Sensitivity Analysis

Today: some cases
The rest: after introducing Duality Theory

Changing coefficients of objective fun.
\[ \text{max } C^T x \]
\[ \text{st } \ A x = b \]
\[ x \geq 0 \]

- Opt: soln: \( x^*, z^* \)
- Suppose \( c_j \) is changed to \( \overline{c}_j \)
- Question:
  - Is feasibility of \( x^* \) changed?
    - \( \text{No} \) because the feasible region is the same.
  - Is optimality of \( x^* \) changed?
    - Can't answer immediately depends on the magnitude of changes.

**Graphically:**

depends on the slope of isoprofit lines

- Thus, the new coefficients of non-basic variables in row (0):

\[
C^b_{\overline{c}} B^* N - C^b_c
= (1 + s, 1)(2 \overline{s} - \frac{1}{2}s)(1 0) - (0, 0)
= (\frac{1}{3} + \frac{3}{2}s, \frac{1}{6} - \frac{1}{2}s)
\]

- The coefficients should be non-negative to maintain optimality:

\[
\begin{align*}
\frac{1}{3} + \frac{3}{2}s & \geq 0 \\
\frac{1}{6} - \frac{1}{2}s & \geq 0 \\
\end{align*}
\]

- If \( s \in [\frac{1}{3}, \frac{2}{3}] \) then \( x^* = (\frac{1}{2}) \) is still optimal.
- Otherwise it is not optimal.
- Get new row (0) and reoptimize.

Note: reoptimize starting from the final tableau of the original problem. By this, we might save a lot of iterations.

**Example:**
max \( (1, 1) x \)
\[ \text{st } \begin{align*}
2x_1 + 1x_2 & \leq 6 \\
3x_1 + 4x_2 & \leq 12 \\
\end{align*} \]
\( x_1, x_2 \geq 0 \)

- What if \( c = (1) \) changes to \( (1 + s) \)?
- \( x^* = (\frac{1}{2}) \) still be optimal?
- Only row (0) in (\( \star \)) is affected by this change.

Recall that current basis is \( B = \{2, 2\} \).
New \( C^b_{\overline{c}} = (1 + s, 1) \)
\[ B^* = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} ; \quad N = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

- \( -\frac{1}{2} \leq s \leq 1 \)
- \( -1 - \frac{1}{2} \leq s \leq 1 + s \)
- \( x^* = (\frac{1}{2}) \) is still optimal if \( c_b \) is in the range: \( [\frac{1}{2}, 2] \).

This is called allowable range to stay optimal for \( c_b \).
- Even if \( x^* \) is still optimal, \( z^* \) might be changed.
- Compute new \( z^* \) by the expression from (\( \star \)):
New \( z^* = C^b_{\overline{c}} B^* b = (1 + s, 1) \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 4 + 2s \)

More observations:
- If a coefficient of a non-basic variable \( x_j \) is changed then need to recompute only \( x_j \)'s coefficient in the final tableau.
- Can do sensitivity analysis also when several coefficients are changed simultaneously.