Deriving the Simple Continued Fraction of $e$
without Integrals

Senior Math Talks

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1 Introduction

The mathematical constant, \( e \), is a crucial component of various fields of mathematics. It is sometimes better known as Euler’s number since Euler is credited with proving the irrationality of \( e \). The way Euler went about proving this was by showing that the simple continued fraction expansion of \( e \) is infinite. Mathematicians across the globe know that \( e \) is irrational, but what might not be as clear is what the simple continued fraction of \( e \) actually is. Without using integrals, we are going to derive the simple continued fraction expansion of \( e \) in a less familiar way.

Before we get started, let’s become acquainted with what a simple continued fraction is by running through an example. For instance, let’s convert the fraction \( \frac{2013}{405} \) into simple continued fraction form.

\[
\frac{2013}{405} = 4 + \frac{131}{135}
\]

Next, take the fractional part \( \frac{131}{135} \) and take its reciprocal and put it underneath 1.

\[
= 4 + \frac{1}{\frac{135}{131}}
\]

Afterwards, divide 135 by 131.

\[
= 4 + \frac{1}{1 + \frac{1}{\frac{135}{131}}}
\]

Once again take the fractional part, take its reciprocal, and put it underneath 1.

\[
= 4 + \frac{1}{1 + \frac{1}{1 + \frac{131}{4}}}
\]

Then, divide 131 by 4.

\[
= 4 + \frac{1}{1 + \frac{1}{\frac{131}{4}}}
\]

One last time take the reciprocal of \( \frac{3}{4} \) and place it underneath 1.

\[
= 4 + \frac{1}{1 + \frac{1}{\frac{131}{32 + \frac{1}{4}}}}
\]

Finally, divide 4 by 3 which gives us the simple continued fraction of \( \frac{2013}{405} \).

\[
= 4 + \frac{1}{1 + \frac{1}{32 + \frac{1}{3 + \frac{1}{3}}}}
\]
This simple continued fraction can also be written in a condensed form like so

\[ 4 + \frac{1}{32 + \frac{1}{1 + \frac{1}{x}}} = [4; 1, 32, 1, 3] \]

To generalize all simple continued fractions can be written in the following form

\[ x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}} \]

where \( a_0 \in \mathbb{Z} \) and \( a_1, a_2, a_3, \ldots \in \mathbb{N} \).

Notice that what makes this a simple continued fraction is the fact that all the numerators are ones. This can be written in the condensed form as

\[ x = [a_0; a_1, a_2, a_3, \ldots] \]

where the semi-colon separates the whole integer from the fractional part.

Now, we want to recall three identities that will be used very shortly.

\[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

\[ \cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} \]

\[ \sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} \]

2 Finding the Simple Continued Fraction of \( \coth(x) \)

In order to find the simple continued fraction for \( e \), we will first find the simple continued fraction for \( \coth(x) \). To do so, let’s first define the following function

\[ \varphi_\nu = \sum_{n=0}^{\infty} \frac{2^\nu (\nu+n)!}{n!(2\nu+2n)!} (\frac{1}{k})^{2n+\nu} \]

where \( k \in \mathbb{N} \) and \( \nu = 0, 1, 2, \ldots \).

Note that \( \varphi_\nu > \varphi_{\nu+1} \) which is true since

\[
\begin{align*}
&\sum_{n=0}^{\infty} \frac{2^\nu (\nu+n)!}{n!(2\nu+2n)!} (\frac{1}{k})^{2n+\nu} > \sum_{n=0}^{\infty} \frac{2^{\nu+1} (\nu+1+n)!}{n!(2\nu+1+2n)!} (\frac{1}{k})^{2n+\nu+1} \\
\Leftrightarrow &\sum_{n=0}^{\infty} \frac{2^\nu (\nu+n)!}{n!(2\nu+2n)!} (\frac{1}{k})^{2n+\nu} > \sum_{n=0}^{\infty} \frac{2^\nu (2\nu+2\nu+n)(\nu+n)!}{n!(2\nu+2\nu+2n)!} (\frac{1}{k})^{2n+\nu} \left(\frac{1}{k}\right) \\
\Leftrightarrow &\sum_{n=0}^{\infty} \frac{2^\nu (\nu+n)!}{n!(2\nu+2n)!} (\frac{1}{k})^{2n+\nu} > \sum_{n=0}^{\infty} \frac{2^\nu (\nu+n)!}{n!(2\nu+2n)!} (\frac{1}{k})^{2n+\nu} \left[ \frac{2(\nu+1+n)}{(2\nu+2\nu+2n)(2\nu+1+2n)} \right] \\
\varphi_\nu > & \varphi_{\nu+1} \left[ \frac{1}{(2\nu+1+2n)k} \right] \\
1 > & \frac{1}{(2\nu+1+2n)k}
\end{align*}
\]
which is strictly greater than except when \( n = \nu = 0 \) and \( k = 1 \), in which case we have equality. Hence, \( \varphi_\nu > \varphi_{\nu+1} \).

