Lecture 2: Abhyankar’s Results on the Newton Polygon

**Definition.** Let \(P(X, Y) = \sum p_{ij}X^iY^j \in \mathbb{C}[X, Y]\). The support of \(P\), \(\mathcal{S}(P)\), is the set \(\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid f_{ij} \neq 0\}\). The Newton polygon of \(P\), \(\mathcal{N}(P)\), is the convex hull in \(\mathbb{R}^2\) of \(\mathcal{S}(P) \cup \{(0, 0)\}\).

In the next lecture, in connection with valuations, we will use a slightly different kind of Newton polygon.

The ordinary degree of \(F(X, Y)\) arises by giving each of \(X\) and \(Y\) degree one. One can also have degree in \(X\) and degree in \(Y\). More generally

**Definition.** Let \((a, b) \in \mathbb{Z} \times \mathbb{Z}\). The \((a, b)\)-degree, \(d = d(a, b)\) on \(\mathbb{C}[X, Y]\) is defined by \(d(P) = \max\{ai + bj \mid p_{ij} \neq 0\}\) if \(P = \sum p_{ij} \neq 0\). By convention, \(d(0) = -\infty\). \(P\) is \((a, b)\)-homogeneous of degree \(r\) if \(ai + bj = r\) for every \((i, j)\) in the support of \(P\).

Thus the \((1,1)\)-degree is the ordinary degree, the \((1,0)\)-degree is the \(X\)-degree, and the \((0,1)\)-degree is the \(Y\)-degree. A polynomial is \((a, b)\)-homogeneous if its support lies on a line of slope \(-b/a\). Note that the zero polynomial is \((a, b)\)-homogeneous of all degrees.

**In the first lecture, \(J(F, G)\) denoted a \(2 \times 2\) matrix of partial derivatives. In this lecture \(J(F, G)\) will denote the determinant of this matrix.**

The starting point of Abhyankar’s work is the observation that

\[
J(X^iY^j, X^rY^s) = (is - jr)X^{i+r-1}Y^{j+s-1}.
\]

Thus the Jacobian of monomials \(X^iY^j\) and \(X^rY^s\) is, up to a scalar factor, their product divided by \(XY\). The scalar factor is zero iff \(is - jr = 0\) iff \(X^iY^j\) and \(X^rY^s\) are powers of the same monomial iff \(X^iY^j\) and \(X^rY^s\) are algebraically dependent. Thus the above observation implies

**Lemma 1.** Let \(d\) be the \((a, b)\)-degree. If \(P\) and \(Q\) are \(d\)-homogeneous and algebraically independent, then \(J(P, Q)\) is \(d\)-homogeneous of degree \(d(P) + d(Q) - d(XY)\).

E. Noether proved a refined version of Lüroth’s Theorem.

**Theorem 2.** Let \(K\) be a subfield of \(\mathbb{C}(X_1, \ldots, X_n)\) which contains \(\mathbb{C}\) and has transcendence degree one over \(\mathbb{C}\).

(a) (J. Lüroth.) There is \(P \in \mathbb{C}(X_1, \ldots, X_n)\) such that \(K = \mathbb{C}(P)\).

(b) (E. Noether.) If \(K\) contains a nonconstant polynomial in \(\mathbb{C}[X_1, \ldots, X_n]\), then \(K = \mathbb{C}(P)\) for some \(P \in \mathbb{C}[X_1, \ldots, X_n]\).

It is easy to see that if \(P \in \mathbb{C}[X, Y]\) and \(Q \in (\mathbb{C}[X, Y] \cap \mathbb{C}(P))\) is non-constant and \(d\)-homogeneous, then \(P\) must be \(d\)-homogeneous. This leads to

**Lemma 3.** If \(P, Q\) are \(d\)-homogeneous and algebraically dependent, then there is a \(d\)-homogeneous polynomial \(H\) such that \(P = aH^r\) and \(Q = bH^s\) for some \(a, b \in \mathbb{C}\), \(r, s \in \mathbb{N}\).

Recall that \(P, Q \in \mathbb{C}[X, Y]\) are algebraically dependent iff \(J(P, Q) = 0\). This leads to the following lemma,
Lemma 4. Suppose $P, Q \in \mathbb{C}[X, Y]$ are $d$-homogeneous and $J(P, Q) = 0$. Then $uP^r = vQ^s$ for some $u, v \in \mathbb{C}$ (not both 0) and $r, s \in \mathbb{N}$.

