“Hello!” – Vladamir Uspenskiy (translated from Russian, this means “HOLD ON TO YOUR SEATS!”)

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1. Exam #1 Material

1.1. Homotopy.

Definition 1.1. Consider two function \( f_0 : X \rightarrow Y \) and \( f_1 : X \rightarrow Y \). Then \( f_0 \) and \( f_1 \) are homotopic (denoted \( f_0 \sim f_1 \)) if there is a function \( f_t : X \rightarrow Y \) with \( 0 \leq t \leq 1 \) such that \( f_t(x) \) depends continuously on \( I \times X \) (and where \( f_0(x) \) and \( f_1(x) \) are the given functions). That is, \( f_0 \sim f_1 \) if and only if there is a continuous function

\[ \phi : X \times [0,1] \rightarrow Y \]

such that for all \( x \in X \), we have \( f_0(x) = \phi(x,0) \) and \( f_1(x) = \phi(x,1) \).

Note 1.2. We informally say that \( f_0 \) and \( f_1 \) are homotopic if we can “deform one continuously into the other.”

Example 1.3. Let \( Y \) be a convex set in a normed linear space (e.g. a Euclidean space). Then for all \( f_0, f_1 : X \rightarrow Y \), we have \( f_0 \sim f_1 \). In essence, for each \( x \in X \), we parametrize the straight line between \( f_0(x) \) and \( f_1(x) \). See Figure 1. The explicit homotopy between \( f_0 \) and \( f_1 \) is given by

\[ \phi(x,t) = f_t(x) = (1-t)f_0(x) + tf_1(x). \]

![Figure 1. The homotopy between X and Y, where Y is convex in a normed linear space.](image)
Note 1.4. Homotopy is an equivalence relation. Reflexivity and symmetry are easy to see. Transitivity, on the other hand, relies on a long a complicated formula, but can be seen intuitively with a simple picture.

Definition 1.5. We denote by \([X,Y]\) the set of homotopic equivalence classes between X and Y.

Example 1.6. We have already shown in Example 1.3 that \(|[X,Y]| = 1\) if Y is convex. As we will argue later, \([Y,X]\) is also trivial if Y is convex.

Note 1.7. As a matter of notation, we denote by \(S^n\) to be the n-dimensional unit sphere in \(\mathbb{R}^{n+1}\). For example, \(S^1\) is the unit circle in \(\mathbb{R}^2\), \(S^2\) is the unit sphere in \(\mathbb{R}^3\), etc.

We note that \([S^n, S^n] \simeq \mathbb{Z}\). The integer assigned to each homotopy class is called the degree of that homotopy class.

Example 1.8. As in the previous Note, we have \([S^1, S^1] \simeq \mathbb{Z}\). (Think of \(S^1 = \{z \in \mathbb{C} : |z| = 1\} \).)

We can write the homotopy class corresponding to the integer n as \(f_n\). The canonical map for \(f_n\) is \(f_n(z) = z^n\). These represent all homotopy classes, and furthermore, \(\text{deg}(f_n) = n\). Returning to the idea of \(S^1\) being the unit circle in the complex plane, we see that the mapping \(f_n(z) = z^n\) sends \(z^n\) around the circle \(n\) times for every time \(z\) goes around the circle. If \(n > 0\) then the orientation is “positive” (thinking in terms of complex analysis) and if \(n < 0\) then the orientation is “negative.”

Given a map \(f_n : S^1 \to S^1\) by \(f_n(z) = z^n\), we note that \(f_n^{-1}(1)\) is the set of points that correspond to the vertices of a regular \(n\)-gon inscribed inside the unit circle. Each of these points is a root of unity.

Example 1.9. Given a map \(f : S^1 \to S^1\), there is a natural way to define a map \(\Sigma f : S^2 \to S^2\). First, fix the poles N and S (for North and South, respectively). Given a point \(z \in S^2\), the “latitude” stays fixed and the “longitude” changes according to the map \(f : S^1 \to S^1\). For instance, if I am on the equator, I will stay on the equator under the mapping \(\Sigma f\), but I may go to any other part of the world that also lies along the equator.

Definition 1.10. We discuss the category of topological spaces, denoted \(\text{Top}\). Objects in this category are, surprisingly enough, topological spaces. The morphisms (arrows) between objects are continuous mappings. This is the category we have been studying in general topology.

Definition 1.11. We also wish to discuss a new category, the homotopy category, denoted \(\text{HTop}\), where the objects are once again topological spaces, but now the morphisms are homotopic classes of continuous maps.

Definition 1.12. Given a category \(\mathcal{C}\), we say that \(X\) and \(Y\) are isomorphic objects if there exist continuous mappings \(f : X \to Y\) and \(g : Y \to X\) such that \(\text{Id}_X = g \circ f\) and \(\text{Id}_Y = f \circ g\). This composition is understood in the category.

Definition 1.13. We say \(X\) and \(Y\) are homotopically equivalent, denoted \(X \sim Y\), if they are isomorphic objects in \(\text{HTop}\).