In order to compute values of \( \varphi_\nu \), we must show that this summation converges.

Observe that

\[
\frac{2^\nu(\nu+n)!}{n!(2\nu+2n)!} \leq \frac{2^\nu}{n!} = \frac{2^\nu}{n!}.
\]

Thus, \( \sum_{n=0}^{\infty} \frac{2^\nu}{n!} (1)_k^{2n+\nu} \) dominates \( \sum_{n=0}^{\infty} \frac{2^\nu(\nu+n)!}{n!(2\nu+2n)!} (1)_k^{2n+\nu} \) which means that each of the terms in the first summation is larger than the respective term in the second summation. This implies that if \( \sum_{n=0}^{\infty} \frac{2^\nu}{n!} (1)_k^{2n+\nu} \) converges, then \( \sum_{n=0}^{\infty} \frac{2^\nu(\nu+n)!}{n!(2\nu+2n)!} (1)_k^{2n+\nu} \) must converge as well which we will prove is true.

Let’s examine \( \sum_{n=0}^{\infty} \frac{2^\nu}{n!} (1)_k^{2n+\nu} \). Since \( k \) and \( \nu \) are constants, we can pull them out. Thus,

\[
\sum_{n=0}^{\infty} \frac{2^\nu}{n!} (1)_k^{2n+\nu} = \frac{2^\nu}{k!} \sum_{n=0}^{\infty} \frac{1}{n!} (\frac{1}{k})^n = (\frac{2}{k})^\nu \sum_{n=0}^{\infty} \frac{1}{n!} (\frac{1}{k})^n.
\]

Recall that \( \sum_{n=0}^{\infty} \frac{1}{n!} = e^{1/k} \). Then,

\[
\sum_{n=0}^{\infty} \frac{2^\nu}{n!} (1)_k^{2n+\nu} = (\frac{2}{k})^\nu e^{1/k}.
\]

Hence, \( \sum_{n=0}^{\infty} \frac{2^\nu}{n!} (1)_k^{2n+\nu} \) converges which means \( \sum_{n=0}^{\infty} \frac{2^\nu(\nu+n)!}{n!(2\nu+2n)!} (1)_k^{2n+\nu} \) converges for all \( \nu \).

Thus, all \( \varphi_\nu \) values exist. Since all \( \varphi_\nu \) exist, we are able to determine the values of \( \varphi_0 \) and \( \varphi_1 \).

\[
\varphi_0 = \sum_{n=0}^{\infty} \frac{2^{0}(0+n)!}{n!(2n)!} (1)_k^{2n} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} (\frac{1}{k})^{2n} = \cosh (\frac{1}{k}).
\]

\[
\varphi_1 = \sum_{n=0}^{\infty} \frac{2^{1}(1+n)!}{n!(2n+2)!} (1)_k^{2n+1} = \sum_{n=0}^{\infty} \frac{2^{1}(1+n)!}{n!(2n+2)!} (\frac{1}{k})^{2n+1} = \sum_{n=0}^{\infty} \frac{1}{(1+2n)!} (\frac{1}{k})^{2n+1} = \sinh (\frac{1}{k}).
\]
Let’s define a new function as

\[ \beta_\nu = \frac{\varphi_\nu}{\varphi_{\nu+1}} \quad \text{where } \nu = 0, 1, 2, \ldots \]

Since \( \varphi_\nu > \varphi_{\nu+1} \), then \( \beta_\nu > 1 \).

Letting \( \nu = 0 \),

\[ \beta_0 = \frac{\varphi_0}{\varphi_1} = \frac{\cosh(\frac{1}{k})}{\sinh(\frac{1}{k})} = \coth\left(\frac{1}{k}\right). \]

