Ore Extensions and $V$-domains

S. K. Jain, T. Y. Lam and A. Leroy

Abstract. We give necessary and sufficient conditions for a skew polynomial ring $K[t;\sigma,\delta]$ over a division ring $K$ to be a left $V$-domain. In particular, when this ring admits a unique simple left module, the conditions obtained include: 1) all polynomials are Wedderburn, 2) all $n \times n$ matrices over $K$ are $(\sigma,\delta)$-similar. We also provide necessary and sufficient conditions for this ring to be both left and right $V$-domain. These results, that are indeed motivated by a long-standing open question whether a left $V$-domain is a right $V$-domain, provide clues towards finding a possible counterexample to this open question or answering it in the affirmative for the ring $K[t;\sigma,\delta]$.

1. Introduction

Skew polynomial rings such as Weyl algebras and quantum groups have been a source of many interesting examples in noncommutative ring theory. In particular, differential polynomial ring over a “universal differential” field was the first example of a simple right-left $V$-domain and $PCI$-domain (A domain such that each proper cyclic right module is injective is called right $PCI$-domain). Although there exists an example of a nondomain which is a left $V$-ring but not a right $V$-ring, the question whether the property of being $V$-domain or $PCI$-domain is left-right symmetric remains open. The examples in the literature that relate to these two properties have been constructed by using either differential polynomial rings or localizations of twisted polynomial rings (Cf. [C],[Os]). In this paper, we obtain necessary and sufficient conditions for $K[t;\sigma,\delta]$ to be a left $V$-domain (equivalently, left $PCI$-domain). We also provide conditions for $K[t;\sigma,\delta]$ to be both right and left $V$-domain ($PCI$-domain).

2000 Mathematics Subject Classification. Primary 16D50, 16S36; Secondary.

The research of the third author was partially done while he was visiting the "center of ring theory and its applications" in Ohio University. He would like to thank all the members of the center for their support and warm hospitality.
2. Notations and Definitions.

Throughout all rings have unity. A ring $R$ is called a left $V$-ring if every simple left $R$-module is injective. A ring $R$ is called a left PCI-ring if each proper cyclic left $R$-module is injective (i.e. Left $R$-modules $R/I$ where $I$ is a nonzero left ideal are injective). It is known that $R$ is a left $V$-ring if and only if each left ideal is an intersection of maximal left ideals (Cf. Cozzens and Faith [CF]). Furthermore, a left PCI-ring $R$ is known to be either semisimple artINian or simple left Noetherian left hereditary domain such that each proper cyclic left $R$-module is semisimple (Faith [F], Damiano [D]).

Let $K$ be a division ring, $\sigma$ be an endomorphism of $K$ and $\delta$ be a $\sigma$-derivation of $K$ (i.e. $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$). Throughout the paper $R$ will denote the ring of left polynomials $K[t;\sigma,\delta]$ whose elements are polynomials of the form $\sum a_it^i$ where addition of polynomials is natural and multiplication of polynomials follows usual multiplication rule with $ta = \sigma(a)t + \delta(a)$. We note that if $\sigma$ is an automorphism then we can also view the ring $K[t;\sigma,\delta]$ as a (right) polynomial ring with multiplication induced by $at = t\sigma^{-1}(a) - \delta(\sigma^{-1}(a))$. If $\delta = 0$ then $K[t;\sigma,\delta]$ is the skew polynomial ring $K[t]$ considered by Jacobson and others and if $\sigma$ is the identity then $K[t;\sigma,\delta]$ is the standard differential polynomial ring $K[t;\delta]$. When $\delta = 0$ and $\sigma =$ identity, $K[t;\sigma,\delta]$ is the classical polynomial ring $K[t]$.

$K[t;\sigma,\delta]$ is a principal left ideal domain and right division algorithm holds, i.e. for any two polynomials $f(t)$ and $g(t)$ there exist polynomials $q(t)$ and $r(t)$ such that $f(t) = q(t)g(t) + r(t)$, where the degree of $r(t)$ is less than the degree of $g(t)$. The evaluation of a polynomial $f \in R$ at $c \in K$, denoted by $f(c)$, is the only element in $K$ satisfying $f(t) - f(c) \in R(t - c)$. The rule for evaluating the product $f(t)g(t)$ of left polynomials $f(t), g(t) \in R = K[t;\sigma,\delta]$ at $c \in K$ is given by:

$$(fg)(c) = 0 \text{ if } g(c) = 0 \text{ and } (fg)(c) = f(c^\sigma(c))g(c) \text{ if } g(c) \neq 0,$$