Note 1.14. In the definition of isomorphic objects, we require \(\text{Id}_X = g \circ f\) and \(\text{Id}_Y = f \circ g\). However, we recall that two maps in \(\text{HTop}\) are the same if they belong to the same homotopy class. Therefore, we merely require \(\text{Id}_X \sim g \circ f\) and \(\text{Id}_Y \sim f \circ g\) for \(X\) and \(Y\) to be isomorphic objects in \(\text{HTop}\). This may happen even if \(X\) and \(Y\) are not homeomorphic, as the next example illustrates.
**Example 1.15.** Let $X$ be the 1-dimensional circle and $Y$ be the 2-dimensional annulus (the shaded region) given in Figure 2. We illustrate that these two spaces are isomorphic objects in the category $\text{HTop}$ (i.e. that they are homotopically equivalent). Let $f : X \to Y$ be given by $f(x) = x$, and let $g : Y \to X$ be given by the natural projection onto $X$, as seen in Figure 2. Then $g \circ f = Id_X$ and $f \circ g = h \sim Id_Y$, where the latter follows from the same idea as the convex argument given in Example 1.3 (that is, the homotopy is given by moving continuously along the straight line between the two points. This is an example of two objects that are isomorphic objects (i.e. homotopically equivalent) but are not homeomorphic.

**Figure 2.** Homotopically equivalent spaces that are not homeomorphic.

**Example 1.16.** We can essentially repeat the preceding argument for the case of a convex set in a normed linear space. Let $Y$ be any convex set in a normed linear space and let $X = \{p\}$ be a one-point space. Then $Y \sim X$. Let $f : X \to Y$ be given by $f(p) = p$ and $g : Y \to X$ be given by $g(y) = p$ for all $y \in Y$. Now $g \circ f = Id_X$ and $f \circ g = h : Y \to Y$, where $h \sim Id_Y$ by the previous argument.

**Definition 1.17.** We say that $Y$ is **homotopically trivial** or **contractible** if and only if $Y \sim \{p\}$.

**Example 1.18.** The unit circle $S^1$ is not contractible. In particular, any compact manifold ($\neq \{p\}$) without boundary is not contractible. For the $S^1$ case, there will be a problem point opposite the point $p$ on the circle. See Figure 3. This point poses a problem for contractibility since either possible path prevents the map from being continuous.

**Example 1.19.** We have already shown that if $Y$ is a compact subset of a normed linear space, then $[X,Y] = \{\ast\}$. But since $Y \simeq \{p\}$, this means that $[X,\{p\}] = \{\ast\}$.

**Example 1.20** (Application to graph theory). From Example 1.18, we see we run into problems regarding contractibility if there are “loops” in our objects. Thus, a connected graph is contractible if and only if it is a tree (i.e. no cycles).

**Note 1.21.** Let $X \subset Y$ and recall that a retraction $r : Y \to X$ is a continuous mapping for which $r|_X = Id_X$. Now consider again Example 1.15. Essentially we retracted $Y$ onto $X$. It is not true in general that a retraction guarantees a homotopic equivalence. For instance, every space can be retracted to the space $\{p\}$, but clearly every space is not homotopically equivalent to $\{p\}$.  

4
Figure 3. An illustration of why $S^1$ is not contractible.

**Definition 1.22.** Let $X$ be a topological space and let $A \subset X$. A map $r : X \rightarrow A$ is a retraction if $r$ is continuous and $r|_A = Id_A$. We say that $r$ is a deformation retraction if $r$ is a retraction and $r \sim Id_X$.

**Note 1.23.** Let $r : X \subset A$ be a deformation retraction. Applying the definition of a deformation retraction to our ideas of homotopy, we let $f_0 = Id_X$ and let $f_1 = r$. We have $f_t(x)$ for $0 \leq t \leq 1$, which depends continuously on $x$ and $t$. In particular, we have $f_0(x) = x$ and $f_1(x) = r(x) \in A$. See Figure 4.

Figure 4. A visualization of a deformation retraction in terms of homotopy.

**Lemma 1.24.** If $A$ is a deformation retract of $X$, then $X \sim A$.

**Proof.** This is essentially immediate from our definitions. For instance, consider the commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{i} & X \\
\downarrow{r} & & \\
A & \xrightarrow{i} & X
\end{array}
$$
where \( i : A \to X \) is the inclusion mapping (that is, \( i(a) = a \) for each \( a \in A \)), and \( r \) is the deformation retraction from \( X \) onto \( A \). Now we have \( r \circ i = Id_A \) and \( i \circ r = r \sim Id_X \).

**Note 1.25.** Lemma 1.24 applies to Example 1.15. In other words, the map \( g : Y \to X \) is a deformation retraction of \( Y \) onto \( X \), and so \( X \sim Y \). More generally, we always have \( X \sim X \times I \); that is, each space can be “fattened,” as the space \( X \) was in the example.

Also, note that if \( X \) is a deformation retract of a space \( Z \), and if \( Y \) is a deformation retract of \( Z \), then \( X \sim Z \sim Y \), and so \( X \sim Y \).