This, however, tells us nothing about the simple continued fraction of \( \coth\left(\frac{1}{k}\right) \), so we need to define \( \beta_0 \) in a different way. To do so, let’s evaluate \( \varphi_\nu - (2\nu + 1)k \varphi_{\nu+1} \) using

\[ \varphi_\nu = \sum_{n=0}^{\infty} \frac{2^n(\nu+n)!}{n!(2\nu+2n)!} \left(\frac{1}{k}\right)^{2n+\nu}. \]

\[ \varphi_\nu - (2\nu + 1)k \varphi_{\nu+1} = \sum_{n=0}^{\infty} \left[ \frac{2^n(\nu+n)!}{n!(2\nu+2n)!} \left(\frac{1}{k}\right)^{2n+\nu} - (2\nu + 1)k \frac{2^{\nu+1}(\nu+1+n)!}{n!(2\nu+2n+2)!} \left(\frac{1}{k}\right)^{2n+\nu+1} \right] \]

\[ = \sum_{n=0}^{\infty} \left[ 2^n(\nu+n)! \left(\frac{1}{k}\right)^{2n+\nu} - (2\nu + 1)k \frac{2^{\nu+1}(\nu+1+n)!}{n!(2\nu+2n+2)!} \left(\frac{1}{k}\right)^{2n+\nu+1} \right] \]

\[ = \sum_{n=0}^{\infty} \left[ 2^n(\nu+n)! \left(\frac{1}{k}\right)^{2n+\nu} - (2\nu + 1)k \frac{2^{\nu+1}(\nu+1+n)!}{n!(2\nu+2n+2)!} \left(\frac{1}{k}\right)^{2n+\nu+1} \right] \]

\[ = \sum_{n=0}^{\infty} \left[ 2^n(\nu+n)! \left(\frac{1}{k}\right)^{2n+\nu} - (2\nu + 1)k \frac{2^{\nu+1}(\nu+1+n)!}{n!(2\nu+2n+2)!} \left(\frac{1}{k}\right)^{2n+\nu+1} \right] \]

\[ = \sum_{n=0}^{\infty} \left[ 2^n(\nu+n)! \left(\frac{1}{k}\right)^{2n+\nu} - (2\nu + 1)k \frac{2^{\nu+1}(\nu+1+n)!}{n!(2\nu+2n+2)!} \left(\frac{1}{k}\right)^{2n+\nu+1} \right] \]

\[ = \sum_{n=0}^{\infty} \left[ 2^n(\nu+n)! \left(\frac{1}{k}\right)^{2n+\nu} - (2\nu + 1)k \frac{2^{\nu+1}(\nu+1+n)!}{n!(2\nu+2n+2)!} \left(\frac{1}{k}\right)^{2n+\nu+1} \right] \]

\[ = \varphi_{\nu+2}. \]

Hence, \( \varphi_\nu - (2\nu + 1)k \varphi_{\nu+1} = \varphi_{\nu+2}. \)

Dividing both sides by \( \varphi_{\nu+1} \) produces

\[ \frac{\varphi_\nu}{\varphi_{\nu+1}} - (2\nu + 1)k = \frac{\varphi_{\nu+2}}{\varphi_{\nu+1}}. \]
Then, recognize on the left hand side that $\frac{\nu}{\nu+1}$ is how we defined $\beta_\nu$, and on the right hand side, take the reciprocal of the fraction and put it underneath 1.

$$\beta_\nu -(2\nu + 1)k = \frac{1}{\frac{\nu}{\nu+1}}$$

Now recognize that $\frac{1}{\frac{\nu}{\nu+1}} = \frac{1}{\beta_{\nu+1}}$, so

$$\beta_\nu -(2\nu + 1)k = \frac{1}{\beta_{\nu+1}}.$$

Finally, add $(2\nu + 1)k$ to the right hand side. This gives us

$$\beta_\nu = (2\nu + 1)k + \frac{1}{\beta_{\nu+1}} \text{ for } k \in \mathbb{N} \text{ and } \nu = 0, 1, 2, \ldots$$

Now, let’s plug in the values $\nu = 0, 1, 2,$ and 3.

When $\nu = 0$,

$$\beta_0 = (2(0) + 1)k + \frac{1}{\beta_1}$$
$$= k + \frac{1}{\beta_1}.$$  

When $\nu = 1$,

$$\beta_1 = (2(1) + 1)k + \frac{1}{\beta_2}$$
$$= 3k + \frac{1}{\beta_2}.$$  

When $\nu = 2$,

$$\beta_2 = (2(2) + 1)k + \frac{1}{\beta_3}$$
$$= 5k + \frac{1}{\beta_3}.$$  

Finally, when $\nu = 3$,

$$\beta_3 = (2(3) + 1)k + \frac{1}{\beta_4}$$
$$= 7k + \frac{1}{\beta_4}.$$  

Next, we can write $\beta_0$ using the values of $\beta_1$, $\beta_2$, and $\beta_3$ since all $\beta_i > 1$, so

$$\beta_0 = k + \frac{1}{\frac{1}{3k} + \frac{1}{5k} + \frac{1}{7k} + \beta_4}$$
$$= [k; 3k, 5k, 7k, \ldots].$$
Previously, we discovered that $\beta_0 = \coth(\frac{1}{k})$. Consequently, $\coth(\frac{1}{k}) = [k; 3k, 5k, 7k, \ldots]$ which is a result of Euler.