where, for $k \in K \setminus \{0\}$, $c^k$ denotes $\sigma(k)c\delta^{-1} + \delta(k)k^{-1}$. It can be verified that $(c^k)^{k'} = c^{kk'}$ for any nonzero elements $k, k' \in K$. An element $b \in K$ is $(\sigma, \delta)$-conjugate to $c$ if there exists $k \in K \setminus \{0\}$ such that $b = c^k$. The $(\sigma, \delta)$ conjugacy is an equivalence relation and we denote by $\Delta(c) = \{\sigma(k)c\delta^{-1} + \delta(k)k^{-1} \ | \ 0 \neq k \in K\}$ the $(\sigma, \delta)$ conjugacy class of $c \in K$. The conjugacy relation defined above is consistent with the classical conjugacy relation (i.e. when $\sigma = \text{Id}, \delta = 0$). In other words, in the classical case $\Delta(c) = \{kck^{-1} \ | \ 0 \neq k \in K\}$. For any polynomial $f \in R$ we denote by $V(f)$ the set of right roots of $f$, i.e. $V(f) = \{a \in K \ | \ f(a) = 0\}$. A monic polynomial $f$ in $R$ is called a Wedderburn polynomial, in short $W$-polynomial, if there exists a finite set $\{a_1, ..., a_n\} \subset K$ such that $Rf = \bigcap_{i=1}^{n} R(t - a_i)$. In other words, $f$ is the monic least left common multiple of the polynomials $t - a_1, ..., t - a_n$. In particular, any monic linear polynomial is a $W$-polynomial. The product formula mentioned above shows that the monic least left common multiple of $t - a, t - b$ with $a \neq b$ is the Wedderburn polynomial $f(t) = (t - b^{a-b})(t - a) = (t - a^{a-b})(t - b)$. Moreover it is known that a monic polynomial $f \in R$ is a $W$-polynomial if and only if $f$ can be written as a product of linear polynomials and every quadratic monic factor of $f$ is a $W$-polynomial. For more information on left polynomial rings $R = K[t;\sigma,\delta]$ the reader is referred to [LL4],[LL5],[LLO]. A ring $S$ is called an $n$-fr if each $n$-generated left ideal is free of unique rank $\leq n$. This property is left-right symmetric. We note that a domain $S$ is a 2-fr if and only if, for all
a, b ∈ S, Sa + Sb is principal whenever Sa ∩ Sb ≠ 0. The left-right symmetry implies that a similar characterization using right principal ideals also holds. Since a skew polynomial ring R = ℜ[t; σ, δ] over a division ring K is always a left principal ideal domain, it is a 2-Dr. In particular, if f, g ∈ R are such that fR ∩ gR ≠ 0 then fR + gR is a principal right ideal. This observation will be used freely in the text.

An element of a ring is called an atom if it is nonunit and cannot be written as a product of nonunits. An integral domain is called atomic if every element other than zero or a unit is a product of atoms. If an element a of a ring S is such that Sa = ∩n1=1Sfi, where f1, . . . , fn are atoms in S then a is called fully reducible. We remark that p ∈ R = ℜ[t, σ, δ] is an atom if and only if p is irreducible.

The idealizer of a left (right) ideal I in a ring R is the largest subring of R in which I is a two-sided ideal. It is denoted Idl(I).

For any ring S, Qrmax(S) (Qlmax(S)) denotes the right (left) maximal quotient ring of S. For a right nonsingular ring S, Qrmax(S) is always von Neumann regular right selfinjective. If, in addition, S is an integral domain then Qlmax(S) is also simple.

3. Preliminaries

Let us start with the following general lemma which may also be of independent interest.

**Lemma 3.1.** Let f, g be nonzerodivisors in a ring R. Then the following are equivalent:

(i) 0 → R/fR −→ R/frfRgR −→ R/Rg −→ 0 splits (where fr stands for the right multiplication by g).
(ii) 1 ∈ Rf + gR.
(iii) 0 → R/gRfRfRgR −→ R/(fR) −→ 0 splits (where fR stands for the left multiplication by f).
(iv) Idl(Rf) ⊆ Rf + gR.
(v) Idl(Rf) ⊆ Rf + gR.