**Example 1.26.** Consider the space \( \mathbb{R}^2 \setminus \{A\} \) for some point \( A \in \mathbb{R}^2 \). Then the circle \( S^1 \) is a deformation retract of \( \mathbb{R}^2 \setminus \{A\} \), and so \( \mathbb{R}^2 \setminus \{A\} \sim S^1 \). In general, \( \mathbb{R}^n \setminus \{A\} \sim S^{n-1} \).

**Note 1.27.** What does it mean that \( \{p\} \) is a deformation retract of a space \( X \)? Consider the mappings \( f_0 : \{p\} \to \{p\} \) and \( f_1 : X \to X \), where \( f_0 \) is obviously an inclusion mapping and \( f_1 \) is a deformation retraction. Then by Lemma 1.24 we have \( X \sim \{p\} \), which is our definition of contractibility. Thus a space \( X \) is contractible if \( \{p\} \) is a deformation retract of \( X \).

**Note 1.28.** As mentioned in Example 1.20, we know that every tree is contractible. We make some observations without a rigorous proof.

1. Every tree can be thought of as a metric space.
2. Any two points on a tree have a unique shortest path between them.
3. Every point of the tree is a deformation retract of the tree, where the deformation retraction is given to be where points move through the shortest path.

**Definition 1.29.** A space \( (X, x_0) \), for \( x_0 \in X \), is a **pointed space** with base point \( x_0 \).

**Definition 1.30.** For the disjoint sum \( (X, x_0) \oplus (Y, y_0) \) of pointed spaces, the **wedge sum** (or simply **wedge**) of \( X \) and \( Y \), denoted \( X \lor Y \), is an equivalence relation on \( X \) and \( Y \) where the only non-trivial class is \( \{x_0, y_0\} \). That is, the wedge is a quotient space where the base points \( x_0 \) and \( y_0 \) are identified.

**Example 1.31.** Let \( \mathcal{E} \) denote the wedge of two circles; that is, \( \mathcal{E} = S^1 \lor S^1 \). This is the figure-eight shape given in Figure 5(a). Note, in particular, that \( \mathcal{E} \sim \mathbb{R}^2 \setminus \{A, B\} \) for distinct points \( A, B \in \mathbb{R}^2 \setminus \{A, B\} \) because \( \mathcal{E} \) is a deformation retract of \( \mathbb{R}^2 \setminus \{A, B\} \). Let \( \mathcal{F} \) denote the object in Figure 5(b). According to these pictures, we see that both \( \mathcal{E} \) and \( \mathcal{F} \) are deformation retracts of the space \( \mathbb{R}^2 \setminus \{A, B\} \), and in fact

\[
\mathbb{R}^2 \setminus \{A, B\} \sim \mathcal{E} \sim \mathcal{F}.
\]

**Note 1.32.** For any graph \( G \) and any tree \( T \subset G \), if we collapse \( T \) to a single point, we do not change the homotopic type of \( G \).

In fact, every connected graph \( G \) has a **spanning sub-graph** (that is, a connected sub-graph that includes all vertices of \( G \)) that is a tree, also known as a **spanning tree**. Such a maximal spanning tree is guaranteed by Zorn’s Lemma. Since the spanning tree contains every vertex of \( G \), when we collapse the spanning tree to a single vertex, the entire graph will be collapsed to a single vertex. The remaining edges will form several possible loops from the remaining vertex to itself. This is called a **bouquet of circles**. What is preserved is the number of loops, which is a reflection of the homotopy type.
1.2. The Brouwer Fixed Point Theorem.

Definition 1.33. Let $X$ be a topological space. Then the cone of $X$, denoted $CX$, is the quotient

$$CX = X \times I / X \times \{1\}.$$ See Figure 6(a) for a visualization of the cone of $X$.

The following result gives equivalent definitions for contractibility. Some of these we either have established or will establish (cf. Lemma 1.37), and some we will not.

Lemma 1.34. The following are equivalent for a space $X$.

1. The space $X$ is contractible.
2. $X \sim \{p\}$.
3. There exists a continuous function $\phi : X \times I \to X$ such that $f_0(x) = \phi(x, 0) = x$ and $f_1(x) = \phi(x, 1) = p \in X$. (Not sure about this one...maybe copied down wrong.)
4. There exists a continuous function $g : CX \to X$ such that $g|_X = \text{Id}_X$.
5. There exists a retraction $r : CX \to X$.
6. $\text{Id}_X \sim \text{constant}$.

Example 1.35. What is $C(S^{n-1})$? In the case of $n = 2$, it is an actual cone, as depicted in Figure 6(a). Note also that we can characterize $B^2$ as follows, seen in Figure 6(b):

$$B^2 = \{(x, t) : x \in S^1, 0 \leq t \leq 1\}.$$ Note that $(x, 1) \sim (y, 1)$ for all $x, y \in S^1$ in this characterization, and so we conclude that $B^2 \simeq C(S^1)$. In general, it is true that $B^n \simeq C(S^{n-1})$.

