq \lambda} \) is \( \lambda \)-directed.

**Proof.** We will show by induction on \( \lambda_0 < \lambda \) that \( \prod A/J^I_{\leq \lambda} \) is \( \lambda_0^+ \)-directed. If \( |F| \leq \text{wsat}(I) < \min(A) \), then we let \( g \) be defined by \( g(i) = \sup\{f(i) : f \in F\} \). Then since each \( \mu_i \) is regular, it follows that \( g \in \prod A \) and \( f \leq g \) everywhere.

By way of induction, assume we have shown for some cardinal \( \lambda_0 \) with \( \text{wsat}(I) < \lambda_0 < \lambda \), that \( \prod A/J^I_{\leq \lambda} \) is \( \lambda_0 \)-directed, and let \( F \subseteq \prod A \) of size \( \lambda_0 \) be given. We first assume that \( \lambda_0 \) is singular. In this case, we can write \( F = \bigcup_{\alpha < cf(\lambda_0)} F_\alpha \) such that \( |F_\alpha| < \lambda_0 \). Then by assumption, we can bound each \( F_\alpha \) by some \( g_\alpha \), and then bound the set \( \{g_\alpha : \alpha < cf(\lambda_0)\} \) by some \( g \in \prod A \). We then have that \( f \leq g \) modulo \( J^I_{\leq \lambda} \) for each \( f \in F \).

So assume that \( \lambda_0 \) is regular. We begin by replacing \( F = \{h_i : i < \lambda_0\} \) with a \( \leq J^I_{\leq \lambda} \)-increasing sequence \( \tilde{f} = (f_i : i < \lambda_0) \). We just let \( f_i \) be an \( \leq J^I_{\leq \lambda} \)-upper bound for \( \{h_j : j \leq i\} \cup \{f_j : j < i\} \). By construction, if we can find a \( g \in \prod A \) such that \( f_i \leq g \) modulo \( J^I_{\leq \lambda} \) for each \( i < \lambda_0 \), then we will be done. At this point, we will proceed by induction on \( \alpha < \text{wsat}(I) \) and attempt to construct a \( \leq J^I_{\leq \lambda} \)-increasing sequence of candidates for bounds of \( \tilde{f} \). As usual, we will show that this construction must terminate at some point, or we will be able to generate a contradiction.

By induction on \( \alpha < \text{wsat}(I) \), we will define functions \( g_\alpha \), ordinals \( \xi(\alpha) \), and sequences \( \langle B^\alpha_\xi : \xi < \lambda_0 \rangle \) with the following properties:

1. \( g_\alpha \in \prod A \) and for all \( \beta < \alpha \), we have that \( g_\beta \leq g_\alpha \);
2. \( B^\alpha_\xi := \{a \in A : f_\xi(a) > g_\alpha(a)\} \);
3. For each \( \alpha < \text{wsat}(I) \), and every \( \xi \in [\xi(\alpha+1), \lambda_0) \), we have that \( B^\alpha_\xi \neq B^{\alpha+1}_\xi \) modulo \( J^I_{\leq \lambda} \).

The construction proceeds as follows. At stage \( \alpha = 0 \), we simply let \( g_0 = f_0 \), and set \( \xi(0) = 0 \) (note that \( \xi(\alpha) \) only matters when \( \alpha = \beta + 1 \) for some ordinal \( \beta < \text{wsat}(I) \)). At limit stages, assume that \( g_\beta \) has been defined for each \( \beta < \alpha \), and define \( g_\alpha \) by setting \( g_\alpha(a) = \sup_{\beta < \alpha} f_\beta(a) \). Note since \( \alpha < \text{wsat}(I) < \min(A) \) and each \( a \in A \) is regular, that \( g_\alpha \in \prod A \).

At successor stages, let \( \alpha = \beta + 1 \), and suppose that \( g_\beta \) has been defined. If \( g_\beta \) is a \( \leq J^I_{\leq \lambda} \)-upper bound for \( \tilde{f} \), then we’re done and we can terminate the induction. Otherwise, note that the sequence \( \langle B^\beta_\xi : \xi < \lambda_0 \rangle \) is \( \leq J^I_{\leq \lambda} \)-increasing and so there is a minimum \( \xi(\alpha) \) for which every \( \xi \in [\xi(\alpha), \lambda_0) \) has the property that \( B^\beta_\xi \notin J^I_{\leq \lambda} \) (else if there is no such \( \xi(\alpha) \), then \( g_\alpha \) was indeed the desired bound). By definition, that means we can find some ultrafilter \( D \), disjoint from \( J^I_{\leq \lambda} \) such that \( \prod A/J^I_{\leq \lambda} \geq \lambda \). Thus is follows that \( \tilde{f} \) must have a \( \leq D \)-upper bound in \( \prod A \), say \( h_\alpha \). We then define \( g_\alpha \in \prod A \) by \( g_\alpha(a) = \max\{g_\beta(a), h_\alpha(a)\} \).
Note that for each $\xi \in \{\xi(\beta + 1), \lambda_0\}$, we have that $B^\beta_\xi \in D$. On the other hand, our definition of $g_\alpha$ gives us that $B^{\beta+1}_\xi \notin D$ since $g_\alpha$ is at least $h_\alpha$ everywhere. Thus, condition 3) is satisfied, as are 1) and 2) trivially by construction.

We claim that this process must have terminated at some stage. Otherwise, we let $\xi(\ast) = \sup\{\xi(\alpha) : \alpha < \text{wsat}(I)\}$, and note that each $B^\alpha_{\xi(\ast)} \notin J^I_{<\lambda}$ since the induction never terminated. Next, we note that condition 1) gives us that we let $\xi$. Thus, condition 3) is satisfied, as are 1) and 2) trivially by construction.

For the other direction, suppose that $J^I_{<\lambda}$ is uninteresting. For the other direction, suppose that $J^I_{<\lambda}$ is uninteresting.

By definition, $\text{cf}(\prod A/D) \geq \lambda$ if and only if $J^I_{<\lambda}[A] \cap D = \emptyset$.

Proof. By definition, $\text{cf}(\prod A/D) \geq \lambda$ implies that $J^I_{<\lambda}[A] \cap D = \emptyset$ so this direction is uninteresting. For the other direction, suppose that $J^I_{<\lambda}[A] \cap D = \emptyset$, and note that $\prod A/D$ is $\lambda$ directed. We then see that $\text{cf}(\prod A/D) \geq \lambda$, since any sequence of length $\kappa < \lambda$ will be bounded above in $\prod A/D$.

In line with the notation of [1], we will isolate the additional hypothesis of Lemma 2.2.

**Definition 2.6.** We say that $A$ is weakly progressive over $I$ if $\text{wsat}(I) < \min(A)$.

**Corollary 2.1.** Suppose $A$ is weakly progressive over $I$, then for every ultrafilter $D$ over $A$ disjoint from $I$, $\text{cf}(\prod A/D) \geq \lambda$ if and only if $J^I_{<\lambda}[A] \cap D = \emptyset$.