Now let $d$ be the $(a, b)$-degree, where $a$ and $b$ are not both zero. Any nonzero $P = P(X, Y)$, $Q = Q(X, Y)$ can be written as sums of homogeneous elements

$$P = P_{i_1} + \cdots + P_{i_k}, \quad Q = Q_{j_1} + \cdots + Q_{j_k},$$

where $P_{i_k}$ is a nonzero homogeneous polynomial of $(a, b)$-degree $i_k$, $k \geq 1$, $i_1 < \cdots < i_k$, and similarly for $Q$. If $P_{i_k}$ and $Q_{j_l}$ are algebraically independent, then $J(P_{i_k}, Q_{j_l})$ will be nonzero and homogeneous of $(a, b)$-degree $d(P) + d(Q) - d(XY)$, and will be the highest degree homogeneous component of $J(P, Q)$. On the other hand, if $P_{i_k}$ and $Q_{j_l}$ are algebraically dependent, then $P^r$ will be a $\mathbb{C}$-multiple of $Q^s$, where $r/s = d(Q)/d(P)$. Thus

Lemma 5. Let $d$ be the $(a, b)$-degree, where $a$ and $b$ are not both zero, and let $P(X, Y)$, $Q(X, Y)$ be nonzero polynomials.

(a) If the highest degree homogeneous components of $P$ and $Q$ are algebraically independent, then $d(J(P, Q)) = d(P) + d(Q) - d(XY)$.

(b) If the highest degree homogeneous components of $P$ and $Q$ are algebraically dependent, then $d(J(P, Q)) < d(P) + d(Q) - d(XY)$. In this case there is $u \in \mathbb{C}$ such that $\deg(P^r - uQ^s) < r \cdot d(P)$, where $r/s = d(Q)/d(P)$.

Let us consider a polynomial $P(X, Y) = \sum p_{ij}X^iY^j$ and its support $S(P)$ and Newton polygon $N(P)$. If we are given a nonzero degree $(a, b)$ and an integer $r$, then the points $\{(i, j) \mid ai + bj = r\}$ lie on a line of slope $-b/a$ (if $a = 0$ we say the slope is $\infty$), and the polynomial

$$\sum \{p_{ij}X^iY^j \mid ai + pj = r\}$$

is the homogeneous component of $P$ of degree $r$.

The boundary of the Newton polygon of $P$ will consist of line segments on the positive sides of the coordinate axes (which may reduce to the point $(0, 0)$) whose endpoints are joined by a polygonal line which goes from the $Y$-axis clockwise to the $X$-axis. Ignoring cases where the Newton polygon reduces to a point or a line, we can traverse its boundary in a clockwise direction by starting at the origin, going up the $Y$-axis, around to the $X$-axis, and back to the origin. With each edge $E$ of the Newton polygon we can associate an edge polynomial

$$P_E(X, Y) = \sum \{p_{ij}X^iY^j \mid (i, j) \in E\}.$$
and assume (as will be the case when trying to prove the Jacobian Conjecture) that \( P_0(Y), Q_0(Y) \neq 0 \), then the edge polynomials \( P_0(Y) \) and \( Q_0(Y) \) will be the highest degree homogeneous components with respect to the \((-1, 0)\)-degree. We will automatically have \( J(P_0(Y), Q_0(Y)) = 0 \), but this will not in itself be of particular use.

Now let us consider a Jacobian pair \((F, G)\), which means that \( J(F, G) \) is a nonzero complex number. If the Newton polygon of \( F \) does not contain the point \((1, 1)\) then either \( F(X, Y) = aX + f(Y) \) or \( F(X, Y) = f(X) + aY \), where \( a \in \mathbb{C} \) and \( f \) is a polynomial in one variable; in either case it is easy to show that \((F, G)\) is not a counterexample to the Jacobian Conjecture. Likewise, it is easy to see that if \( F(X, Y) = F_0(X) + F_1(X)Y + \cdots \), and \( F_0(X) \) is a constant, then \((F, G)\) is not a counterexample to the Jacobian Conjecture.

Next suppose that \((F, G)\) is a counterexample to the Jacobian Conjecture. By the remarks above, both \( \mathcal{N}(F) \) and \( \mathcal{N}(G) \) contain the point \((1, 1)\) and points on both axes other than the origin. Suppose \( d = (a, b) \) is a degree with at least one of \( a, b \) positive. Then \( d(F) \geq d(XY) \) because \((1, 1) \in \mathcal{N}(F) \) and \( d(G) > 0 \) because \( \mathcal{N}(G) \) contains points \((t, 0)\) and \((0, u)\) with \( t, u > 0 \). This leads to

**Theorem 6.** Suppose that \((F, G)\) is a counterexample to the two-variable Jacobian Conjecture. (i.e. \( J(F, G) \in \mathbb{C}^* \), but \( \mathbb{C}[F, G] \neq \mathbb{C}[X, Y] \).) Then

(a) Let \( d = (a, b) \) be a degree with at least one of \( a, b \) positive, and let \( P, Q \) be the highest degree components of \( F \) and \( G \) with respect to \( d \). Let \( d(G)/d(F) = d(Q)/d(P) = r/s \). Then there is \( c \in \mathbb{C}^* \) such that \( P^r = cQ^s \).