Lemma 1.36. Let $f : S^{n-1} \to X$ be continuous. Then the following are equivalent:
Figure 6. (a) The cone \( CX \) of an arbitrary space \( X \). (b) A visual depiction of \( B^2 \).

(1) \( f \sim \text{constant} \).
(2) \( f \) has a continuous extension over \( B^n \); that is, there is a continuous function \( \overline{f} : B^n \to X \) such that \( \overline{f}|_{S^{n-1}} \equiv f \).

Proof. We have \( f \sim \text{constant} \) if and only if there is a continuous function \( \phi : S^{n-1} \times [0, 1] \to X \) with \( \phi(x, 0) = f(x) \) and \( \phi(x, 1) = p \in X \) for all \( x \in S^{n-1} \). Now we can describe \( B^n \) as
\[
B^n = \{(x, t) : x \in X^{n-1}, 0 \leq t \leq 1\}
\]
as in Example 1.35, where \( (x, 1) \sim (y, 1) \) for all \( x, y \in S^{n-1} \). Thus we let \( \overline{f} : B^n \to X \) be defined by \( \overline{f}(x, t) = \phi(x, t) \), which is well-defined since \( \phi(x, 1) = p \) for all \( x \in S^{n-1} \) (i.e. for all members of the equivalence class \( S^{n-1} \times \{1\} \)). It is now clear that \( \overline{f} \) satisfies the properties of being a continuous extension of \( f \) over \( B^n \) since \( S^{n-1} = \{(x, 0) : x \in S^{n-1}\} \).

Lemma 1.37. A space \( X \) is contractible if and only if \( \text{Id}_X \sim \text{constant} \).

Proof. First, suppose \( X \) is contractible. Then \( X \sim \{p\} \) for some \( p \in X \). This means that there are continuous maps \( f : X \to \{p\} \subset X \) and \( g : \{p\} \to X \) such that \( \text{Id}_X \sim g \circ f \) and \( \text{Id}_{\{p\}} \sim f \circ g \). Note that \( g \circ f \) is a constant map, and so this direction is established.

Conversely, suppose \( \text{Id}_X \sim \text{constant} \). Let \( r : X \to \{p\} \subset X \) be the constant map for which \( \text{Id}_X \sim r \). Now clearly \( r \) is continuous since \( r^{-1}(U) = X \) or \( \emptyset \), depending on whether \( p \in U \). Also, we have \( r|_{\{p\}} = \text{Id}_{\{p\}} \), and so \( r \) is a retraction. Now by assumption since \( r \sim \text{Id}_X \), it follows that \( r \) is a deformation retraction, and by Lemma 1.24 it follows that \( X \sim \{p\} \), so \( X \) is contractible.

Theorem 1.38. The following are equivalent:
(1) \( S^{n-1} \) is not contractible.
(2) \( S^{n-1} = \partial B^n \) is not a retract of \( B^n \).
(3) (Brouwer Fixed Point Theorem.) For every continuous map \( f : B^n \to B^n \) there is an \( x \in B^n \) such that \( f(x) = x \).

Proof. Note that by Lemma 1.37 we see that \( S^{n-1} \) is contractible if and only if \( \text{Id}_{S^{n-1}} \sim \text{constant} \). Thus, let \( f : S^{n-1} \to S^{n-1} \) be the identity map. We apply Lemma 1.36 to show that \( \text{Id}_{S^{n-1}} = f \sim \text{constant} \) if and only if there is a continuous function \( r : B^n \to S^{n-1} \) such that \( r|_{S^{n-1}} \equiv f = \text{Id}_{S^{n-1}} \); that is, \( S^{n-1} \) is contractible if and only if \( S^{n-1} \) is a retract of \( B^n \). Hence, (1) \( \iff \) (2).

We thus need to show (2) \( \iff \) (3). Suppose (3) holds, and suppose to the contrary that \( r : B^n \to S^{n-1} \) is a retraction. Then take \( -r : S^{n-1} \to S^{n-1} \) where \( x \mapsto -r(x) \), which is a continuous mapping. Then \( -r \circ r : B^n \to B^n \) is a continuous mapping with no fixed points, contradicting (3). Hence, we have (3) \( \Rightarrow \) (2).

Conversely, suppose (2) holds and suppose to the contrary that (3) fails. Thus we may find a continuous function \( f : B^n \to B^n \) for which \( f(x) \neq x \) for all \( x \in B^n \). We define a map

\[
r : B^n \to S^{n-1}
\]

where \( r(x) \) is the point on \( S^{n-1} \) that is co-linear with \( x \) and \( f(x) \), where \( x \) is between \( f(x) \) and \( r(x) \). See Figure 7. This map is well-defined because \( f(x) \neq x \) for any \( x \in B^n \). Furthermore, it easily follows that \( r|_{S^{n-1}} = \text{Id}_{S^{n-1}} \) and also that \( r \) is continuous. (For continuity, note that since \( f \) is continuous, we know that if \( x \) changes a little, then \( f(x) \) will change a little, and consequently \( r(x) \) will only change a little.) Hence, \( r \) is a retraction of \( B^n \) onto \( S^{n-1} \), contradicting (2). Thus (2) \( \Rightarrow \) (3). This completes the proof. \( \square \)

Now that we have established Theorem 1.38, we simply need to show that \( S^{n-1} \) is not contractible to establish the Brouwer Fixed Point Theorem. We do this in a number of ways.