Proof. By definition, $\text{cf}(\prod A/D) \geq \lambda$ implies that $J^I_{<\lambda}[A] \cap D = \emptyset$ so this direction is uninteresting. For the other direction, suppose that $J^I_{<\lambda}[A] \cap D = \emptyset$, and note that $\prod A/D$ is $\lambda$ directed. We then see that $\text{cf}(\prod A/D) \geq \lambda$, since any sequence of length $\kappa < \lambda$ will be bounded above in $\prod A/D$.

This corollary tells us that $\text{cf}(\prod A/D) = \lambda$ if and only if $D$ has non-empty intersection with $J^I_{<\lambda^+}$, but misses $J^I_{<\lambda}$. So $\text{cf}(\prod A/D) = \lambda$ if and only if $\lambda$ is the first cardinal for which $D$ has nontrivial intersection with $J^I_{<\lambda^+}$. It follows that we can associate to each $\lambda \in \text{pcf}_I(A)$ a set $X_\lambda \in J^I_{<\lambda^+} \setminus J^I_{<\lambda}$, which gives us an injection from $\text{pcf}_I(A)$ to $\mathcal{P}(A)$. The idea then is that, if we can get some control over how we generate $J^I_{<\lambda^+}$ from $J^I_{<\lambda}$, we should be able to say something more about the size of $\text{pcf}_I(A)$. Also, note that each of these sets $X_\lambda$ is $I$-positive.

**Corollary 2.2.** If $A$ is weakly progressive over $I$, then $\max \text{pcf}_I(A)$ exists.
Proof. Let $J = \bigcup\{J^I_{<\lambda} : \lambda \in \text{pcf}_I(A)\}$, and note that $J$ is a proper ideal since it is the union of an ascending chain of proper ideals. Let $D$ be an ultrafilter disjoint from $J$ and let $\kappa = cf(\prod A/D)$. As $D \cap J^I_{<\lambda} = \emptyset$ for all $\lambda \in \text{pcf}_I(A)$, it follows that $\kappa \in \text{pcf}_I(A)$ but $\kappa \geq \lambda$ for all $\lambda \in \text{pcf}_I(A)$. Hence, $\kappa = \max \text{pcf}_I(A)$. □

At this point, we would like to make good on the promise of getting more control over the generators of $J^I_{<\lambda}$. The following is implicit in [5], but we will instead mimic the proof of Lemma 2.2. The proof here is also very similar to the proof that universal cofinal sequences exist given in [1]

**Theorem 2.1.** If $A$ is weakly progressive over $I$, then every $\lambda \in \text{pcf}_I(A)$ has a universal cofinal sequence.

**Proof.** The proof of this will be very similar to the proof of Lemma 2.2, insofar as we will proceed by induction on $\alpha < \text{wsat}(I)$, and suppose that we fail to get a universal cofinal sequence at each stage. From this we will be able to produce a contradiction to weak saturation. We begin by noting that, if $\lambda = \min(A)$, then we can define $\bar{f} = \langle f_\xi : \xi < \lambda \rangle$ by setting $f_\xi(a) = \xi$ which is an everywhere increasing sequence. So, it trivially is a universal sequence. In that vein, we may assume that $\text{wsat}(I) < \min(A) < \lambda$.

We will proceed by induction on $\alpha < \text{wsat}(I)$, and construct candidate universal sequences $\bar{f}^\alpha = \langle f_\xi^\alpha : \xi < \lambda \rangle$. Now, we want to come up with sets $B_\xi^\alpha$ that are $\subseteq$-increasing in the $\alpha$ coordinate but differ from each other modulo $J^I_{<\lambda}$ (and hence $I$). So we will ask that not only is the collection $\langle f_\xi^\alpha : \alpha < \text{wsat}(I), \xi < \lambda \rangle$ strictly increasing modulo $J^I_{<\lambda}$ in the $\xi$ coordinate, but that it is $\subseteq$-increasing in the $\alpha$ coordinate. With that in mind, we will use $\lambda$-directedness to inductively construct these sequences.

At stage $\alpha = 0$, we let $\bar{f}^0 = \langle f_\xi^0 : \xi < \lambda \rangle$ be any $<_{J^I_{<\lambda}}$-increasing sequence in $\prod A$. We can create such a sequence inductively as follows: let $f_\eta^0$ be arbitrary, and then assume that $f_\eta^0$ have been defined for $\eta < \xi$. By $\lambda$-directedness, we can find $g \in \prod A$ such that $f_\eta^0 \leq f_\eta^0 g$ for all $\eta < \xi$, and let $f_\xi^0 = g + 1$.

At limit stages, let $\gamma < \lambda$ and assume that $\bar{f}^\alpha$ has been defined for each $\alpha < \gamma$. We inductively define $\bar{f}^\gamma = \langle f_\xi^\gamma : \xi < \lambda \rangle$ as follows: let $f_\gamma^0 = \sup\{f_\eta^0 : \alpha < \gamma\}$, which is in $\prod A$ since $\gamma < \text{wsat}(I) < \min(A)$. Now suppose that $f_\eta^0$ has been defined for each $\eta < \xi$, and let $g = \sup\{f_\eta^0 : \alpha < \gamma\}$. Again $g \in \prod A$, and let $h$ be such that $f_\eta^0 \leq f_\eta^0 h$ for all $\eta < \xi$ by $\lambda$-directedness. Then define $f_\xi^\gamma$ by $f_\xi^\gamma(a) = \max\{g(a), h(a)\} + 1$, which is as desired.

At successor stages suppose that $\bar{f}^\alpha$ has been defined. If $\bar{f}$ is a universal sequence, then we can terminate the induction. If not, we inductively define $\bar{f}^\alpha+1 = \langle f_\xi^\alpha+1 : \xi < \lambda \rangle$ as follows: Since $\bar{f}^\alpha$ is not universal, we can find an
ultrafilter $D_\alpha$ over $A$ with the property that $\text{cf}(\prod A/D_\alpha) = \lambda$, but $\vec{f}^\alpha$ is $<_{D_\alpha}$-dominated by some $h \in \prod A/D_\alpha$ (note that $D_\alpha$ is disjoint from $J^I_{<\lambda}$). Let $\vec{g} = \langle g_\xi : \xi < \lambda \rangle$ be a cofinal sequence in $\prod A/D_\alpha$. We define $f_{0}^{\alpha+1}$ by setting $f_{0}^{\alpha+1}(a) = \max\{h(a), f_{0}^{\alpha}(a), g_\alpha(a)\}$. Now suppose that $f_{\eta}^{\alpha+1}$ has been defined for each $\eta < \xi$, and let $h$ be such that $f_{\eta}^{\alpha+1} \leq J^I_{<\lambda} h$ for all $\eta < \xi$ by $\lambda$-directedness. Then define $f_{\xi}^{\alpha+1}$ by $f_{\xi}^{\alpha+1}(a) = \max\{f_{\xi}^{\alpha}(a), h(a), g_\xi(a)\} + 1$, which is as desired. Note that $f_{\xi}^{\alpha+1}$ is cofinal in $\prod A/D_\alpha$.