Now that we know the simple continued fraction of $\coth(\frac{1}{k})$, there are two different ways that we can derive the simple continued fraction expansion of $e$. The first way involves a proof by induction followed by taking the limit of a sequence, and the second way involves a result discovered by Euler and has been coined the easy way.

### 3 Deriving the Continued Fraction for $e$ Through Induction

With the first way, let
\[
\alpha = [2; 1, 2, 1, 4, 1, 1, 6, 1, \ldots] = [2; \overline{1, 2 \lambda}]_{\lambda=1}.
\]

Let the $\nu^{th}$ convergent of $\alpha$ be $\frac{A_\nu}{B_\nu}$.

Recall from our previous work that $\coth(\frac{1}{2}) = [k; 3k, 5k, 7k, \ldots]$. If $k = 2$, then $\coth(\frac{1}{2}) = [2; 6, 10, 14, \ldots]$ which is the same as $\frac{e+1}{e-1} = [2; 6, 10, 14, \ldots]$. Let the $\nu^{th}$ convergent of $\frac{e+1}{e-1}$ be $\frac{C_\nu}{D_\nu}$.

We can create convergents’ tables for both $\alpha$ and $\frac{e+1}{e-1}$.

The table of convergents for $\alpha$ is

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
<th>$7$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$2$</td>
<td>$1$</td>
<td>$2$</td>
<td>$1$</td>
<td>$1$</td>
<td>$4$</td>
<td>$1$</td>
<td>$1$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$A_\nu$</td>
<td>$0$</td>
<td>$1$</td>
<td>$2$</td>
<td>$3$</td>
<td>$8$</td>
<td>$19$</td>
<td>$87$</td>
<td>$193$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$B_\nu$</td>
<td>$1$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
<td>$3$</td>
<td>$7$</td>
<td>$32$</td>
<td>$71$</td>
<td>$\ldots$</td>
</tr>
</tbody>
</table>

Notice that $\frac{A_0}{B_0} = \frac{2}{1}$, $\frac{A_1}{B_1} = \frac{3}{1}$, $\frac{A_2}{B_2} = \frac{8}{3}$, $\frac{A_3}{B_3} = \frac{11}{4}$, and $\frac{A_4}{B_4} = \frac{19}{7}$.

The table of convergents for $\frac{e+1}{e-1}$ is

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
<th>$7$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{e-1}{e+1}$</td>
<td>$2$</td>
<td>$6$</td>
<td>$10$</td>
<td>$14$</td>
<td>$18$</td>
<td>$22$</td>
<td>$26$</td>
<td>$30$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$C_\nu$</td>
<td>$0$</td>
<td>$1$</td>
<td>$2$</td>
<td>$13$</td>
<td>$132$</td>
<td>$1861$</td>
<td>$33630$</td>
<td>$741721$</td>
<td>$19318376$</td>
</tr>
<tr>
<td>$D_\nu$</td>
<td>$1$</td>
<td>$0$</td>
<td>$1$</td>
<td>$6$</td>
<td>$61$</td>
<td>$860$</td>
<td>$15541$</td>
<td>$342762$</td>
<td>$8927353$</td>
</tr>
</tbody>
</table>
Notice that $C_0 \over D_0 = 2 \over 1$, $C_1 \over D_1 = 13 \over 6$, $C_2 \over D_2 = 132 \over 61$, $C_3 \over D_3 = 1861 \over 860$, and $C_4 \over D_4 = 33630 \over 15541$.

From these tables we can observe that

\[
\begin{align*}
A_1 &= C_0 + D_0 & A_4 &= C_1 + D_1 \\
B_1 &= C_0 - D_0 & B_4 &= C_1 - D_1
\end{align*}
\]

In general, we claim that

**Theorem 1**

\[
\begin{align*}
A_{3\nu+1} &= C_\nu + D_\nu \\
B_{3\nu+1} &= C_\nu - D_\nu
\end{align*}
\]

is true for all $\nu \geq 0$.

**Proof.** We will show that it is true by induction. Let’s begin with two base cases by showing it is true for $\nu = 0, 1$.

When $\nu = 0$, $A_{3(0)+1} = C_0 + D_0$, and we know $A_1 = 3$ and $C_0 + D_0 = 2 + 1 = 3$.

Similarly, when $\nu = 0$, $B_1 = C_0 - D_0$, and from the convergents’ table, $B_1 = 1$ and $C_0 - D_0 = 2 - 1 = 1$. Hence, $A_{3\nu+1} = C_\nu + D_\nu$ and $B_{3\nu+1} = C_\nu - D_\nu$ are true when $\nu = 0$.