**Proof.** (i)⇒(ii). By hypothesis there exists a map φ : R/fRfRgR −→ R/Rf such that φ ◦ gR = IdlR(Rf). Let y ∈ R be such that φ(1 + Rfg) = y + Rf. We then have (φ ◦ gR)(1 + Rf) = gy + Rf = 1 + Rf, i.e. gy − 1 ∈ Rf. This gives that there exists x ∈ R such that gy + xf = 1; proving (ii).

(ii)⇒(iii). If x, y ∈ R are such that xf + gy = 1, we define φ : R/gRfRfRgR : 1 + fgR −→ x + gR. This map is well defined since xf + gy = g(1 − y)gR. Moreover (φ ◦ fi)(u + gR) = xf(u + gR) = xfu + gR = (1 − gy)u + gR = u + gR. This means that φ is a splitting of fi, as desired.

(iii)⇒(ii) and (ii)⇒ (i) have similar proofs.

(ii)⇒(iv). Let us write 1 = xf + gyR for some x, y ∈ R. If r ∈ Idl(Rf) then there exists r′ ∈ R such that fr = r′f and we get r = xfr + gyr = xfr′f + gyr ∈ Rf + gR. This shows that Idl(Rf) ⊆ Rf + gR. The reverse implication is clear since 1 ∈ Idl(Rf).

(ii)⇒(v) is proved similarly. □

**Theorem 3.2.** Let R be an atomic left principal ideal domain. Then the following are equivalent:
(i) $R$ is a left PCI domain.
(ii) $R$ is a left V-domain.
(iii) $R = Rp + gR$ for every atom $p \in R$ and every element $g \in R \setminus \{0\}$.
(iv) $R = Rf + gR$ for every $f, g \in R \setminus \{0\}$.
(v) $R = Rf + I$ for every $f \neq 0$ and every nonzero right ideal $I$.
(vi) $1 \in Rf + I$ for every $f \neq 0$ and every nonzero right ideal $I$.
(vii) $1 \in Rf + gR$ for every $f, g \in R \setminus \{0\}$.
(viii) $R/Rfg \cong R/Rf \times R/Rg$ for every $f, g \in R \setminus \{0\}$.
(ix) $R/fgR \cong R/fR \times R/gR$ for every $f, g \in R \setminus \{0\}$.
(x) $1 \in Rp + qR$ for every atoms $p, q \in R$.
(xi) $1 \in Rp + I$ for every nonzero right ideal $I$ and every atom $p \in R$.
(xii) $\text{Idl}(Rp) \subseteq Rp + gR$ for every atoms $p, q \in R$.
(xiii) $\text{Idl}(qR) \subseteq Rp + qR$ for every atoms $p, q \in R$.
(xiv) All products $pq$, where $p, q$ are atoms, are fully reducible.
(xv) All nonunit elements of $R$ that are nonzero are fully reducible.

Proof. (i)$\Rightarrow$(ii) is obvious.

(ii)$\Rightarrow$(iii). Since $R$ is a left V-domain, left simple modules are injective, and so divisible. On the other hand, since $R$ is an atomic left principal ideal domain, the simple left modules are of the form $R/Rp$, where $p$ is an atom. But then, for every nonzero $g \in R$, $g(R/Rp) = R/Rp$. This yields $gR + Rp = R$, proving statement (iii).

(iii)$\Rightarrow$(iv). Let $p, r$ be atoms and $g$ be a nonzero element of $R$. By hypothesis we have $R = Rp + gR$ and $R = Rp + gR$. Using these two equalities we have $R = (Rp + gR)r + gR = Rp + gR$. Since $R$ is atomic, an induction on the length of $f$ implies that $Rf + gR = R$ for all $f, g \in R$.

(iv)$\Rightarrow$(i). $Rf + Rg = R$ implies that $g(R/Rf) = R/Rf$. Thus every cyclic module $R/Rf$ is divisible and hence injective because $R$ is a left principal ideal domain. Since the equivalence (iv)$\Rightarrow$(v) is obvious. We have thus shown that statements (i) to (v) are equivalent.

The implication (v)$\Rightarrow$(vi) is clear.

The equivalences (vi)$\Leftrightarrow$(vii)$\Leftrightarrow \ldots \Leftrightarrow$(ix) are either obvious or direct consequences of Lemma 3.1.

(vii)$\Rightarrow$(x) is obvious.

(x)$\Rightarrow$(vii) can be obtained in the same way as the implication (iii)$\Rightarrow$(iv).

The equivalences (x)$\Leftrightarrow$(xi)$\Leftrightarrow \ldots \Leftrightarrow$(xiii) are either obvious or direct consequences of Lemma 3.1.