We claim that we must have terminated the induction at some stage. Otherwise, with the property that

\[
\eta < \xi
\]

we let $\eta < \xi$. Let $\vec{f}$ be the $\vec{f}$-dominated by some $\vec{g}$.

We define $\vec{f}^\alpha = \langle f_\xi^\alpha : \xi < \lambda \rangle$ which are $J^I_{<\lambda}$-increasing in the $\xi$ coordinate,

and $\leq$-increasing in the $\alpha$ coordinate.

(2) Ultrafilters $D_\alpha$ disjoint from $J^I_{<\lambda}$ such that $\vec{f}^\alpha$ is $<_{D_\alpha}$ dominated by $f_{0}^{\alpha+1}$, and $f_{\xi}^{\alpha+1}$ is cofinal in $\prod A/D_\alpha$.

We will use this to derive a contradiction. We begin by letting $h \in \prod A$ be defined by setting $h(a) = \sup\{f_{\xi}^{\alpha}(a) : \alpha < \text{wsat}(I)\}$ (recall that $\text{wsat}(I) < \min(A)$). By condition 2) above, for every $\alpha < \text{wsat}(I)$, there exists an index $\xi(\alpha) < \lambda$ such that $h <_{D_\alpha} f_{\xi(\alpha)}^{\alpha+1}$. Since $\text{wsat}(I) < \min(A) < \lambda$ for $\lambda$ regular, it follows that $\xi(\alpha) = \sup\{\xi(\alpha) : \alpha < \text{wsat}(I)\}$ is below $\lambda$. So, for each $\alpha < \text{wsat}(I)$, we have that $h <_{D_\alpha} f_{\xi(\alpha)}^{\alpha+1}$. Now define the sets

\[
B_\alpha = \{a \in A : h(a) \leq f_{\xi(\alpha)}^{\alpha}(a)\}.
\]

By construction, we have that $B_\alpha \notin D_\alpha$ since $f_{\xi(\alpha)}^{\alpha+1} <_{D_\alpha} f_{0}^{\alpha+1} \leq h$. On the other hand, $B_{\alpha+1} \in D_\alpha$ since $h <_{D_\alpha} f_{\xi(\alpha)}^{\alpha+1}$. So, it follows that $B_\alpha \neq B_{\alpha+1}$ modulo $J^I_{<\lambda}$ (hence modulo $I$). But since $f_{\xi(\alpha)}^{\alpha} \leq f_{\xi(\alpha)}^{\alpha+1}$, we have that $B_\alpha \subseteq B_{\alpha+1}$ (in fact $\beta < \alpha$ implies that $B_\beta \subseteq B_\alpha$) and so we are in the same position as the proof of Lemma 2.2. That is, $\langle B_\alpha \setminus B_{\alpha+1} : \alpha < \text{wsat}(I)\rangle$ is a collection of $I$-positive sets which are disjoint, contradiction weak saturation. Therefore, the induction must have halted at some stage and we are done.

The above statement is equivalent to what is referred to as the weak pcf theorem in [5]. We pause for a second to extract some corollaries.

**Corollary 2.3.** If every $\lambda \in \text{pcf}_I(A)$ carries a universal sequence, then $\text{cf}(\prod A/I) = \max \text{pcf}_I(A)$.

**Proof.** Let $\kappa = \max \text{pcf}_I(A)$, and note that since there is an ultrafilter $D$ disjoint from $I$ with the property that $\text{cf}(\prod A/D) = \kappa$, then $\text{cf}(\prod A/I) \geq \kappa$. So it suffices to show that there exists a cofinal subset of $\prod A/I$ with size $\kappa$. For each $\lambda \in \text{pcf}_I(A)$, we let $\vec{f}^\lambda = \langle f_\xi^\lambda : \xi < \lambda \rangle$ be a universal sequence for $\lambda$. Let $F \subseteq \prod A$ be the
collection of functions of the form \( g_{X,Y} \) for \( X = \{ \lambda_1, \ldots, \lambda_n \} \) a finite subset of \( \text{pcf}_I(A) \) and \( Y = \{ \xi_1, \ldots, \xi_n \} \) a collection of indices such that \( \xi_i < \lambda_i \) where \( g_{X,Y} \) is defined as follows

\[
 g_{X,Y}(a) = \max\{ f_{\xi_i}^b(a) : i \leq n \}. 
\]

Note then that the set \( F \) has size \( \kappa \), so the proof is complete once we show that \( F \) is cofinal in \( \prod A/I \). In this vein, let \( g \in \prod A \) and for each \( f \in F \) define the set \( B_f = \{ a \in A : f(a) > g(a) \} \). Then the collection \( J = \{ B_f : f \in F \} \) is closed under finite unions and so can be extended to a maximal ideal. If \( A \in J \) modulo \( I \), then we are done since that would mean there is some \( f \in F \) that dominates \( g \) with respect to \( <_I \). Otherwise, we can extend \( J \) to a proper maximal ideal \( M \) and let \( \lambda_0 = cf(\prod A/M) \). Then we see that \( \bar{f} \lambda_0 \) must be cofinal in \( \prod A/M \), but at the same time is \( \leq_M \)-bounded by \( g \). This is obviously absurd, so there is some \( f \in F \) such that \( g <_I f \). \( \square \)

It turns out that, under the appropriate assumptions, we can use Theorem 2.1 to gain some control over generators of \( J^{I}_{< \lambda^{+}} \) from \( J^{I}_{< \lambda} \). We first need a number of definitions.

**Definition 2.7.** Let \( F \) be a collection of functions from \( A \) to \( \text{ON} \). We say that \( f : A \to \text{ON} \) is an exact upper bound for \( F \) if \( f \) is a least upper bound of \( F \), and \( F \) is cofinal in \( \{ g \in \prod A/\text{ON} : g <_I f \} \).

One reason to isolate the hypothesis of exactness is that it allows us to build products with true cofinality. For example, if \( \lambda \) is regular and \( \bar{f} = \langle f_\xi : \xi < \lambda \rangle \) is a \( <_I \)-increasing sequence of functions in \( \prod A/\text{ON} \) which has an exact upper bound \( f \), then \( tc_f(\prod_{a \in A} f(a)/I) = \lambda \) as witnessed by \( \bar{f} \).

**Definition 2.8.** Let \( \lambda \in \text{pcf}_I(A) \). We say that \( B \) is a generator of \( J^{I}_{< \lambda^{+}}[A] \) (written \( J^{I}_{< \lambda^{+}}[A] = J^{I}_{< \lambda}[A] + B \)) if the ideal \( J^{I}_{< \lambda}[A] \) is generated by \( J^{I}_{< \lambda}[A] \cup \{ B \} \).

The pcf theorem (in the classical theory) is the statement that, for every \( \lambda \in \text{pcf}(A) \), we can find a generator. We now show how to extract generators from universal cofinal sequences under the appropriate conditions.