When $\nu = 1$, $A_4 = C_1 + D_1$ and $B_4 = C_1 - D_1$ which are both true seeing that $A_4 = 19$ and $C_1 + D_1 = 13 + 6 = 19$ while $B_4 = 7$ and $C_1 - D_1 = 13 - 6 = 7$.

Thus, $A_{3\nu+1} = C_\nu + D_\nu$ and $B_{3\nu+1} = C_\nu - D_\nu$ hold true for $\nu = 1$.

Note that

\[
\begin{align*}
C_\nu &= (2 + 4\nu)C_{\nu-1} + C_{\nu-2} \\
D_\nu &= (2 + 4\nu)D_{\nu-1} + D_{\nu-2}
\end{align*}
\]

for $\nu \geq 2$.

Now, expanding on the original convergents’ table for $\alpha$, we get
From this table we can gather that

\[
\begin{align*}
A_{3\nu - 3} &= A_{3\nu - 4} + A_{3\nu - 5} \\
A_{3\nu - 2} &= A_{3\nu - 3} + A_{3\nu - 4} \\
A_{3\nu - 1} &= 2\nu A_{3\nu - 2} + A_{3\nu - 3} \\
A_{3\nu} &= A_{3\nu - 1} + A_{3\nu - 2} \\
A_{3\nu + 1} &= A_{3\nu} + A_{3\nu - 1}
\end{align*}
\]

\[
\begin{align*}
B_{3\nu - 3} &= B_{3\nu - 4} + B_{3\nu - 5} \\
B_{3\nu - 2} &= B_{3\nu - 3} + B_{3\nu - 4} \\
B_{3\nu - 1} &= 2\nu B_{3\nu - 2} + B_{3\nu - 3} \\
B_{3\nu} &= B_{3\nu - 1} + B_{3\nu - 2} \\
B_{3\nu + 1} &= B_{3\nu} + B_{3\nu - 1}
\end{align*}
\]

Let’s begin with

\[
A_{3\nu + 1} = A_{3\nu} + A_{3\nu - 1}.
\]

After substituting in for both \(A_{3\nu}\) and \(A_{3\nu - 1}\) produces

\[
\begin{align*}
&= A_{3\nu - 1} + A_{3\nu - 2} + 2\nu A_{3\nu - 2} + A_{3\nu - 3} \\
&= A_{3\nu - 1} + (1 + 2\nu)A_{3\nu - 2} + A_{3\nu - 3}.
\end{align*}
\]

Then, substitution is made for \(A_{3\nu - 1}\).

\[
\begin{align*}
&= 2\nu A_{3\nu - 2} + A_{3\nu - 3} + (1 + 2\nu)A_{3\nu - 2} + A_{3\nu - 3} \\
&= (1 + 4\nu)A_{3\nu - 2} + A_{3\nu - 3} + A_{3\nu - 3}
\end{align*}
\]

Finally, we are going to substitute in \(A_{3\nu - 4} + A_{3\nu - 5}\) for one of the \(A_{3\nu - 3}\) terms and for the other rearrange \(A_{3\nu - 2} = A_{3\nu - 3} + A_{3\nu - 4}\) so that \(A_{3\nu - 3} = A_{3\nu - 2} - A_{3\nu - 4}\).
\[
(1 + 4\nu)A_{3\nu-2} + A_{3\nu-4} + A_{3\nu-5} + A_{3\nu-2} - A_{3\nu-4} = (2 + 4\nu)A_{3\nu-2} + A_{3\nu-5}
\]

Now, let’s do a similar procedure for

\[
B_{3\nu+1} = B_{3\nu} + B_{3\nu-1}.
\]

Now, substitute in for both \(B_{3\nu}\) and \(B_{3\nu-1}\).

\[
\begin{align*}
&= B_{3\nu-1} + B_{3\nu-2} + 2\nu B_{3\nu-2} + B_{3\nu-3} \\
&= B_{3\nu-1} + (1 + 2\nu)B_{3\nu-2} + B_{3\nu-3}.
\end{align*}
\]

Next, substitution is made for \(B_{3\nu-1}\).

\[
\begin{align*}
&= 2\nu B_{3\nu-2} + B_{3\nu-3} + (1 + 2\nu)B_{3\nu-2} + B_{3\nu-3} \\
&= (1 + 4\nu)B_{3\nu-2} + B_{3\nu-3} + B_{3\nu-3}
\end{align*}
\]

Finally, we are going to substitute in \(B_{3\nu-4} + B_{3\nu-5}\) for one of the \(B_{3\nu-3}\) terms and for the other rearrange \(B_{3\nu-2} = B_{3\nu-3} + B_{3\nu-4}\) so that \(B_{3\nu-3} = B_{3\nu-2} - B_{3\nu-4}\).