(b) The Newton polygons of \( F \) and \( G \) are similar. More precisely, the constants \( c \) and \( r/s \) in (a) are independent of the particular degree \( d \). Except for the edge polynomials on the coordinate axes, the edge polynomials of \( F^s \) and \( cG^s \) are equal.

**Proof.** (a) We have observed that \( d(P) \geq d(XY) \) and \( d(Q) > 0 \). If \( P \) and \( Q \) were algebraically independent, then Lemma 5(a) would give the contradiction

\[ 0 = d(J(P, Q)) = d(P) + d(Q) - d(XY) > 0. \]

Hence \( P \) and \( Q \) are algebraically dependent, and then Lemma 4 implies that there is \( c \in \mathbb{C}^* \) such that \( P^r = cQ^s \), where \( r/s = d(Q)/d(P) \).

(b) Proceeding clockwise from the \( Y \)-axis to the \( X \)-axis on the boundary of \( \mathcal{N}(F) \) or \( \mathcal{N}(G) \), the endpoint of an edge is the initial point of the next edge. Thus the constant \( c \) and the ratio \( r/s \) are independent of the particular edge, which implies that \( P^r = cQ^s \) for every corresponding pair of edge polynomials of \( F \) and \( G \) except those on the coordinate axes. In particular, \( \mathcal{N}(F) \) and \( \mathcal{N}(G) \) are similar.

**A Resolution Process for Edge Polynomials**

The next step in Abhyankar’s program is to study the edge polynomials which arise if \((F, G)\) is a counterexample to the Jacobian Conjecture. It turns out that they satisfy some unusual “Diophantine” conditions. Polynomials satisfying these conditions are rare, but they do exist. Thus, the Jacobian Conjecture cannot be established by showing that these Diophantine conditions lead to a contradiction.

What Abhyankar is able to show is that if \((F, G)\) is a counterexample to the Jacobian Conjecture and the Newton polygon of \( F \) has an edge of negative slope, then there is an automorphism \( \varphi \) of \( \mathbb{C}[X, Y] \) such that
where }D(P) = d_X(P) + d_Y(P),\text{ the sum of the }X\text{-degree and the }Y\text{-degree. This leads to Abhyankar’s main result, which says that if the two-variable Jacobian Conjecture is false, then there is a counterexample } (F, G) \text{ for which the Newton polygons of } F \text{ and } G \text{ have no edges with negative slope.}

Suppose that } P, Q \text{ are nonzero polynomials in } C, \text{ and } d \text{ is the } (a, b)\text{-degree, where at least one of } a, b \text{ is positive. By Lemma 5(b),}

\[ d(P) + d(Q) - d(XY) - d(J(P, Q)) \geq 0, \]

with equality if the highest degree homogeneous components of } P \text{ and } Q \text{ are algebraically independent. We now define a nonnegative integer } \Delta(P, Q) \text{ by}

\[ \Delta(P, Q) = d(P) + d(Q) - d(XY) - d(J(P, Q)). \]

Now suppose further that } (F, G) \text{ is a Jacobian pair which is a counterexample to the Jacobian Conjecture. Then we know that } \Delta(F, G) > 0. \text{ Set}

\[ F_1 = F, f_1 = d(F_1), P_1 = \text{highest degree } d\text{-homogeneous component of } F_1, \]

\[ g = d(G), Q = \text{highest degree } d\text{-homogeneous component of } G, \]

and define polynomials } F_2, F_3, \ldots \text{ with highest degree } d\text{-homogeneous components } P_2, P_3, \ldots \text{ and degrees } f_2, f_3, \ldots \text{ inductively as follows:

