SECTION 8.8

1. Four colors

3. Three colors

5, 7, 9, 11, 13. Graphs with no edges

15. If \( n \) is even, 4 if \( n \) is odd

17. Period 1: Math 115, Math 185; period 2: Math 116, CS 473; period 3: Math 195, CS 101; period 4: CS 102; period 5: CS 273


23. 5 25. The set of vertices with one of the colors is one of the parts, and the set of vertices with the other color is the other part. Since no edge can join vertices of the same color, there are no edges between vertices in the same part.

27. Color 1: \( e, f, d, \) color 2: \( c, a, i, g, \) color 3: \( h, b, j \)

29. Color \( C_6 \)

31. a) 6 b) 7 c) 9 d) 11 33. Represent frequencies by colors and zones by vertices. Join two vertices with an edge if the sets of vertices represent interference with one another. Then a \( k \)-tuple coloring is precisely an assignment of frequencies that avoids interference.

35. We use induction on the number of vertices of the graph. Every graph with five or fewer vertices can be colored with five or fewer colors, since each vertex can get a different color. That takes care of the basis case(s). So we assume that all graphs with \( k \) vertices can be 5-colored and consider a graph \( G \) with \( k + 1 \) vertices. By Corollary 2 in Section 8.7, \( G \) has a vertex \( v \) with degree at most 5. Remove \( v \) to form the graph \( G' \). Since \( G' \) has only \( k \) vertices, we 5-color it by the inductive hypothesis. If the neighbors of \( v \) do not use all five colors, then we can 5-color \( G \) by assigning to \( v \) a color not used by any of its neighbors. The difficulty arises if \( v \) has five neighbors, and each has a different color in the 5-coloring of \( G' \). Suppose that the neighbors of \( v \), when considered in clockwise order around \( v \), are \( a, b, c, m, \) and \( p \). (This order is determined by the clockwise order of the curves representing the edges incident to \( v \).) Suppose that the colors of the neighbors are azure, blue, chartreuse, magenta, and purple, respectively. Consider the azure-chartreuse subgraph (i.e., the vertices in \( G \) colored azure or chartreuse and all the edges between them). If \( a \) and \( c \) are not in the same component of this graph, then in the component containing \( a \) we can interchange these two colors (make the azure vertices chartreuse and vice versa), and \( G' \) will still be properly colored. That makes a chartreuse, so we can now color \( v \) azure, and \( G \) has been properly colored. If \( a \) and \( c \) are in the same component, then there is a path of vertices alternately colored azure and chartreuse joining \( a \) and \( c \). This path together with edges \( av \) and \( vc \) divides the plane into two regions, with \( b \) in one of them and \( m \) in the other. If we now interchange blue and magenta on all the vertices in the same region as \( b \), we still will have a proper coloring of \( G' \), but now blue is available for \( v \). In this case, too, we have found a proper coloring of \( G \). This completes the inductive step, and the theorem is proved.

SUPPLEMENTARY EXERCISES

1. 2500 3. Yes 5. Yes 7. \( \sum_{i=1}^{m} n_i \), vertices, \( \sum_{i<j} n_i n_j \) edges

9. a) \( b \) b) \( c \) c) \( b \)

11. Complete subgraphs containing the following sets of vertices: \( \{b, c, e, f\}, \{a, b, g\}, \{a, d, g\}, \{d, e, g\}, \{b, e, g\} \)

13. Complete subgraphs containing the following sets of vertices: \( \{b, c, d, j, k\}, \{a, b, j, k\}, \{e, f, g, l\}, \{a, b, i\}, \{a, i, j\}, \{b, d, e\}, \{b, e, i\}, \{b, i, j\}, \{g, h, i\}, \{h, i, j\} \)

15. \( \{c, d\} \) is a minimum dominating set.

19. a) 1 b) 2 c) 3

21. a) A path from \( u \) to \( v \) in a graph \( G \) induces a path from \( f(u) \) to \( f(v) \) in an isomorphic graph \( H \). b) Suppose \( f \) is an isomorphism from