\[
\begin{align*}
&= (1 + 4\nu)B_{3\nu-2} + B_{3\nu-4} + B_{3\nu-5} + B_{3\nu-2} - B_{3\nu-4} \\
&= (2 + 4\nu)B_{3\nu-2} + B_{3\nu-5}
\end{align*}
\]

From this arithmetic, we know

\[
\begin{align*}
A_{3\nu-1} &= (2 + 4\nu)A_{3\nu-2} + A_{3\nu-5} \\
B_{3\nu+1} &= (2 + 4\nu)B_{3\nu-2} + B_{3\nu-5}.
\end{align*}
\]
We can rewrite these two equations like so

\[
A_{3\nu+1} = (2 + 4\nu)A_{3(\nu-1)+1} + A_{3(\nu-2)+1} \\
B_{3\nu+1} = (2 + 4\nu)B_{3(\nu-1)+1} + B_{3(\nu-2)+1}.
\]

Since we know \(A_{3\nu+1} = C_\nu + D_\nu\) is true by our induction hypothesis,

\[
A_{3\nu+1} = (2 + 4\nu)(C_{\nu-1} + D_{\nu-1}) + C_{\nu-2} + D_{\nu-2} \\
= (2 + 4\nu)C_{\nu-1} + C_{\nu-2} + (2 + 4\nu)D_{\nu-1} + D_{\nu-2} \\
= C_\nu + D_\nu.
\]

Similarly, because \(B_{3\nu+1} = C_\nu - D_\nu\) is true by our induction hypothesis,

\[
B_{3\nu+1} = (2 + 4\nu)(C_{\nu-1} - D_{\nu-1}) + C_{\nu-2} - D_{\nu-2} \\
= (2 + 4\nu)C_{\nu-1} + C_{\nu-2} - (2 + 4\nu)D_{\nu-1} - D_{\nu-2} \\
= C_\nu - D_\nu.
\]

Thus, \(A_{3\nu+1} = C_\nu + D_\nu\) and \(B_{3\nu+1} = C_\nu - D_\nu\) are true for all \(\nu \geq 0\). □

Now, recall that \(\alpha = [2; 1, 2\lambda, 1]_{\lambda=1}^\infty\). Since \(2; 1, 2, 1, 1, 4, 1, 1, ...\) is an infinite sequence of integers,

\[
[2; 1, 2\lambda, 1]_{\lambda=1}^\infty = \lim_{\nu \to \infty} [2; 1, 2\lambda, 1]_{\lambda=1}^\nu.
\]

Earlier we let the \(\nu^{th}\) convergent of \(\alpha\) be \(A_\nu/B_\nu\), so we can rewrite the limit as

\[
[2; 1, 2\lambda, 1]_{\lambda=1}^\infty = \lim_{\nu \to \infty} \frac{A_\nu}{B_\nu}.
\]

We know that a subsequence of a convergent sequence converges to the same limit, so

\[
\lim_{\nu \to \infty} \frac{A_{3\nu+1}}{B_{3\nu+1}}.
\]

Through induction, we proved \(A_{3\nu+1} = C_\nu + D_\nu\) and \(B_{3\nu+1} = C_\nu - D_\nu\),

\[
\lim_{\nu \to \infty} \frac{C_\nu + D_\nu}{C_\nu - D_\nu}.
\]
Dividing each term by $D_\nu$ produces

$$= \lim_{\nu \to \infty} \frac{C_\nu + 1}{C_\nu}. \quad (1)$$

Earlier we let the $\nu^{th}$ convergent of $\frac{e+1}{e-1}$ be $\frac{C_\nu}{D_\nu}$, so we can substitute the two.

$$= \lim_{\nu \to \infty} \frac{\frac{e+1}{e-1} + 1}{\frac{e+1}{e-1}}$$

$$= \frac{e+2}{e}$$

$$= \frac{2e}{e-1}$$

$$= e$$

Thus, $e = [2; 1, 2\lambda, 1]_{\lambda=1}^\infty$.

4 The Easy Way

The second approach to finding the simple continued fraction of $e$ is completely different.

Let’s begin with the identity of $\coth \left( \frac{1}{k} \right)$.

$$\coth \left( \frac{1}{k} \right) = \frac{e_+ + e_-}{e_+ - e_-} \quad (2)$$

Multiplying the top and bottom of the identity by $\frac{e_+}{e_-}$ produces $\frac{e_+^2 + 1}{e_+^2 - 1}$.