(xiii)$\Rightarrow$(xiv). Since $(xi)$ is equivalent to (x) it is enough to show that if $1 \in Rp + qR$ then $pq$ is fully reducible. Therefore, suppose there exist $u, v \in R$ such that $up + vq = 1$. Notice that this implies that $qv \notin Rp$ and $v \notin Rp$. Left multiplying the equality $up + vq = 1$ by $p$ we get $pup + pqv = p$. Hence $pqv \in Rp \cap Rqv = Rpqv$ for some $p' \in R$. We thus have $p \in Rp'$. Observe that, since $qv \notin Rp$, $p'$ cannot be a unit. Since $p$ is an atom we conclude that $Rp = Rp'$. Let us put $Rp \cap Rv = Rp'v$ for some $p'' \in R$. Note that $p''$ is an atom. We then have $Rp'vq = Rp \cap Rqv = Rp \cap Rv \cap Rqv = Rp'v \cap Rqv = (Rp' \cap R)v$. This leads to $Rpq = Rp'q = Rp'' \cap Rq$. This shows that $pq$ is fully reducible, as required.

(xiv)$\Rightarrow$(xv). Let $f$ be a nonunit element of $R \setminus 0$. Since $R$ is assumed to be an atomic principal ideal domain, $f$ can be written as $f = up_1p_2 \cdots p_n$, where $p_1, \ldots, p_n$ are atoms. The number $n$ of atoms appearing in such a factorization of an element $f$ is
Let $R$ be a left domain if and only if it is a right domain.

For all nonzero elements $f, g \in R$, the following statements are equivalent:

(i) $R$ is a left domain and all irreducible polynomials are linear.

(ii) All monic polynomials are Wedderburn polynomials.

Corollary 3.3. Let $R$ be an atomic left and right principal ideal domain. Then $R$ is a left $V$-domain if and only if it is a right $V$-domain.

Proof. This is clear from the fact that statement (x) in Theorem 3.2 is left-right symmetric.

Corollary 3.4. Let $R = K[t; \sigma, \delta]$ be an Ore extension over a division ring $K$. Then the following are equivalent:

(i) $R$ is a left $V$-domain and all irreducible polynomials are linear.

(ii) All monic polynomials are Wedderburn polynomials.

4. $K[t; \sigma, \delta]$ as Left $V$-Domain.

In this section we will obtain necessary and sufficient conditions for an Ore extension $R = K[t; \sigma, \delta]$ to be a left $V$-domain.

Lemma 4.1. For $R = K[t; \sigma, \delta]$ the following statements are equivalent:

(i) For all nonzero elements $f \in R$, $(fR + Rf) \cap K \neq 0$.

(ii) $R$ is simple.

(iii) For all nonzero elements $f, g \in R$, $(fR + Rg) \cap K \neq 0$.

Proof. (i)⇒(ii). Let $f$ be a nonzero element in $R$. By hypothesis there exists a nonzero element $\alpha \in (fR + fR) \cap K$. Thus $\alpha \in fR$ and we get $Rf = R$.

(ii)⇒(iii). If $R$ is simple then, for nonzero elements $f, g \in R$, we have $R = RgR \subseteq (fR + Rg)R$. This gives that $(fR + Rg)R = R$ and hence $fR + Rg$ must contain a polynomial of degree 0.

(iii)⇒(i) is clear.

Let $f = \sum_{i=0}^{n} a_i t^i \in R = K[t; \sigma, \delta]$ be a monic polynomial ($a_n = 1$). In order to characterize the injectivity of the cyclic left $R = K[t; \sigma, \delta]$-module $R/Rf$, let us denote by $C_f$ the usual companion matrix of the monic polynomial $f = \sum_{i=0}^{n} a_i t^i$.

We thus have

\[
C_f = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-a_0 & -a_1 & -a_2 & \cdots & -a_{n-1}
\end{pmatrix}
\]

Considering $R/Rf$ as a left $K$-vector space, the element $\sum_{i=0}^{n-1} v_i t^i + Rf \in R/Rf$ is represented by the vector $v = (v_0, v_1, \ldots, v_{n-1}) \in K^n$ and the action of $t$ on
Let $R/Rf$ be a pseudo linear transformation i.e. $T_f$ is additive and we have $T_f(\alpha v) = \sigma(\alpha) T_f(v) + \delta(\alpha)v$ where $\alpha \in K$. For example, $T_{t-a} : K \rightarrow K$ is given by $x \mapsto \sigma(x)a + \delta(x)$. Notice that $T_{t} = \delta$ and, in case $\delta = 0$, $T_{t-1} = \sigma$. For $g = \sum_{i=0}^{l} a_i t^i \in R$ and $f \in R$, we also consider $g(T_f) = \sum_{i=0}^{l} a_i (T_f)^i$. For more details on pseudo linear transformations the reader is referred to [L]. Let us now mention the following technical result to be used later.