**Lemma 2.3.** Suppose that \( \lambda \in \text{pcf}_I(A) \) has a universal cofinal sequence \( \bar{f} = \langle f_\xi : \xi < \lambda \rangle \) with an exact upper bound \( f \). Then the set \( B = \{ a \in A : f(a) = a \} \) is a generator for \( J^{I}_{< \lambda^{+}}[A] \).

**Proof.** Let \( \lambda \in \text{pcf}_I(A) \), \( \bar{f} \), and \( f \) be as desired. Note that since the identity function on \( A \) is obviously an upper bound for \( \bar{f} \), we may assume that \( f(a) \leq a \) for each \( a \in A \) by modifying \( f \) on a \( J^{I}_{< \lambda^{+}} \)-null set if necessary. In order to show that \( B \) is a generator as desired, we first show that \( B \in J^{I}_{< \lambda^{+}}[A] \). From there, we will show that if \( D \) is an ultrafilter over \( A \) disjoint from \( I \) with \( cf(A/D) = \lambda \) then \( B \)
must be in $D$. We will complete the proof by showing that these two properties imply that $B$ is a generator.

With this in mind, let $D$ be an ultrafilter containing $B$. If $D \cap J_{\lambda}^{I} \neq \emptyset$, then we are done since then $\text{cf}(\prod A/D) < \lambda$, so we suppose that $D \cap J_{\lambda}^{I} = \emptyset$. In this case $f$ is still an exact upper bound for the $<_{D}$-increasing sequence $\vec{f}$ in $\prod A/D$ since $D$ extends the dual of $J_{\lambda}^{I}$. Since $B \in D$ and $f$ is exact, it follows that $\vec{f}$ is cofinal in $\prod A/D$ and so it follows that $B \in J_{\lambda}^{I}$.

Now let $D$ be an ultrafilter over $A$ disjoint from $I$ containing $X \setminus B$. Since $X \in J_{\lambda}^{I}$ and $\vec{f}$ is a universal sequence for $\lambda$, it follows that $\vec{f}$ has an upper bound in $\prod A/D$ and so it is not cofinal there. But, this contradicts the fact that $\vec{f}$ is a universal sequence for $\lambda$. Thus it must be the case that $B \in D$.

With both of these properties in hand, we now show that $B$ is a generator for $J_{\lambda}^{I}$. Certainly we have that $J_{\lambda}^{I} \supseteq J_{\lambda} + B$ since $B \in J_{\lambda}^{I}$ so we only need to show the reverse inclusion. For this it suffices to show that for every $X \in J_{\lambda}^{I}$, we have $X \setminus B \in J_{\lambda}^{I}$. So let $D$ be an ultrafilter over $A$ disjoint from $I$ containing $X \setminus B$. Since $X \in J_{\lambda}^{I}$, we know that $\text{cf}(\prod A/D) \leq \lambda$. On the other hand, we know that $B \notin D$ and so $\text{cf}(\prod A/D) \neq \lambda$ and thus $X \setminus B \in J_{\lambda}^{I}$. Therefore we have the desired equality $J_{\lambda}^{I}[A] = J_{\lambda}^{I}[A] + B$. $\square$

The following is immediate.

**Theorem 2.2** (The pcf Theorem). If $A$ is weakly progressive over $I$ and each $\lambda \in \text{pcf}_{I}(A)$ carries a universal sequence with an exact upper bound, then for every $\lambda \in \text{pcf}_{I}(A)$, there exists a $B_{\lambda} \subseteq A$ such that $J_{\lambda}^{I}[A] = J_{\lambda}^{I}[A] + B_{\lambda}$.

In order to get control over generators and move from the weak pcf theorem to the pcf theorem, we need to develop the theory of exact upper bounds.

### 2.2. Exact Upper Bounds

Our goal in this section is to work towards a proof of Shelah’s trichotomy theorem for our setting. From there, we will give a relatively easy way of manufacturing nice exact upper bounds for our desired setting. As before we will fix:

1. A singular cardinal $\mu$;
2. A collection $A$ of regular cardinals cofinal in $\mu$.
3. An ideal $I$ over $A$;

In order to develop the theory of exact upper bounds, we will need stronger hypotheses on $A$ and $I$ than simply asking that $A$ is weakly progressive over $I$.

**Definition 2.9.** Let $I$ be an ideal over a set $A$. We say that $I$ is $\theta$-irregular if, given any collection $\langle A_{i} : i < \theta \rangle$ of $I$-positive sets, there exists some $H \in [\theta]^{\theta}$ such that $\bigcap_{i \in H} A_{i} \neq \emptyset$. We let $\text{reg}(I)$ denote the least cardinal $\theta$ such that $I$ is $\theta$-irregular.
Now, let $\lambda$ be regular, and suppose that $\vec{f} = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of functions from $A$ to ON. Consider the following properties of $\vec{f}$ for a regular cardinal $\theta \leq \lambda$:

1. **Good**: $\vec{f}$ has an exact upper bound $f \in \prod A/I$ with $\{a \in A : cf(f(a)) < \theta\} \in I$.

2. **Bad**: There are sets $S(a)$ for each $a \in A$ such that $\|S(a)\| < \theta$ and an ultrafilter $D$ over $A$ disjoint from $I$ such that, for all $\xi < \lambda$, there exists some $\hat{h}_\xi \in \prod_{a \in A} S(a)$ and some $\eta < \lambda$ such that $f_\xi <_D \hat{h}_\xi <_D f_\eta$.

3. **Ugly**: There is a function $g : A \to \text{ON}$ such that, letting $t_\xi = \{a \in A : f_\xi(a) > g(a)\}$, the sequence $\vec{t} = \langle t_\xi : \xi < \lambda \rangle$ (which is $\subseteq I$-increasing) does not stabilize modulo $I$. That is, for every $\xi < \lambda$, there is some $\xi < \eta < \lambda$ such that $t_\xi \setminus t_\eta \notin I$.

The above properties will be one of our tools in the analysis of exact upper bounds. Our presentation follows that of [3].

**Lemma 2.4.** Let $\lambda$ be regular and $\vec{f} = \langle f_\xi : \xi < \lambda \rangle$ be $<_I$-increasing. If $\vec{f}$ is not Ugly, then every least upper bound of $\vec{f}$ is an exact upper bound.

**Proof.** Suppose otherwise, that $\vec{f}$ is not Ugly, but there is some least upper bound $f$ for $\vec{f}$ which is not exact. So there is some $g : A \to \text{ON}$ such that $g <_I f$, but for all $\xi < \lambda$, the set $t_\xi = \{a \in A : f_\xi(a) > g(a)\} \notin I^*$. As Ugly fails, the sequence $\vec{t} = \langle t_\xi : \xi < \lambda \rangle$ stabilizes (modulo $I$) at some ordinal $\eta < \lambda$. Now we define a function $\hat{f} : A \to \text{ON}$ by $\hat{f}(a) = f(a)$ for $a \in t_\eta$, and $\hat{f}(a) = g(a)$ for $a \in \mu \setminus t_\eta$.