Hence,

$$\frac{e_+^2 + 1}{e_+^2 - 1} = [k; 3k, 5k, 7k, \ldots] = k + \frac{1}{3k + \frac{1}{5k + \frac{1}{7k + \ldots}}}.$$ 

Subtracting 1 from both sides produces

$$\frac{2}{e_+^2 - 1} = (k - 1) + \frac{1}{3k + \frac{1}{5k + \frac{1}{7k + \ldots}}}.$$ 

Since we are looking for the continued fraction expansion for $e$, we want to let $k = 2$.

$$\frac{2}{e-1} = (2 - 1) + \frac{1}{6 + \frac{1}{10 + \frac{1}{14 + \ldots}}.$$ 

Then, we want to take the reciprocal of both sides.

$$\frac{e-1}{2} = \frac{1}{1 + \frac{1}{6 + \frac{1}{10 + \frac{1}{14 + \ldots}}}}.$$
Finally, multiplying both sides by 2 and adding 1 gives us

\[ e = 1 + \frac{2}{1 + \frac{1}{6 + \frac{1}{10 + \frac{1}{14 + \cdots}}}} = [1; 1^*, 6, 10, 14, \ldots] \]

where the * stands for 2 over the number starred.

Notice that this is not the simple continued fraction of \( e \), because all the numerators are not ones. Thus, our goal is to get rid of the *. I want to introduce three transformations that will help derive Euler’s formula. They are as follows:

\[
[1; 0, 1, \beta] = [2; \beta]
\]

\[
[\ldots, a^*, b, y, \ldots] = [\ldots, \frac{a}{2}, 2b, y^*\ldots] \text{ when } a \text{ is even}
\]

\[
[\ldots, a^*, b, y, \ldots] = [\ldots, \frac{a-1}{2}, 1, 1, (b-1)^*, y, \ldots] \text{ when } a \text{ is odd}
\]

I am going to show how to derive \([1; 0, 1, \beta] = [2; \beta]\); the other two follow similarly.

The derivation of \([1; 0, 1, \beta]\) consists of simple algebra:

\[
[1; 0, 1, \beta] = 1 + \frac{\frac{1}{\beta}}{1 + \frac{1}{\beta}}
\]

\[
= 1 + \frac{\frac{1}{\beta}}{\frac{1}{\beta} + 1}
\]

\[
= 1 + \frac{1}{\beta + 1}
\]

\[
= 1 + \frac{\beta + 1}{\beta}
\]

\[
= \frac{\beta + \beta + 1}{\beta}
\]

\[
= \frac{2\beta + 1}{\beta}
\]

\[
= 2 + \frac{1}{\beta}
\]

\[
= [2; \beta]
\]

Recall that \( e = [1; 1^*, 6, 10, 14, 18, 22, \ldots] \). Since 1 is odd, we are going to use the \([\ldots, a^*, b, y, \ldots] = [\ldots, \frac{a-1}{2}, 1, 1, (b-1)^*, y, \ldots] \) transformation where \( a^* = 1^*, b = 6, \) and \( y = 10, 14, 18, 22, \ldots \).

\[
e = [1; 1^*, 6, 10, 14, 18, 22, \ldots]
\]

\[
= [1; 0, 1, 1, (6 - 1)^*, 10, 14, 18, 22, \ldots]
\]

Then, using \([1; 0, 1, \beta] = [2; \beta]\) where \( \beta = 1, 5^*, 10, 14, 18, 22, \ldots \), it becomes

\[
= [2; 1, 5^*, 10, 14, 18, 22, \ldots].
\]
Five is an odd number, so we are going to use \([\ldots, a^*, b, y, \ldots] = [\ldots, \frac{a-1}{2}, 1, 1, (b-1)^*, y, \ldots]\) where \(a^* = 5^*, b = 10, \) and \(y = 14, 18, 22, \ldots\)

\[
\begin{align*}
&= [2; 1, \frac{5-1}{2}, 1, 1, (10 - 1)^*, 14, 18, 22, \ldots] \\
&= [2; 1, 2, 1, 1, 9^*, 14, 18, 22, \ldots]
\end{align*}
\]

One last time we are going to use \([\ldots, a^*, b, y, \ldots] = [\ldots, \frac{a-1}{2}, 1, 1, (b-1)^*, y, \ldots]\) since 9 is odd where \(a^* = 9^*, b = 14, \) and \(y = 18, 22, \ldots\)

\[
\begin{align*}
&= [2; 1, 2, 1, 1, \frac{9-1}{2}, 1, 1, (14 - 1)^*, 18, \ldots] \\
&= [2; 1, 2, 1, 1, 4, 1, 1, 13^*, 18, \ldots] \\
e &= \left[\frac{[2;1,2\lambda,1]_{\lambda=1}^\infty}{[2;1,1,1,1,1,1,1,\ldots]}\right] = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, \ldots]
\end{align*}
\]

which is once again Euler’s result.