**Lemma 4.2.** Let $R = K[t; \sigma, \delta]$ be an Ore extension over a division ring $K$. Then for $f \in R$, and $0 \neq y \in K$, $f(\delta)(y) = f(0^y)y$.

**Proof.** Evaluating $f(t)y$ at $0$ and using the product formula stated in the Notations and Definitions (Section 2), we get $(f(t)y)(0) = f(0^y)y$. On the other hand the evaluation at $0$ is also the independent term $f(\delta)(y)$ of the polynomial $f(t)y$. This completes the proof. □

**Lemma 4.3.** Let $f$ be a nonzero element in $R = K[t; \sigma, \delta]$. Then the following are equivalent:

(i) $R/Rf$ is injective.
(ii) $g(T_f)$ is onto for all nonzero polynomials $g \in R$.

**Proof.** (i)$\Leftrightarrow$(ii). Since $R$ is a left principal domain, the injectivity of $R/Rf$ is equivalent to its divisibility. Now, the divisibility of $R/Rf$ is equivalent to the fact that for every nonzero $g \in R$, $g.R/Rf = R/Rf$. This means that the action of $g(t) = g(t.)$ is onto. □

Let us recall that the inner $\sigma$-derivation on $K$ induced by $a$, denoted $\delta_{a,\sigma}$ is defined by $\delta_{a,\sigma}(x) = ax - \sigma(x)a$, for $x \in K$.

**Lemma 4.4.** Let $f, a$ be elements in $R$ and $K$ respectively. Then

(a) $a \in fR + Rt$ iff $a \in \text{Im} f(\delta)$.
(b) $Rt + fR = R$ iff $f(\delta)$ is onto.
(c) $R/Rt$ is divisible iff for any $f \in R$, $f(\delta)$ is onto. In particular, if $\delta = 0$, $R/Rt$ is never divisible.
(d) $R/R(t-a)$ is divisible iff for any $f \in R$, $f(\delta - \delta_{a,\sigma})$ is onto.

**Proof.** (a) Suppose $a \in fR + Rt$ and let $g, h \in R$ be such that $fg + ht = a$. Comparing independent terms on both side we get $f(\delta)(g_0) = a$ where $g_0$ stands for the independent term of $g$. Conversely, suppose there exists $c \in K$ such that $f(\delta)(c) = a$. This implies that the independent term of $f(t)c$ is $a$, i.e. $f(t)c - a \in Rt$, as required.
(b) Since $Rt + fR = R$ if and only if $K \subseteq Rt + fR$, the statement (a) above gives us immediately the conclusion.
(c) We know $R/Rt$ is divisible if and only if for any $f \in R$, $fR + Rt = R$ and (b) above give us the first statement. The particular case is clear.
(d) This is a consequence of (c) using the fact that for any $a \in K$ one has $K[t; \sigma, \delta] = K[t-a; \sigma, \delta - \delta_{a,\sigma}]$ □

Let us give some examples:

**Example 4.5.** A well known example of Cozzens (cf.,[C] or [CF], chapter 5) is the following: let $k$ be a commutative field and $\delta$ be derivation of $k$. Assume that $k$ is differentially closed. Then $R = k[t; \text{Id.}, \delta]$ is a left and right $V$-domain.
Of course, not every simple skew polynomial ring over a division ring is a V-domain.

Example 4.6. Let \( K = k(x) \) be the field of rational functions over a field \( k \) of characteristic zero. \( R = k(x)[t; \text{Id.}, d/dx] \) is a simple noetherian domain. Since \( d/dx \) is not onto on \( k(x) \), \( R/Rt \) is not injective (Cf. Lemma 4.4) and hence \( R \) is not a left V-domain.

Theorem 4.7. Assume \( R = K[t; \sigma, \delta] \) is a left V-domain. Then

1. For any \( a, b, c \in K \), the \((\sigma, \delta)\) metro equation
   
   \[ bx - \sigma(x)a - \delta(x) = c \]

   has a solution in \( K \).

2. For any \( n \in \mathbb{N} \), the map \( \delta + n\sigma \) is onto.

3. \( \text{End}_R(R/Ra) \cong \text{End}(R/aR) \). These rings are division rings if and only if \( a \in R \) is an atom.