Theorem 1.39. \( S^n \) is not contractible.

Note 1.40. We present four proofs, each of which has details missing and none of which do we feel very comfortable with.
Proof 1 (Theorem 1.39). Recall that \([S^n, S^n] \simeq \mathbb{Z}\) where \(f \mapsto \deg(f)\). For \(f, g \in [S^n, S^n]\), we have \(f \sim g\) if and only if \(\deg(f) = \deg(g)\). Note that \(\deg(\text{Id}_{S^n}) = 1\) and \(\deg(\text{constant}) = 0\). Therefore, we have \(\text{Id}_{S^n} \not\sim \text{constant}\), and hence \(S^n\) is not contractible by Lemma 1.37. □

Note 1.41. Let \(X\) be any topological space. Then we can define an object \(H_n(X)\) for \(n \geq 0\), called the homology group of \(X\), which is an abelian group. We do not define this group here, but merely indicate some of its properties. Given a map \(f : X \to Y\), we have another map \(f_* : H_n(X) \to H_n(Y)\), which is a homomorphism of abelian groups. If also \(g : X \to Y\) with \(f \sim g\), then \(f_* = g_*\).

Example 1.42. We have \(H_n(S^n) \simeq \mathbb{Z}\) and \(H_n(\{p\}) = \{0\}\), the trivial group, for \(n > 0\). This will form another proof that \(S^n\) is not contractible.

Proof 2 (Theorem 1.39). Let \(f = \text{Id}_{S^n} : S^n \to S^n\), and so \(f_* : H_n(S^n) \to H_n(S^n)\) is the identity homomorphism from \(\mathbb{Z} \to \mathbb{Z}\). Now let \(g : S^n \to \{p\} \hookrightarrow S^n\) be the constant map. Then \(g_* : H_n(S^n) \to H_n(\{p\}) \hookrightarrow H_n(S^n)\) must pass through the trivial group, and so cannot be the identity homomorphism. Hence, \(f_* \neq g_*\), and in particular, \(f \not\sim g\); that is, the identity map on \(S^n\) is not homotopic to a constant map, and so by Lemma 1.37 it follows that \(S^n\) is not contractible. □

Proof 3 (Theorem 1.39). Suppose we have a smooth (i.e. infinitely differentiable) retraction \(r : B^n \to S^{n-1}\) with \(r|_{S^{n-1}} = \text{Id}_{S^{n-1}}\). Let \(x \in S^{n-1}\) be a regular value. Then \(r^{-1}(x)\) is a 1-dimensional compact manifold with boundary. Now a theorem (cf. appendix in Milnor) states that each compact 1-dimensional manifold must be either a circle or an interval. See Figure 8. The endpoint referred to by the question mark in Figure 8 gives the contradiction since one end point is on the circle and the other is not. Indeed, the other endpoint cannot be on the circle since we require the retraction to be the identity on the circle. This proof has a lot of missing details, such as needing Sard’s Theorem (cf. Milnor) to show the existence of regular values. □

Proof 4 (Theorem 1.39). We argue there is no piecewise linear retraction of the triangle onto the boundary. We can then pass to the case of the circle. Suppose to the contrary there is a retraction
$r : T \rightarrow \partial T$ from the triangle $T$ onto its boundary. Then for $x \in \partial T$ for which $r^{-1}(x)$ does not coincide with any vertices of the triangulation of $T$, we see that $r^{-1}(x)$ is the union of “lines” inside the triangle. See Figure 9. In Figure 9(a), we have the problem of $r^{-1}(x)$ containing more than one point on the boundary, violating the retraction being the identity on the boundary. In Figure 9(b), we have the problem that the inverse image $r^{-1}(x)$ contains a vertex of the triangulation; in the picture, this does not seem to be the case, but if three segments of $r^{-1}(x)$ meet, that means that three distinct triangles of the triangulation share a common point on their boundary. This can only happen if this common point is a vertex of the triangulation. In Figure 9(c), we have the problem that $r^{-1}(x)$ cannot stop somewhere since the border of two adjacent triangles shares this point; since the map is piecewise linear, there would be a segment of $r^{-1}(x)$ in the bordering triangle as well under the triangulation. All cases lead to contradictions, so we are done. \[\square\]
1.3. The Fundamental Group and Winding Numbers. We now assume all spaces are path-wise connected.