As $t_\eta \notin I^*$, we have that $\hat{f} \neq f$. Further, we see that for all $\xi < \lambda$, $f_\xi \leq \hat{f}$ by definition, while $\hat{f} \leq_I f$, contradicting the fact that $f$ was a least upper bound. $\square$

**Lemma 2.5.** Suppose that $\lambda$ and $\theta \leq \lambda$ are regular, and that $\vec{f} = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of functions from $\mu$ to ON. If $\vec{f}$ has an exact upper bound $f$ such that $\{a \in A : cf(f(a)) < \theta\} \notin I$ then $\vec{f}$ satisfies Bad.$\theta$.

**Proof.** Let $f$ be an upper bound for $\vec{f}$ with $B = \{a \in A : cf(f(a)) < \theta\} \notin I$, and let $D$ be an ultrafilter over $A$ disjoint from $I$ such that $B \in D$. Next for each $a \in B$, let $S(a)$ be cofinal in $f(a)$ with $\|S(a)\| = cf(f(a)) < \theta$, and let $S(a) = \{0\}$ for each $a \notin B$. For each $\xi < \lambda$, let $f_\xi^+$ be defined by $f_\xi^+(a) = \min(S(a) \setminus f_\xi(a))$ for $i \in A$ and $f_\xi^+(a) = 0$ otherwise.

Now for any $\xi < \lambda$ we have that $f_\xi <_D f_{\xi+1}^+$ where $f_{\xi+1}^+ \in \prod_{a \in A} S(a)$. On the other hand, $f$ is exact and since $S(a)$ is cofinal in $f$ $D$-almost everywhere, it follows
that there is some \( \eta \in \lambda \) such that \( f_{\xi + 1}^\eta <_D f_\eta \) and so \( \vec{f} \) is bad as witnessed by \( \langle S(a) : a \in A \rangle \) and \( D \). \hfill \Box

**Theorem 2.3** (Trichotomy). Let \( \lambda > \text{reg}(I) \) and \( \theta \in [\text{reg}(I), \lambda] \) be regular. If \( \vec{f} = \langle f_\xi : \xi < \lambda \rangle \) is a \( <_I \)-increasing sequence of functions from \( \mu \) to ON, then at least one of Good\( _\theta \), Bad\( _\theta \), or Ugly must hold.

**Proof.** We will show that, assuming \( \vec{f} \) is not Ugly, then we can either find a witness to Bad\( _\theta \) or find a least upper bound \( f \) for \( \vec{f} \). By Lemma 2.4, this least upper bound is actually exact and so by Lemma 2.5 we can either find a witness to Bad\( _\theta \) or \( f \) witnesses Good\( _\theta \). We proceed by induction on \( \alpha < \text{reg}(I) \), and at each stage create a candidate for a least upper bound. We will terminate at successor stages if we have found a least upper bound, and at limit stages if we can construct a witness to Bad\( _\theta \). At the end, we will show that we must have terminated at some \( \alpha < \text{reg}(I) \), else we will be able to derive a contradiction. At each stage \( \alpha < \text{reg}(I) \), we will define:

1. Functions \( g_\alpha : A \to \text{ON} \) which are upper bounds for \( \vec{f} \) such that, for each \( \beta < \alpha \), we have \( g_\alpha \leq_I g_\beta \) but \( g_\alpha \neq_I g_\beta \).
2. Sets \( S^\alpha(a) = \{g_\beta(a) : \beta < \alpha\} \).
3. Functions \( h_\alpha^\xi : A \to \text{ON} \) defined by \( h_\alpha^\xi(a) = \min(S^\alpha(a) \setminus f_\xi(a)) \).

Note here that 2) and 3) depend on how we defined 1). Further, the sequence \( \langle h_\alpha^\xi : \xi < \lambda \rangle \) is \( \leq_I \)-increasing in \( \prod_{a \in A} S^\alpha(a) \).

Stage \( \alpha = 0 \): Here we let \( g_0 \) be any \( I \)-upper bound of \( \vec{f} \), for example \( g(a) = \sup\{f_\xi(a) : \xi < \lambda\} + 1 \) works.

Stage \( \alpha + 1 \): Assume that \( g_\alpha \) has been define. If \( g_\alpha \) is a \( I \)-least upper bound for \( \vec{f} \), then we can terminate the induction. Otherwise as \( g_\alpha \) is not a least upper bound, and so there is some \( I \) upper bound \( g_{\alpha + 1} \) such that \( g_{\alpha + 1} \leq_I g_\alpha \) but \( g_{\alpha + 1} \neq_I g_\alpha \).

Stage \( \gamma \) limit: Suppose that \( \gamma < \text{reg}(I) \) is a limit ordinal and that \( g_\alpha \) has been defined for each \( \alpha < \gamma \). Now consider the functions \( h_\gamma^\xi \) and the sets

\[ t_\gamma^\xi := \{a \in A : h_\gamma^\xi(a) < f_\eta(a)\} \]

Fixing the \( \xi \) coordinate, the function \( h_\gamma^\xi \) is fixed while we run through each \( f_\xi \) and so the sequence \( t_\gamma^\xi = \langle t_\gamma^{\eta} : \eta < \lambda \rangle \) is \( \subseteq_I \)-increasing since the sequence \( \vec{f} \) is \( <_I \)-increasing. Fixing the \( \eta \) coordinate on the other hand, we fix \( f_\eta \) and run through the functions \( h_\gamma^\xi \) and so the sequence \( t_\gamma^\eta = \langle t_\gamma^{\xi} : \xi < \lambda \rangle \) is \( \subseteq_I \)-decreasing.
Since \( \vec{f} \) is not Ugly, it follows that each sequence \( \vec{t}_\xi \) stabilizes modulo \( I \) at some ordinal \( \eta(\xi) \). That is, for all \( \eta > \eta(\xi) \), we have that \( t_\xi^{\eta(\xi)} = t_\xi^{\eta} \). We have two cases: either \( \eta(\xi) \notin I \) for each \( \xi < \lambda \), or there is a club subset \( X \) of \( \lambda \) such that \( \eta(\xi) \in I \) for each \( \xi \in X \). To see this, simply note that if there is some \( \xi \in \lambda \) for which \( t_\xi^{\eta(\xi)} \in I \), then it follows that \( t_\xi^{\eta(\xi)} \in I \) for each \( \xi' > \xi \) since \( t_\xi^{\eta(\xi)} \in I \) and \( t_\xi^{\eta(\xi)} \subseteq t_\xi^{\eta} \).

Assume that the former happens (i.e. \( \eta(\xi) \notin I \) for each \( \xi < \lambda \)), and consider the sequence \( \{t_\xi^{\eta(\xi)} : \xi < \lambda \} \). Note that this sequence is \( \subseteq I \)-decreasing, and so \( I^* \cup \{t_\xi^{\eta(\xi)} : \xi < \lambda \} \) has the finite intersection property. So let \( D \) be an ultrafilter over \( A \) extending \( I^* \cup \{t_\xi^{\eta(\xi)} : \xi < \lambda \} \), and note that \( \text{Bad}_\theta \) is witnessed by \( D \) and \( \prod_{\alpha \in A} S^\gamma(\alpha) \). By construction we know that \( f_\xi <_D h_\xi^\gamma \) for each \( \xi < \lambda \). On the other hand, we have that \( h_\xi^\gamma \in D \) since \( t_\xi^{\eta(\xi)+1} \in D \). If this happens, we can terminate the induction.