Proof. (1) Since \( R \) is a left V-domain, we have for all \( a, b \in R \), \( t - b \) divides \( R/R(t - a) \) i.e. \( R(t - a) + (t - b)R = R \). So, in particular, for any \( c \in K \) there exists \( u, v \in R \) such that \( u(t - a) + (t - b)v = c \). Dividing \( v \) on the right by \( t - a \) and writing \( v = q(t - a) + d \) for some \( q \in R \) and \( d \in K \). We may thus assume that \( v \in K \). Comparing the degrees we then conclude that \( u \in K \) as well. The above equality gives us that \((u + \sigma(v))t + \delta(v) - ua - bv - c = 0 \). This leads to \( \sigma(v)a + \delta(v) - bv - c = 0 \). This shows that \( v \) is a solution of the metro equation.

(2) This follows from (1) by putting \( a = n \) and \( b = 0 \).

(3) Let us recall that \( \text{Idl}(aR) \) and \( \text{Idl}(Ra) \) stand for the idealizers of \( aR \) and \( Ra \) respectively. It is well known and easy to prove that if \( R \) is a domain then \( \text{Idl}(Ra)/Ra \cong \text{Idl}(aR)/aR \). This shows that \( \text{End}_R(R/Ra) \cong \text{End}_R(R/aR) \). Schur’s lemma and the fact that \( R \) is a left principal ideal domain imply that \( \text{End}_R(R/Ra) \) is a division ring if and only if \( a \) is an atom. Conversely if \( a = p_1 \ldots p_n \), \( n > 1 \), is a product of atoms then Theorem 3.2 (\textit{viii}) implies that \( R/Ra \) is a product of simple modules \( R/Rp_i \), \( 1 < i \leq n \) but \( R/Ra \) is not simple. Hence \( \text{End}_R(R/Ra) \) cannot be a division ring.

Our methods give criteria for certain cyclic left modules to be injective modules.

B. Osofsky considered injective modules over skew polynomial rings \( k[t; \sigma] \), where \( k \) is a commutative field \( \text{[Os]} \). We conclude this section with a generalization of one of her results to the case when \( k \) is a division ring.

Proposition 4.8. Let \( \sigma \) be an endomorphism of a division ring \( K \) and let \( R \) denote the skew polynomial ring \( K[t; \sigma] \). Then \( R/R(t - 1) \) is injective if and only if for any \( p \in K[t; \sigma] \), \( p(\sigma) \) is onto.

Proof. It is clear from the paragraph preceding Lemma 4.2 that \( T_{t-1} = \sigma \).

5. \( K[t; \sigma, \delta] \) as a V-Domain With Unique Simple Left Module

We will now give characterizations of Ore extensions \( R = K[t; \sigma, \delta] \) which are left V-domains and have a unique simple left module (up to isomorphisms). This
will explain some features of examples given by Cozzens in ([C]). We assume throughout the section that \( \delta \neq 0 \).

First, we give general characterizations of Ore extensions \( R \) having a unique simple left \( R \)-module.

**Proposition 5.1.** Let \( R = K[t; \sigma, \delta] \) be an Ore extension over a division ring \( K \) where \( \delta \neq 0 \). Then the following are equivalent:

(i) \( R \) admits, up to isomorphisms, a unique simple left module.

(ii) Every monic irreducible polynomial is of the form \( t - 0^y \) for some \( y \in K \setminus \{0\} \).

(iii) For any nonconstant polynomial \( f \in R \), \( \text{Ker}(f(\delta)) \neq 0 \), that is, each linear differential equation over \( K \) has a nonzero solution in \( K \).

(iv) For any irreducible polynomial \( p \), \( \text{Ker}(p(\delta)) \neq 0 \).

**Proof.** (i) \( \Rightarrow \) (ii). Every simple left \( R \)-module is of the form \( R/Rp \) for some irreducible polynomial \( p \). The uniqueness of the isomorphic class of simple left \( R \)-modules gives \( R/Rp \cong R/Rt \). Comparing the dimensions as left \( K \) vector spaces we see that \( p = t - a \) for some \( a \in K \). Write \( \varphi(1+Rp) = \varphi(1+R(t-a)) = y + Rt, y \in R \). Note that \( y \neq 0 \). Then \( (t-a)y \in Rt \) i.e. \( \sigma(y)t + \delta(y) - ay \in Rt \) and so \( \delta(y) - ay = 0 \). Since \( y \neq 0 \) we obtain that \( p = t - a = t - \delta(y)y^{-1} \).