**Definition 1.43.** Let \((X, x_0)\) be a pointed space. Then the **fundamental group** of \((X, x_0)\), denoted \(\pi_1(X, x_0)\), is the “space of loops.” More formally, let \(f_0 : [0, 1] \to X\) and \(f_1 : [0, 1] \to X\) be such that \(f_0(0) = f_0(1) = x_0\) and \(f_1(0) = f_1(1) = x_0\) be loops starting and ending at \(x_0\). Then \(f_0 \sim f_1\) if and only if for all \(0 \leq t \leq 1\) we have a map \(f_t : [0, 1] \to X\) with \(f_t(0) = f_t(1) = x_0\). (This condition is in place to ensure that \(f_t\) is a loop for all \(t \in [0, 1]\).) For \(0 \leq s \leq 1\), we may think of this as \((t, s) \mapsto f_t(s)\), so that we have a continuous map \([0, 1] \times [0, 1] \to X\). Informally, we think of “deforming one loop continuously into the other.” Then \([f_0]\) is an equivalence class under homotopy. Then

\[
\pi_1(X, x_0) = \{[f] : f \text{ is a loop at } x_0\}.
\]

**Note 1.44.** The fundamental group is a group in the algebraic sense. We describe how the group axioms are satisfied. Note, however, that to perform these operations, we need to re-parametrize to fit the operation into a function defined on \([0, 1]\); for instance, the elements \(f\) and \(g\) are both defined on \([0, 1]\), and so should the composition \(f \ast g\) (where we use \(f\) first and then \(g\)), so in order for this to happen, we make \(f\) and \(g\) both go “twice as fast.” That is, re-parametrize \(f\) to \([0, 1/2]\) and \(g\) to \([1/2, 1]\). Now clearly \(\pi_1(X, x_0)\) is closed under this operation because \(f(0) = x_0\) and \(g(1) = x_0\), so \(f \ast g\) is a loop.

For associativity, we note that \((f \ast g) \ast h\) clearly corresponds to the same path as \(f \ast (g \ast h)\). Indeed, the only difference is the re-parametrization, as in the former we see that \(f\) is defined on \([0, 1/4]\), \(g\) is defined on \([1/4, 1/2]\), and \(h\) is defined on \([1/2, 1]\), whereas in the latter \(f\) is defined on \([0, 1/2]\), \(g\) is defined on \([1/2, 3/4]\), and \(h\) is defined on \([3/4, 1]\).

The identity loop is \(e : [0, 1] \to x_0\), which is the constant map. Again, the only difference between \(f \ast e\), \(e \ast f\), and \(f\) is the parametrization.

The inverse of \(f\) is traveling along \(f\) with the opposite orientation. If \(f : [0, 1] \to X\) is a loop, then \(f^{-1} : [0, 1] \to X\) is a loop given by \(f^{-1}(t) = f(1-t)\). We can then describe a homotopy \(f \ast f^{-1} \sim f^{-1} \ast f \sim e\), the identity.

We now consider examples of the fundamental group for certain spaces.

**Example 1.45.** Let \(s_0 \in S^1\) and note that \(\pi_1(S^1, s_0) \simeq \mathbb{Z}\). The equivalence classes of loops are characterized by the number of turns around the circle (with orientation), which is why this fundamental group is isomorphic to \(\mathbb{Z}\).

**Example 1.46.** Consider again the wedge \(X = S^1 \cup S^1\); i.e., the figure-eight space seen in Figure 10. Now \(a\) and \(b\) are the loops in the positive orientation, so \(a^{-1}\) and \(b^{-1}\) are loops in the negative orientation. In this case, \(\pi_1(X, s_0) \simeq F_2\), the free group on 2 letters.

**Example 1.47.** Note that \(\pi_1(S^n, x_0) \simeq \{1\}\), the trivial group, for \(n > 1\), because any loop can be continuously deformed into the identity (i.e. constant) loop at \(x_0\).

**Definition 1.48.** A space \(X\) is called **simply connected** if \(X\) is path-connected and \(\pi_1(X, x_0) = \{1\}\).

**Lemma 1.49.** If \(X\) is path-connected and \(x_0 \neq x_1\), then \(\pi_1(X, x_0) \simeq \pi_1(X, x_1)\).
Figure 10. The fundamental group of $S^1 \vee S^1$.

Proof. This proof is visual. See Figure 11. There is no canonical isomorphism between the fundamental groups, but we see intuitively from the picture that you form a path between $x_0$ and $x_1$ and then form loops for $x_0$ that move through $x_1$. □

Example 1.50. Note that $\mathbb{R}^3 \setminus \{\text{line}\} = (\mathbb{R}^2 \setminus \{\text{point}\}) \times \mathbb{R} \sim S^1$, and so $\pi_1(\mathbb{R}^3 \setminus \{\text{line}\}, x_0) \simeq \mathbb{Z}$.

Example 1.51. Note that $\mathbb{R}^3 \setminus \{\text{point}\} \sim S^2$, and so $\pi_1(\mathbb{R}^3 \setminus \{\text{point}\}, x_0) = \{1\}$.