Otherwise, suppose that there is some club subset \( X \) of \( \lambda \) such that \( \eta(\xi) \in I \) for each \( \xi \in X \). Let \( \xi(\gamma) = \min X \), and define \( g_\gamma := h_{\xi(\gamma)}^\gamma \). Note then that \( g_\gamma \) is an \( I \)-upper bound of \( \vec{f} \) by construction, and so we only need to verify that \( g_\gamma \leq_I g_\alpha \) while \( g_\gamma \neq_I g_\alpha \) for each \( \alpha < \gamma \). Recall that \( S^\gamma(a) = \{g_\alpha(a) : \alpha < \gamma \} \) while \( g_\gamma = \min(S^\gamma(a) \setminus f_{\xi(\gamma)}(\alpha)) \) and \( \langle g_\alpha : \alpha < \gamma \rangle \) is \( \subseteq_I \)-decreasing. Fix \( g_\alpha \), then \( \{i < \mu : g_i(a) > g_\alpha(a)\} \in I \) by definition of the sets \( S^\gamma(a) \) since for \( \gamma > \beta > \alpha \), the set \( \{i < \mu : g_i(a) > g_\alpha(a)\} \in I \). Similarly, it follows that \( g_\gamma \neq_I g_\alpha \). It is worth noting that, in this case the functions \( h_\xi^\gamma \) stabilize modulo \( I \) again by definition.

We claim that this induction must have terminated. Otherwise, for each \( \alpha \in \text{acc}(\text{reg}(I)) \), we have defined:

1. Functions \( g_\alpha : A \to \text{ON} \) which are upper bounds for \( \vec{f} \) such that, for each \( \beta \in \text{acc}(\text{reg}(I)) \) with \( \alpha < \beta \), we have \( g_\alpha \leq_I g_\beta \) but \( g_\alpha \neq_I g_\beta \).
2. Ordinals \( \xi(\alpha) \) such that \( g_\alpha = h_{\xi(\alpha)}^\alpha =_I h_{\xi}^\alpha \) for each \( \xi \geq \xi(\alpha) \).

Since \( \text{reg}(I) < \lambda \) with \( \lambda \) regular, we can see that \( \xi(\ast) = \sup\{\xi(\alpha) : \alpha \in \text{acc}(\text{reg}(I))\} \) is still below \( \lambda \). Note that \( g_\alpha =_I h_{\xi(\ast)}^\alpha \) for each \( \alpha \in \text{acc}(\text{reg}(I)) \), so letting \( H_\alpha = h_{\xi(\ast)}^\alpha \) we have that \( H_\alpha \) enjoys the same properties as \( g_\alpha \). Now for each \( \alpha \in \text{acc}(\text{reg}(I)) \), let \( \alpha' = \min(\text{acc}(\text{reg}(I)) \setminus \alpha + 1) \) be the successor of \( \alpha \) in \( \text{acc}(\text{reg}(I)) \). Define the sets \( B_\alpha := \{a \in A : H_{\alpha'}(a) < H_\alpha(a)\} \) for each \( \alpha \in \text{acc}(\text{reg}(I)) \). Now we, have that, since \( S^\alpha(a) \subseteq S^\beta(a) \) for \( \alpha < \beta \in \text{acc}(\text{reg}(I)) \) and so \( H_\alpha \leq H_\beta \). On the other hand, by construction each \( B_\alpha \notin I \), and so the sequence \( \langle B_\alpha : \alpha < \text{acc}(\text{reg}(I)) \rangle \) has the property that, for some \( H \subseteq \text{acc}(\text{reg}(I)) \) with \( |H| = \text{reg}(I) \), the intersection \( \bigcap_{\alpha \in H} B_\alpha \) is non-empty. Letting \( a \) be in this intersection, we see that for all \( \alpha < \beta \in H \):
Thus, we have an infinite descending sequence of ordinals, which is a contradiction. Therefore the induction must have terminated and the theorem follows. □

So in order to manufacture nice exact upper bounds, it suffices to produce a sequence which is neither Bad nor Ugly. Implicit in [4] is the fact that one can manufacture sequences with a property that is referred to as \((\ast)_\theta\) in [1], and furthermore this property is equivalent to Good\(_\theta\). For us, it will suffice to show one direction.

**Definition 2.10.** Let \(X\) be a set of ordinals, and let \(\vec{f} = \langle f_\xi : \xi \in X \rangle\) be a \(<_I\)-increasing sequence of functions from \(A\) to ON. We say that \(\vec{f}\) is strongly increasing if there are sets \(Z_\xi \in I\) for each \(\xi \in I\) such that, for any \(\eta < \xi \in X\), we have that \(f_\eta(a) < f_\xi(a)\) for all \(a \in A \setminus (Z_\eta \cup Z_\xi)\).

The idea behind strongly increasing sequences is that the sets \(Z_\xi\) serve as canonical witnesses that the sequence is \(<_I\)-increasing.

**Definition 2.11.** Let \(\lambda\) be a regular cardinal, and let \(\vec{f} = \langle f_\xi : \xi < \lambda \rangle\) be a sequence of \(<_I\)-increasing functions from \(A\) to ON. Letting \(\theta \leq \lambda\) be a regular cardinal, we say that \(\vec{f}\) satisfies \((\ast)_\theta\) if for every \(X \subseteq \lambda\) unbounded in \(\lambda\), there exists a set \(X_0 \subseteq \lambda\) of size \(\theta\) such that \(\langle f_\xi : \xi \in X_0 \rangle\) is strongly increasing.

We should note that satisfying \((\ast)_\theta\) for a sequence of functions is somewhat analogous to satisfying \(\text{reg}(I) \leq \theta\) for an ideal \(I\).

**Lemma 2.6.** Let \(\lambda > \text{reg}(I)\) be a regular cardinal, \(\vec{f} = \langle f_\xi : \xi < \lambda \rangle\) be a sequence of \(<_I\)-increasing functions from \(A\) to ON, and \(\theta \in [\text{reg}(I), \lambda] \cap \text{REG}\). If \(\vec{f}\) satisfies \((\ast)_\theta\), then \(\vec{f}\) is not Ugly.