(ii) \( \Rightarrow \) (iii). This is clear since, by (ii), every nonconstant polynomial, say \( f \in R \), has a right factor of the form \( t - 0^y \). Lemma 4.2 then implies that \( f(\delta)(y) = 0 \).

(iii) \( \Rightarrow \) (iv) is obvious.

(iv) \( \Rightarrow \) (i). Since any polynomial is a product of irreducible factors, we conclude that for any \( f \in R \) there exists \( y \neq 0 \) such that \( f(\delta)(y) = 0 \). By Lemma 4.2 we get that \( f(0^y) = 0 \) and this shows that \( t - 0^y \) divides \( f \) on the right. In particular, the irreducible polynomials are of the form \( t - 0^y \). Since \( R/R(t - 0^y) \cong R/Rt \) under the mapping \( T \mapsto \varphi \), we conclude that the left simple modules are all isomorphic to \( R/Rt \). \( \square \)

**Corollary 5.2.** Suppose \( R = K[t; \sigma, \delta] \) has a unique simple left module (up to isomorphisms), then \( K = \Delta(0) = \{\delta(x)x^{-1} | x \in K \setminus \{0\}\} \).

**Proof.** This is clear from the above proposition 5.1 (ii). \( \square \)

Under the hypothesis that \( R \) admits a unique simple left \( R \)-module the following lemma gives a criterion for determining when \( R \) is a left \( V \)-domain.

**Lemma 5.3.** Suppose \( R = K[t; \sigma, \delta] \) has a unique simple left \( R \)-module. Then the following are equivalent:

(i) \( R \) is a left \( V \)-domain.

(ii) \( \delta \) is onto.

(iii) For every \( c \in K \), \( (t - c)(\delta) \) is onto.

**Proof.** (i) \( \Rightarrow \) (ii). If \( R \) is a left \( V \)-domain, Theorem 4.7 shows that \( \delta \) is onto.

(ii) \( \Rightarrow \) (i). By hypothesis \( R \) admits a unique simple left module and \( \delta \) is onto. Let us first show that \( Rt + tR = R \). Since every \( f \in R \) can be written as \( f = qt + a, q \in R, a \in K \), it is enough to show that \( K \subseteq Rt + tR \). Let \( a \) be an element in \( K \). Since \( \delta \) is onto, there exists \( b \in K \) such that \( \delta(b) = a \). So, we get \( tb - \sigma(b)t = \delta(b) = a \in Rt + tR \). This yields that \( t \) divides the simple left \( R \) module \( R/Rt \). Hence \( t \) divides all the simple modules. In other words \( tR + Rp = R \) for any irreducible polynomial.
Let every irreducible monic polynomial is of the form $R$ is a left $\text{V}$-domain.

Every square matrix over $\text{V}$ that $f$ divides $(\text{Frob})$, we can write a monic irreducible polynomials are linear and hence they are of the form $R/tR$ i.e. $(R/tR)p = R/tR$. Let $q \in R$ be another irreducible polynomial.

By hypothesis, we have $R/tR \cong R/Rq$ and hence also $R/qR \cong R/tR$ (Cf. [Co1]). So $p$ divides the right module $R/qR$ i.e. $(R/qR)p = R/qR$. This gives $Rp + qR = R$ for every irreducible polynomials $p, q \in R$. Theorem 3.2 shows that $R$ is a left $\text{V}$-domain.

(iii) $\Leftrightarrow$ (iii). By hypothesis, $R$ has a unique simple left $R$-module (up to isomorphism) hence for any $c \in K$, $R/Rt \cong R/(t-c)R$. As noticed in the proof of (ii) $\Leftrightarrow$ (i) above, this implies that $R/(t-c)R \cong R/(t-c)R$ for all $c \in K$. Now, Lemma 4.4 shows that $(t-c)(\delta)$ is onto if and only if $R = (t-c)R + Rt$ which is equivalent to saying that $t$ divides $R/(t-c)R \cong R/tR$. Using Lemma 4.4 again we conclude that $\delta$ is onto if and only if $(t-c)(\delta)$ is onto.

Let us recall that two matrices $A, B \in M_n(K)$ are $(\sigma, \delta)$ similar, denoted by $B = AP$, if there exists an invertible matrix $P \in M_n(K)$ such that $B = \sigma(P)AP^{-1} + \delta(P)P^{-1}$.