Let $O$ be the origin and let $A = (r_1, \theta_1), B = (r_2, \theta_2) \in \mathbb{R}^2$. Furthermore, let $\gamma : [a, b] \to \mathbb{R}^2$ be a curve such that $\gamma(a) = A$ and $\gamma(b) = B$. We seek to define the angle from the ray $OA$ to the ray $OB$ along the curve $\gamma$. Note that this is not difficult if the curve $\gamma$ lies entirely inside a certain half-plane. See Figure 12. The problem occurs when $\gamma$ may loop around the origin several times, for instance.

Definition 1.52. In the previous discussion, we noted that there was no problem computing the angle traversed by a curve $\gamma$ supposing that $\gamma$ lies entirely in a half-plane. Therefore, given an arbitrary curve $\gamma : [a, b] \to \mathbb{R}^2 \setminus \{0\}$, form a partition $P$ of $[a, b]$ where

$$P = \{a = t_1 < t_2 < \cdots < t_n = b\}$$
where $\gamma : [t_i, t_{i+1}] \to \mathbb{R}^2 \setminus \{0\}$ lies in some open half-plane and where the angle between $\gamma(t_i)$ and $\gamma(t_{i+1})$ is $\theta_i$ for each $i \in \{1, \ldots, n-1\}$. Then we define the angle from $OA$ to $OB$ along $\gamma$ to be

$$\alpha_{A,B,\gamma} = \sum_{i=1}^{n-1} \theta_i.$$ 

The winding number; that is, the number of times $\gamma$ travels around the origin $O$, is defined to be $W(\gamma, O) = \frac{1}{2\pi} \alpha_{A,B,\gamma}$.

**Example 1.53.** For the unit circle on the interval $[0, 2\pi]$ traveling from $A = (1, 0)$ back to itself, we must partition $[0, 2\pi]$ into three intervals to ensure each curve is an open half-plane. Therefore, let us use the partition $\left\{ 0 < \frac{2\pi}{3} < \frac{4\pi}{3} < 2\pi \right\}$. From Figure 13, we see this makes $\alpha_{A,A,\gamma} = 3 \cdot \frac{2\pi}{3} = 2\pi$. 

**Figure 12.** The curve $\gamma$ connecting $A$ and $B$ in $\mathbb{R}^2$.

**Figure 13.** Calculating the angle around the unit circle.
**Definition 1.54.** Two maps $\gamma : [a, b] \to X$ and $\gamma' : [a, b] \to X$ are **homotopic with respect to endpoints** (still denoted $\gamma \sim \gamma'$) if $\gamma(a) = \gamma'(a) = A$ and $\gamma(b) = \gamma'(b) = B$, and we can continuously deform $\gamma$ into $\gamma'$ by a continuous map $\phi : [a, b] \times [0, 1] \to X$, given by $(x, t) \mapsto \gamma_t(x)$, where $\gamma_0 = \gamma$ and $\gamma_1 = \gamma'$, and where $\gamma_t(a) = A$ and $\gamma_t(b) = B$ for all $t \in [0, 1]$.

**Note 1.55.** Note that the homotopy described in the fundamental group is homotopy with respect to endpoints. This is a special case where the endpoints are the same.

**Lemma 1.56.** If $\gamma : [a, b] \to \mathbb{R}^2 \setminus \{0\}$ and $\gamma' : [a, b] \to \mathbb{R}^2 \setminus \{0\}$ are two curves connecting $A$ and $B$ for which $\gamma \sim \gamma'$ with respect to endpoints, then $W(\gamma, 0) = W(\gamma', 0)$.

**Proof.** It is sufficient to show that the map $t \mapsto W(\gamma_t, 0)$ is a continuous map; indeed, since $W(\gamma_t, 0)$ is a discrete set, showing this map is continuous will show that this map is constant, meaning that $W(\gamma, 0) = W(\gamma_0, 0) = W(\gamma_1, 0) = W(\gamma', 0)$. However, continuity is clear for the following reason. Since $\phi : [a, b] \times [0, 1] \to \mathbb{R}^2 \setminus \{0\}$ is a continuous map, changing $t$ by just a little will change the function $\gamma_t$ by just a little. Consequently, the angle traversed around 0 will change just a little, making the map $t \mapsto W(\gamma_t, 0)$ continuous. \hfill $\square$

**Note 1.57.** The previous result is actually an if and only if statement, but the other direction is more difficult to prove. We defer this direction until later.