**Proof.** Suppose otherwise, and let \(g : A \rightarrow \text{ON}\) witness that \(\vec{f}\) is Ugly. That is, letting \(t_\xi = \{a \in A : f_\xi(a) > g(a)\}\) for each \(\xi < \lambda\), the sequence \(\vec{t} = \langle t_\xi : \xi < \lambda \rangle\) does not stabilize modulo \(I\). So for each \(\xi < \lambda\), there is some \(\eta > \xi\) such that \(t_\eta \setminus t_\xi \notin I\). Using this, we can find an unbounded \(X \subseteq \lambda\) such that, for all \(\xi, \eta \in X\) with \(\xi < \eta\), we have \(t_\eta \setminus t_\xi \notin I\). Next, we use the fact that \(\vec{f}\) satisfies \((\ast)_\theta\) to fix a set \(X_0 \subseteq X\) of size \(\theta\) such that \(\langle f_\xi : \xi \in X_0 \rangle\) is strongly increasing as witnessed by \(Z_\xi\) for each \(\xi \in X_0\).

For each \(\xi \in X_0\), let \(\xi' = \min(X_0 \setminus (\xi + 1))\) be the successor of \(\xi\) in \(X_0\), and let
\[
A_\xi = (t_{\xi'} \setminus t_\xi) \cap (A \setminus (Z_{\xi'} \cup Z_\xi)).
\]
Note that $A_{\xi} \notin I$ for each $\xi \in X_0$, and so we can find some $H \subseteq X_0$ of size $\text{reg}(I) \leq \theta$ such that $\bigcap_{\xi \in H} A_{\xi} \neq \emptyset$. Let $a$ be in this intersection, and let $\xi, \eta \in H$ with $\xi < \eta$. Then we have that

$$g(a) \geq f_\eta(a) \geq f_\xi(a) > g(a).$$

Note that we get the first inequality from the fact that $a \notin t_\eta$, while the second inequality comes from the fact that $a \notin Z_0 \cup Z_{\xi'}$ with $\eta \geq \xi' > \xi$, and the final inequality comes from the fact that $a \in t_{\xi'}$. This gives us that $g(a) > g(a)$, which is of course a contradiction. □

**Lemma 2.7.** Let $\lambda > \text{reg}(I)$ be a regular cardinal, $\vec{f} = \langle f_\xi : \xi < \lambda \rangle$ be a sequence of $<_I$-increasing functions from $A$ to ON, and let $\theta \leq \lambda$ be regular such that $I$ is $\theta$-irregular. If $\vec{f}$ satisfies $(\ast)_\theta$, then $\vec{f}$ is not $\text{Bad}_\theta$.

*Proof.* Suppose otherwise, and let $S = \langle S(a) : a \in A \rangle$ with $|S(a)| < \theta$ and $D$ be an ultrafilter over $A$ disjoint from $I$ witness that $\text{Bad}_\theta$ holds. Let $X \subseteq \lambda$ be unbounded such that for all $\xi, \eta \in X$ with $\xi < \eta$, there is a function $h_\xi$ such that $f_\xi <_D h_\xi <_D f_\eta$. Using the fact that $\vec{f}$ satisfies $(\ast)_\theta$, let $X_0 \subseteq X$ be of size $\theta$ such that $\langle f_\xi : \xi \in X_0 \rangle$ is strongly increasing as witnessed by $Z_\xi \in I$ for each $\xi \in X_0$.

As before, for each $\xi \in X_0$, we let $\xi' = \min(X_0 \setminus (\xi + 1))$ be the successor of $\xi$ in $X_0$. For each $\xi \in X_0$, let $B_\xi = \{a \in A : h_\xi(a) < f_{\xi'}(a)\}$ and define

$$A_\xi = B_\xi \cap (A \setminus (Z_{\xi'} \cup Z_\xi)).$$

Note that $B_\xi \in D$, and so each $A_\xi$ is $I$-positive. So we can find some $H \subseteq X_0$ of size $\theta$ such that $\bigcap_{\xi \in H} A_\xi$, so let $a$ be in this intersection. Then for every $\xi, \eta \in X$ with $\xi < \eta$, we have that

$$h_\xi(a) < f_{\xi'}(a) \leq f_\eta(a) < h_\eta(a).$$

The first inequality follows from $a \in B_\xi$, while the third follows from the fact that $a \in B_\eta$. The second inequality comes from the fact that $\xi' \leq \eta$ and $a \notin Z_{\xi'} \cup Z_\eta$. But then the sequence $\langle h_\xi(a) : \xi \in X_0 \rangle$ is strictly increasing along $S(a)$ while $|S(a)| < \theta = |X_0|$ which is absurd. □

So for our purposes, it suffices to be able to construct sequences satisfying $(\ast)_\theta$ for appropriate $\theta$. We now quote a result from [1] which gives us conditions for constructing such sequences.

**Lemma 2.8.** Suppose that

1. $I$ is an ideal over $A$;
2. $\theta$ and $\lambda$ are regular cardinals such that $\theta^{++} < \lambda$;
(3) $\vec{f} = (f_\xi : \xi < \lambda)$ is a $<_I$-increasing sequence of functions from $A$ to $\text{ON}$ such that for every $\delta \in S^\lambda_{\theta^+}$, there is a club $E_\delta \subseteq \delta$ such that for some $\delta' < \delta$ in $\lambda$, we have

$$\sup \{ f_\alpha : \alpha \in E_\delta \} <_I f_{\delta'};$$

Then $(*)_\theta$ holds for $\vec{f}$.

It turns out that, while the above lemma looks technical, constructing sequences with the above properties is itself easy.

**Theorem 2.4.** Let $\lambda > \text{reg}(I)$ be a regular cardinal such that $\prod A/I$ is $\lambda$-directed, and let $\vec{f} = (f_\xi : \xi < \lambda)$ be any sequence of $<_I$-increasing functions in $\prod A$. Then there exists a sequence $\vec{g} = (g_\xi : \xi < \lambda)$ such that:

1. $\vec{g}$ is $<_I$-increasing;
2. for each $\xi < \lambda$, we have $f_\xi < g_{\xi+1};$
3. for every $\theta < \lambda$ regular such that $\theta^+ < \lambda$, $\{ a \in A : a \leq \theta^+ \} \in I$, and $I$ is $\theta$-irregular, we have that $\vec{g}$ is Good$_\theta$.

**Proof.** By Lemma 2.6 and Lemma 2.7, it suffices to produce a sequence which satisfies $(*)_\theta$ for every appropriate $\theta$. In other words, we only need to produce a sequence satisfying the last condition in Lemma 2.8. We proceed by induction on $\xi < \lambda$.

At stage 0, we simply let $g_0$ be any function in $\prod A/I$. At successor stages, suppose that $g_\xi$ has been defined and let $g_{\xi+1}$ be defined by

$$g_{\xi+1}(a) = \max \{ g_\xi(a), f_\xi(a) \} + 1.$$ 

At limit stages $\delta$, we have two cases to deal with. In the first case, we suppose that $cf(\delta) = \theta^+$ for $\theta$ as in condition (3), and let $E_\delta \subseteq \delta$ be club or order type $\theta^+$. Define

$$g_\delta = \sup \{ g_\xi : \xi \in E_\delta \},$$

and note that $g_\delta(a) < a$ whenever $a > \theta^+$ and so $g_\delta \in \prod A/I$. In the other case, simply let $g'_\delta$ be a $\leq_I$-upper bound of $\{ g_\xi : \xi < \delta \}$ and set $g_\delta = g'_\delta + 1$.