### 5 Conclusion

We have derived the simple continued fraction of \(\coth(\frac{1}{k})\) which then assisted us in finding the simple continued fraction expansion of \(e\). We showed two different methods of how to derive the simple continued fraction expansion of \(e\) without the use of integrals which is the more common method. The first way involved a proof by induction which led us to Euler’s result that \(e = [2; 1, 2\lambda, 1]_{\lambda=1}^\infty\). The second way led us to the same result although this process can be used to find simple continued fractions of \(e\) to varying rational powers.

For instance, a similar technique can be used to find the simple continued fraction of \(e^2\).

Recall the previous equation

\[
\frac{2}{e^2 - 1} = (k-1) + \frac{1}{\frac{3k+1}{5k+1} + \frac{1}{\frac{3k+1}{5k+1} + \frac{1}{\ddots}}}
\]

Since we are looking for \(e^2\), let \(k = 1\). Then,

\[
\frac{2}{e^2 - 1} = (1 - 1) + \frac{1}{\frac{1}{\frac{5}{5} + \frac{1}{\frac{5}{5} + \frac{1}{\ddots}}}}
\]
Next, we want to take the reciprocal of both sides.

\[
\frac{e^2-1}{2} = \frac{1}{9 + \frac{1}{3 + \frac{1}{7 + \frac{1}{11 + \ddots}}}}
\]

Lastly, multiply both sides by 2 and add 1.

\[
e^2 = 1 + \frac{2}{9 + \frac{1}{3 + \frac{1}{7 + \frac{1}{11 + \ddots}}}} = [1; 0^*, 3, 5, 7, 9, \ldots]
\]

In order to get rid of the * so that all the numerators are ones, we are going to use three transformations. They are as follows:

\[
[1; 0, 6, \beta] = [7; \beta]
\]

\[
[\ldots, a^*, b, y, \ldots] = [\ldots, \frac{a}{2}, 2b, y^*, \ldots] \text{ when } a \text{ is even}
\]

\[
[\ldots, a^*, b, y, \ldots] = [\ldots, \frac{a-1}{2}, 1, 1, (b - 1)^*, y, \ldots] \text{ when } a \text{ is odd}
\]

The equation we are starting with is

\[
e^2 = [1; 0^*, 3, 5, 7, 9, \ldots].
\]

Since 0 is even, we are going to use the \([\ldots, a^*, b, y, \ldots] = [\ldots, \frac{a}{2}, 2b, y^*, \ldots]\) transformation where \(a^* = 0, b = 3,\) and \(y = 5, 7, 9, \ldots\)

\[
e^2 = [1; \frac{0}{2}, 2(3), 5^*, 7, 9, \ldots]
\]

\[
= [1; 0, 6, 5^*, 7, 9, \ldots]
\]

Now, using \([1; 0, 6, \beta] = [7; \beta]\) where \(\beta = 5^*, 7, 9, \ldots,\) it becomes

\[
= [7; 5^*, 7, 9, \ldots].
\]

Next, 5 is an odd number, so we are going to use \([\ldots, a^*, b, y, \ldots] = [\ldots, \frac{a-1}{2}, 1, 1, (b - 1)^*, y, \ldots]\) where \(a^* = 5, b = 7,\) and \(y = 9, 11, 13, \ldots.\)
= \[7; \frac{5-1}{2}, 1, 1, (7-1)^*, 9, 11, 13\ldots\]
= \[7; 2, 1, 1, 6^*, 9, 11, 13\ldots\]

Let’s do this one more time. Since 6 is even, we are going to use \([\ldots, a^*, b, y, \ldots]\) = \([\ldots, \frac{a}{2}, 2b, y^*, \ldots]\) where \(a^* = 6, b = 9,\) and \(y = 11, 13,\ldots\)

= \[7; 2, 1, 1, \frac{6}{2}, 2(9), 11^*, 13\ldots\]
= \[7; 2, 1, 1, 3, 18, 11^*, 13\ldots\]

After this process is repeated several times, we find that

\[e^2 = 7; 2, 1, 1, 3, 18, 5, 1, 1, 6, 30, \ldots.\]

The same process can be used to show that

\[e^2 = [7, 3n - 1, 1, 1, 3n, 6(2n + 1)]_{n=1}^\infty\]
\[e^k = [1, \frac{1}{2}((6n + 1)k - 1), 6k(2n + 1), \frac{1}{2}((6n + 5)k - 1), 1]_{n=0}^\infty\]

for any \(k \geq 3\) that is odd.

6 References