By Theorem 6, there is } c_1 \in C^* \text{ such that } F_1^g = c_1 Q^{f_1}. \text{ This implies that}

\[ d(F_1^g - c_1 G^{f_1}) < d(F_1^g) = gd(F_1) = gf_1, \]

and we set } F_2 = F_1^g - c_1 G^{f_1}, \text{ and } d(F_2) < d(F_2) = d(F_2) - d(F_2) > 0. \text{ Then}

\[ J(F_2, G) = J(F_1^g - c_1 G^{f_1}, G) = gF_1^{g-1}, \text{ and} \]

\[ \Delta(F_2, G) = d(F_2) + d(G) - d(XY) - d(J(F_2, G)) = (gf_1 - \delta_1) + g - d(XY) - (g - 1)f_1 = f_1 + g_1 - d(XY) - \delta_1 = \Delta(F_1, G) - \delta_1. \]

It may happen that } \Delta(F_2, G) = 0, \text{ in which case the procedure terminates. On the other hand, if } \Delta(F_2, G) > 0, \text{ then there is a } c_2 \in C^* \text{ such that } F_3 = F_2^g - c_2 G^{f_2} \text{ satisfies}

\[ d(F_3) = gd(F_2) - \delta_2, J(F_3, G) = (F_1 F_2)^{g-1}, \text{ and} \]

\[ \Delta(F_3, G) = \Delta(F_1, G) - \delta_1 - \delta_2. \]

As long as } \Delta(F_n, G) > 0 \text{ we continue, letting } F_{n+1} = F_n^g - c_n G^{f_n}, \text{ where}

\[ J(F_{n+1}, G) = (F_1 \cdots F_n)^{g-1} \text{ and } \Delta(F_{n+1}, G) > \Delta(F_2, G) > \cdots > \Delta(F_{n+1}, G). \]

Since } \Delta(F_{n+1}, G) = \Delta(F_1, G) - \delta_1, \text{ where } \delta_1 \text{ is a positive integer, the process must terminate with } \Delta(F_{n+1}, G) = 0 \text{ for some } n. \text{ Consider the highest degree } d\text{-homogeneous components of } G, F_1, F_2, \ldots, F_n. \text{ Since } \Delta(F_1, G), \ldots, \Delta(F_n, G) \text{ are all positive, they are all } C\text{-multiples of powers of some } d\text{-homogeneous polynomial } H, \text{ which we may assume is not a proper power (using the fact that } C[X, Y] \text{ is a unique factorization domain). Letting } K \text{ be the highest degree } d\text{-homogeneous component of } F_{n+1}, \text{ the equation}

\[ J(F_{n+1}, G) = (F_1 \cdots F_n)^{g-1} \]

implies that } J(K, H^t) = c H^t \text{ for some positive integers } s \text{ and } t \text{ and } c \in C^*. \text{ An easy calculation shows that } s \leq t \text{ and that } J(K, H) = c' H^u \text{ for } u = t - s + 1 \text{ and some } c' \in C^*. \text{ This yields}
Theorem 7. Suppose that $(F,G)$ is a counterexample to the two-variable Jacobian Conjecture, and $d = (a,b)$ is a degree with $a, b \neq 0$ such that $N(F)$ has an edge of slope $-b/a$, and let $H = H(X,Y)$ be a $d$-homogeneous polynomial such that $H$ is not a proper power and the $d$-homogeneous component of $F$ is a power of $H$. Then there is a $d$-homogeneous polynomial $K = K(X,Y)$ and a positive integer $s$ such that $J(K,H) = H^s$.

Homogeneous polynomial solutions of the equation $J(K,H) = H^s$ do exist, although in a sense they are rare. They are of four essentially different types:

1. Those corresponding to edges of negative slope, or degree functions $d = (a,b)$ with both $a$ and $b$ positive.
2. Lower edges of slope $> 1$ ($d = (a,b)$, with $a > 0 > b$, $a > -b$) or upper edges of slope $< 1$ ($d = (a,b)$, with $a < 0 < b$, $b > -a$).
3. Lower edges of slope $< 1$ ($d = (a,b)$, with $a > 0 > b$, $a < -b$), or upper edges of slope $> 1$ ($d = (a,b)$, with $a < 0 < b$, $b < -a$).
4. Lower edges of slope $1$ ($d = (1,-1)$), or upper edges of slope $1$ ($d = (-1,1)$).

Although all four types of solutions exist, it is the first type which has so far proved the most useful. This is because of the following result.

Lemma 8. Let $d = (a,b)$ be a degree with $a,b$ positive, and let $H$ be a $d$-homogeneous polynomial in $\mathbb{C}[X,Y]$ which is not a monomial. Suppose that there is a $d$-homogeneous polynomial $K_0$ and a positive integer $s$ such that $J(K_0,H) = H^s$. Then

(a) There is a $d$-homogeneous polynomial $K$ such that $J(K,H) = H$.