By construction, the sequence $\vec{g}$ satisfies the hypotheses of Lemma 2.8, and so we are finished. $\square$

This allows us to finish what we started in the previous section, and immediately derive the following theorem.

**Theorem 2.5 (The pcf Theorem).** If $I$ satisfies $\text{reg}(I) < |A|$, then for each $\lambda \in \text{pcf}_I(A)$, there exists a $B_\lambda \subseteq A$ such that $J^I_{\lambda^+} [A] = J^I_{\lambda^+} [A] + B_\lambda$. 
3. An Application: Pseudo Powers

Throughout this section, we will let \( \mu \) denote a singular cardinal with the property that \( \mu \neq \aleph_\mu \). We begin with a definition.

**Definition 3.1.** \( \text{PP}(\mu) \) is the collection of all cardinals \( \lambda \) of the form \( \lambda = \text{cf}(\prod A/D) \) where

1. \( A \subseteq \mu \cap \text{REG} \);
2. \( \sup A = \mu \);
3. \( D \) extends the ideal of bounded subsets of \( A \).

We say that \( \text{pp}(\mu) = \text{sup} \text{PP}(\mu) \).

One of the central ideas of [4] is that pseudo-powers (\( \text{pp}(\mu) \)) are the correct continuum operators. That is, normally \( 2^\mu \) is very sensitive to what is happening below \( \mu \), but \( \text{pp}(\mu) \) isn’t. In the case where \( \mu \) is strong limit singular, it turns out that \( \text{pp}(\mu) = 2^\mu \) and so the idea is that \( \text{pp}(\mu) \) carries only the relevant cardinal arithmetic information. We will not go into much detail on precisely what is happening here, but we would like to take the opportunity here to connect pseudo powers to what we have done with \( \text{pcf}_I(A) \). We begin by noting that if \( |A|^+ < \text{min}(A) \), then the trivial ideal \( I = \{\emptyset\} \) satisfies the hypothesis that \( \text{reg}(I) < \text{min}(A) \).

**Lemma 3.1.** If \( \mu \) is singular with \( \mu \neq \aleph_\mu \), then there is an interval \( A \) of regular cardinals below \( \mu \) such that \( \text{pp}(\mu) = \text{cf}(\prod A/J^{bd}[A]) \).

**Proof.** We first note that we may express

\[
\text{PP}(\mu) = \bigcup \{ \text{pcf}_{J^{bd}[A]}(A) : A \in \mathcal{P}(\mu \cap \text{REG}) \setminus J^{bd}[\mu \cap \text{REG}] \}. 
\]

Next note that, by Lemma 2.1, for every \( A \in \mathcal{P}(\mu \cap \text{REG}) \setminus J^{bd}[\mu \cap \text{REG}] \), we have that \( \text{pcf}_{J^{bd}[A]}(A) \subseteq \text{pcf}_{J^{sd}[\mu \cap \text{REG}]}(\mu \cap \text{REG}) \). Thus, we get that \( \text{PP}(\mu) = \text{pcf}_{J^{sd}[\mu \cap \text{REG}]}(\mu \cap \text{REG}) \). Now let \( A \) be an unbounded interval of regular cardinals in \( \mu \) such that \( |A|^+ < \text{min}(A) \) (here we use the fact that \( \mu \) is not an \( \aleph \)-fixed point) and note that \( A = J^{sd}[\mu \cap \text{REG}] \cap \mu \cap \text{REG} \). So again by Lemma 2.1 we get that

\[
\text{PP}(\mu) = \text{pcf}_{J^{bd}[\mu \cap \text{REG}]}(\mu \cap \text{REG}) = \text{pcf}_{J^{sd}[A]}(A).
\]

As \( J^{bd}[A] \) is \( |A|^+ \)-irregular (and weakly saturated), our previous work tells us that \( \max \text{pcf}_{J^{bd}[A]}(A) \) exists and is equal to \( \text{cf}(\prod A/J^{bd}[A]) \). But then

\[
\text{pp}(\mu) = \text{sup} \text{PP}(\mu) = \sup \text{pcf}_{J^{bd}[A]}(A) = \max \text{pcf}_{J^{bd}[A]}(A) = \text{cf}(\prod A/J^{bd}[A]).
\]

\[\square\]
The cov vs. pp theorem of [4] tells us more about how pseudo-powers relate to pcf numbers by way of covering numbers. It turns out that the above is weaker (in some sense) than the cov vs. pp theorem, but it will suffice for our purposes. For example, we can use the above to relate the existence of scales to failure/success of GCH (again, this is a known result).

**Lemma 3.2.** If \( \mu \) is a singular cardinal with \( \mu \neq \aleph_\mu \), then \( \text{pp}(\mu) = \mu^+ \) if and only if there is an interval of regular cardinals \( A \) unbounded below \( \mu \) such that \( \text{tcf}(\prod B/\mathbf{J}^{bd}[B]) = \mu^+ \) for every unbounded \( B \subseteq A \) (i.e. for every such \( B \), there is a sequence \( \vec{f} \) of functions in \( \prod B/\mathbf{J}^{bd}[B] \) such that \( (B, \vec{f}) \) is a scale).

**Proof.** The reverse direction is a direct consequence of the previous lemma, so we only need to show the forward direction. Suppose that \( \text{pp}(\mu) = \mu^+ \), then the above lemma tells us that there is an unbounded interval \( A \) of regular cardinals below \( \mu \) such that \( \text{cf}(\prod A/\mathbf{J}^{bd}[A]) = \mu^+ \). So let \( B \subseteq A \) be unbounded, and note that since \( \prod B/\mathbf{J}^{bd}[B] \) is \( \mu^+ \) directed, \( \max \text{pcf}_{\mathbf{J}^{bd}[B]}(B) \geq \mu^+ \). But by Lemma 2.1, this is in fact equal to \( \mu^+ \) and so \( \text{tcf}(\prod B/\mathbf{J}^{bd}[B]) = \mu^+ \). So let \( F = \{ g_\xi : \xi < \mu^+ \} \) be an enumeration of a cofinal subset of \( \prod B/\mathbf{J}^{bd}[B] \).

Inductively define a sequence \( \langle \vec{f}_\xi : \xi < \mu^+ \rangle \) which is \( <_{\mathbf{J}^{bd}[B]} \)-increasing in \( \prod B/\mathbf{J}^{bd}[B] \) and cofinal as follows. At stage 0, we let \( f_0 \) be any function which extends \( g_0 \). If \( f_\delta \) has been defined for each \( \delta < \xi \), let \( \vec{f}_\xi \) be a \( \leq_{\mathbf{J}^{bd}[B]} \)-upper bound of \( \{ g_\delta : \delta \leq \xi \} \cup \{ f_\delta : \delta < \xi \} \). Then \( \vec{f} = \langle f_\xi : \xi < \mu^+ \rangle \) witnesses that \( \text{tcf}(\prod B/\mathbf{J}^{bd}[B]) = \mu^+ \). □

**References**


