Phase 1 for Network Simplex

- How to get initial BFS?
  - Add an artificial node T.
  - b_T = 0 (no demand or supply for T)
  - Add artificial arcs from any supply node to T: A → T, B → T.
  - Add artificial arcs from T to any demand node: T → C, T → D, T → E.
  - The costs of original arcs are set to 0.
  - The costs of artificial arcs are set to 1.

- The artificial problem defined this way is Min-Cost Flow problem with obvious initial BFS:
  - \( x_{ij}^* = b_i \) for any supply node \( i \)
  - \( x_{ij}^* = -b_j \) for any demand node \( j \)
  - \( x_{ij}^* = 0 \) for all other arcs.

2) Suppose the original problem is infeasible.

- Then the optimal soln of the artificial problem will contain artificial arcs with positive flow.
- The reverse is also true:
  - If in the optimal soln of the artificial problem there are artificial arcs with positive flow then the original problem is infeasible.

Sensitivity Analysis for Network Simplex

Recall our problem:

\[
\begin{align*}
\text{Maximize} & \quad z = c_{12} x_{12} + c_{13} x_{13} + c_{23} x_{23} + c_{24} x_{24} + c_{31} x_{31} \\
\text{Subject to} & \quad x_{12} + x_{13} = b_1 \\
& \quad x_{23} + x_{24} = b_2 \\
& \quad x_{31} = b_3 \\
& \quad x_{ij} \geq 0 \quad \forall i,j
\end{align*}
\]

and its optimal soln:

\[
\begin{align*}
x_{12} &= 10 \\
x_{13} &= 0 \\
x_{23} &= 30 \\
x_{24} &= 50 \\
x_{31} &= 20
\end{align*}
\]

2) What if \( c_{ij} \) is changed for some \( i \rightarrow j \)?

2 cases:

1) \( i \rightarrow j \) is a nonbasic arc.
- Find allowable range to stay optimal for \( c_{ij} \).
- In this case, node potentials \( \pi_i^* \)'s of the optimal soln are the same.
  - Just need to ensure that \( \pi_i^* - \pi_j^* \leq c_{ij} \)

Ex: Let \( i \rightarrow j = A \rightarrow D \). Then

\[
\begin{align*}
\pi_A^* &- \pi_D^* \leq c_{AD} \\
6 &- 0 \leq c_{AD}
\end{align*}
\]

\( c_{AD} \geq 6 \) is allowable range to stay optimal for \( c_{AD} \).

2) \( i \rightarrow j \) is a basic arc.
- In this case, node potentials depend on the change of \( c_{ij} \). Recompute those starting from a root.