(b) The support of $K$ either consists of the point $(1,1)$ and one other point on either the $X$-axis or the $Y$-axis (but not $(0,0)$), or is contained in the set $\{(1,0), (1,1), (0,1)\}$. Thus, up to multiplication by an element of $\mathbb{C}^*$, $K$ is equal to $X Y + r Y^{n+1} - X Y + r X^n$, $X Y + r X^{n+1} = X(Y + r X^n)$, or $r(X + s Y)(X + t Y)$, where $r, s, t \in \mathbb{C}^*$. Moreover, the slope, $-a/b$, of $H$ and $K$ is either a negative integer or the reciprocal of a negative integer.

Proof. (a) is proved by a direct calculation which we omit.

(b) is simply the observation that if a line segment lies in the first quadrant, has negative slope, contains the point $(1,1)$, and has endpoints with integer coordinates, then it has the indicated form.

Now suppose that $(F,G)$ is a counterexample to the two-variable Jacobian Conjecture, and the Newton polygon of $F$ has an edge of negative slope, and let $H$ and $K$ be the polynomials given by Theorem 7. Let $d_X = (1,0)$ and $d_Y = (0,1)$ denote the ordinary and $X$ and $Y$-degrees, respectively. We now apply Lemma 8, which says that one of four possibilities occurs.

(a) $K = X(Y + r X^n)$, where $r \in \mathbb{C}^*$ and $n \in \mathbb{N}$, $n \geq 1$. $n \geq 1$. The equation $J(K,H) = H$ implies that $H = q X^m (Y + r X^n)$ for some $q, r \in \mathbb{C}^*$. Let $\varphi = (X,Y-r X^n)$, an elementary automorphism. $d_Y(\varphi(F)) = d_Y(F)$ and $d_X(\varphi(F)) < d_X(F)$, so

$$d_X(\varphi(F)) + d_Y(\varphi(F) < d_X(F) + d_Y(F).$$

(b) $K = Y(X + r Y^n)$. Proceed as in (a) to reduce $d_X(F) + d_Y(F)$. 

(c) \( K = r(X + sY)^2 \) for some \( r, s \in \mathbb{C}^* \). Then \( \varphi = (X - sY, Y) \) reduces \( d_X(F) + d_Y(F) \) as in (b).

(d) \( K = r(X + sY)(X + tY) \) for some \( r, s, t \in \mathbb{C}^* \), where \( s \neq t \). The equation \( J(K, H) = H \) shows that \( H = u(X + sY)^i(X + tY)^j \) for some positive integers \( i \) and \( j \), and \( u \in \mathbb{C}^* \). Consider the elementary automorphism \( \varphi \), where \( \varphi^{-1} = (X + sY, X + tY) \). Since \( \varphi(H) = uX^iY^j \), we see that \( d_X(\varphi(F)) < d_X(F) \) and \( d_Y(\varphi(F)) < d_Y(F) \).

Note that none of the above transformations increase \( d(F) \), where \( d = (1, 1) \) is the ordinary degree. The following theorem sums up most of Abhyankar’s results on the Newton polygon.

**Theorem 9 (S. S. Abhyankar).** Suppose that the two-variable Jacobian Conjecture is false. Then there is a counterexample \( (F, G) \) with the following properties.

(a) The Newton polygons of \( F \) and \( G \) are similar, and have no edges of negative slope. (I.e. There exist positive integers \( a, b \) such that \( (a, b) \in N(F) \) and \( N(F) \subset [0, a] \times [0, b] \).

(b) \( d(F) \) and \( d(F) + d(G) \) are minimal among all counterexamples, where \( d \) is the ordinary degree.

(c) Neither of \( d(F) \), \( d(G) \) divides the other.

**Proof.** (a) follows from the discussion above.

(b) follows from the fact that the automorphisms used to achieve (a) do not increase \( d(F) \) or \( d(G) \).

(c) Assume for definiteness that \( d(G) = rd(F) \) for some positive integer \( r \). Then, by Theorem 6(b), there would be \( c \in \mathbb{C} \) such that \( d(G - cF^r) < d(G) \), and the pair \( (F, G - cF^r) \) would reduce \( d(F) + d(G) \) while leaving \( d(F) \) unchanged.

R. Heitmann, J. Lang and M. Nagata have independently refined Theorem 9(a) as follows: Assume that \( a \leq b \). Then \( N(F) \) is contained in the quadrilateral with vertices \( (0, 0), (0, b-a), (a, b), (a, 0) \). In other words, the upper edges of \( N(F) \) have slope \( \geq 1 \).