**Note 1.58.** Consider Figures 14(a) and 14(b), the graphs of closed curves in $\mathbb{R}^2 \setminus \{0\}$. Note that

![Figure 14](image.png)

**Figure 14.** The winding number does not change for any point inside of a connected component of $\mathbb{R}^2 \setminus \Gamma$, where $\Gamma$ is the graph of a closed curve $\gamma$ in $\mathbb{R}^2$.

the winding number does not change for any other point inside each connected component of $\mathbb{R}^2 \setminus \Gamma$, where $\Gamma = \{(x, \gamma(x)) : x \in [a, b]\}$ is the graph of $\gamma : [a, b] \to \mathbb{R}^2 \setminus \{0\}$. This rule is true in general. In fact, the winding number changes only by 1 for any “bordering” connected components; that
is, connected components that lie on opposite sides of the curve from each other, not requiring to pass through a point of self-intersection to get from one component to the other. To see why this is, consider that we can move $p$ and $q$ arbitrarily close together while leaving the curve $\gamma$ between the two points, as in Figure 15(c). We can separate $\gamma$ into two segments: a small segment that approaches a line near $p$ and $q$ (as pictured in the figure), which we will call $\gamma_0$, and the rest $\gamma_1 = \gamma \setminus \gamma_0$. Now as $p$ and $q$ move closer to each other, the angle approaches $\pi$ for $p$ and $-\pi$ for $q$ on $\gamma_0$, as depicted in the figure. The rest of the curve $\gamma_1$ will have very similar angle changes for $p$ and $q$ since we move $p$ and $q$ arbitrarily close to each other. Hence, the winding number will only be affected by the angle change on $\gamma_0$, and this difference is $2\pi$ (or $-2\pi$, depending on the orientation of $\gamma$). Thus, when dividing by $2\pi$, the winding number will be either 1 greater or 1 less when we move from $p$ to $q$ across $\gamma$.

**Example 1.59.** For another example, see Figure 16 and note that the rule for changing winding numbers changes appropriately as we move out of the curve through the indicated ray; that is, the rule is consistent with our requirement that connected components preserve the winding number.

![Figure 15](image15.png)

**Figure 15.** (a) Another example of a winding number by connected component for a closed curve. (b) The rule governing whether to add 1 or subtract 1, depending on the orientation. (c) A visual explanation why bordering components differ in winding number by 1.

![Figure 16](image16.png)

**Figure 16.** The rules for governing the change in winding number preserves the winding number on connected components.
**Note 1.60.** So far we have considered only closed curves, but we may consider non-closed curves as well. Let $\gamma : [a, b] \to \mathbb{R}^2 \setminus \{0\}$ be a curve without the requirement that $\gamma(a) = \gamma(b)$. As in the case when we defined the angle between $\gamma(a)$ and $\gamma(b)$, we form a partition $P$ of $[a, b]$ by

$$P = \{a = t_1 < t_2 < \cdots < t_n = b\}$$

where the graph of $\gamma|_{[t_i, t_{i+1}]}$ lies entirely in some open half-plane for every $i \in \{1, \ldots, n + 1\}$. Now we want to parametrize $\gamma$ by the rule $\gamma(t) = (r(t), \theta(t))$, but as before, we have a problem that $\theta : [a, b] \to \mathbb{R}$ may not be continuous. However, we can adjust the values of $\theta(t)$ by integer multiples of $2\pi$ if necessary to make this function continuous. Thus, assuming we have made this adjustment for $\theta$ to make it continuous, we can reformulate $\gamma$ by the rule $\gamma(t) = r(t)e^{i\theta(t)}$, which is now continuous.

**Note 1.61.** Consider the exponential-related function $E : \mathbb{R} \to S'$ given by

$$\theta \mapsto e^{i\theta} = \cos \theta + i \sin \theta = (\cos \theta, \sin \theta).$$

Now suppose we also have a continuous function $\gamma : [a, b] \to S^1$. We argue that there is a continuous mapping such that the following diagram commutes.

![Diagram](image)

We show how to construct this function $g$. See Figure 17. For each segment of our partition

**Figure 17.** The construction of the continuous function $g : [a, b] \to \mathbb{R}$.

$P = \{a = t_1 < t_2 < \cdots < t_n = b\}$, we choose a segment of $\mathbb{R}$ that maps to $\gamma([t_i, t_{i+1}]) \subseteq S^1$. Note that there are countably many choices. We do this for each sub-interval, and then after that each selection may need to be adjusted by an integer multiple of $2\pi$ to maintain continuity, much as $\theta$ was adjusted in Note 1.60. This illustrates our next definition.
**Definition 1.62.** For a map $p : X \to Y$ and $U \subset Y$, we say that $U$ is **evenly covered by** $p$ if $p^{-1}(U) \subset X$ can be written as a union of pairwise disjoint open pieces $p^{-1}(U) = \bigcup_{\beta \in B} V_{\beta}$, where for all $\beta \in B$, the map $p|_{V_{\beta}} : V_{\beta} \to U$ is a homeomorphism. See Figure 18.

![Diagram](image.png)

**Figure 18.** An illustration of $U \subset Y$ being evenly covered by $p : X \to Y$.

**Example 1.63.** In Note 1.61, any open interval segment of $S^1$ is evenly covered by $E$, as intuitively seen in Figure 17.

**Definition 1.64.** A map $p : X \to Y$ is a **covering map** if $Y = \bigcup_{\alpha \in A} U_{\alpha}$, where $\{U_{\alpha}\}_{\alpha \in A}$ is a family of open subsets in $Y$ and where $U_{\alpha}$ is evenly covered by $p$ for all $\alpha \in A$. 

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