**Theorem 5.4.** Let $R = K[t; \sigma, \delta]$ be an Ore extension over a division ring. Then the following are equivalent:

(i) $R$ is a left $\text{V}$-domain with a unique simple left module.

(ii) Every monic polynomial is Wedderburn and $K = \Delta(0)$.

(iii) Every irreducible monic polynomial is of the form $t - \delta(y)y^{-1}$ for some $y \in K \setminus \{0\}$ and every quadratic polynomial is Wedderburn.

(iv) Every monic nonconstant linear differential equation has a nonzero solution and $(t-c)(\delta)$ is onto for every $c \in K$.

(v) Every square matrix over $K$ is $(\sigma, \delta)$ similar to a diagonal matrix with coefficients in $\Delta(0)$.

(vi) Every square matrix over $K$ is $(\sigma, \delta)$ similar to the zero matrix of the same size.

(vii) For every $n \geq 1$, for all $A, B \in M_n(K)$, $A$ is $(\sigma, \delta)$ similar to $B$.

**Proof.** (i) $\Rightarrow$ (ii). By Corollary 5.2 $K = \Delta(0)$. Let $f$ be a monic polynomial. By Theorem 3.2 and Proposition 5.1 we can write $Rf = \cap_{i=1}^n R(t - a_i)$. This shows that $f$ is Wedderburn.

(ii) $\Rightarrow$ (iii). Since Wedderburn polynomials split into linear factors, it is clear by (ii) that the monic irreducible polynomials are linear and hence they are of the form $t - \delta(y)y^{-1}$, for some $y \in K \setminus \{0\}$.

(iii) $\Rightarrow$ (iv). Let $g(\delta) = 0$ be a monic linear differential equation. The corresponding polynomial $g(t)$ has a right factor of the form $t - \delta(y)y^{-1}$, for some $y \in K \setminus \{0\}$. In this case, Lemma 4.2 shows that $g(\delta)(y) = 0$. Let $x$ be an element of $K$. To show that $(t-c)(\delta)$ is onto we produce an element $a$ such that $(\delta)(a) - ca = x$. Consider the polynomial $f(t) = (t - \delta(x)x^{-1})(t-c)$. By hypothesis $f$ is a Wedderburn polynomial and hence $f$ has a right root $d$ different from $c$. By invoking the product formula we get $0 = f(d) = (d^{t-c} - \delta(x)x^{-1})(d-c)$ and so $d^{t-c} - \delta(x)x^{-1} = 0$. Our hypothesis implies that there exists $y \in K \setminus \{0\}$ such that $d = 0^y$. We then obtain $0^y(x-cy) = 0^y$. This implies $(0^y(x-cy)x^{-1} = 0^y)x^y = 0^y$. This gives $0^{-1}(\delta(y) - cy) = 0$. Hence, $x^{-1}(\delta(y) - cy) \in \text{Ker}\delta$ and so, $\delta(y) - cy = xb$ for some $b \in \text{Ker}\delta$. Since $c \neq 0^y$, $b$ cannot be zero. Putting $a = yb^{-1}$ we get $\delta(a) - ca = x$. This shows that $(t-c)(\delta)$ is onto.
(iv) $\Rightarrow$ (i). Proposition 5.1 shows that $R = K[t; \sigma, \delta]$ admits a unique simple left $R$-module (up to isomorphisms) and that the irreducible polynomials are linear. Lemma 5.3 implies that $R$ is indeed a left $V$-domain, as desired.

(ii) $\Leftrightarrow$ (v). This equivalence is clear since it has been proved in [LLO] that a matrix is $(\sigma, \delta)$ diagonalizable if and only if the invariant factor of the highest degree is Wedderburn.

(v) $\Leftrightarrow$ (vi). An easy computation shows that if $y_1, \ldots, y_n$ are nonzero elements from $K$, then $\text{diag}(y_1^n, \ldots, y_n^n) = (0_n)^P$ where $P = \text{diag}(y_1, \ldots, y_n)$ and $0_n$ denotes the square zero matrix of size $n \times n$. This yields the equivalence.

(vi) $\Leftrightarrow$ (vii). This is obvious because of the transitivity of the $(\sigma, \delta)$-similarity.

Remarks 5.5. 

a) The conditions (iii) and (iv) in the above theorem do not explicitly refer to $\sigma$. This may, mistakenly be looked as a condition that is independent of $\sigma$. But we remark that $\delta$ is a $\sigma$ derivation (with a unique $\sigma$).

b) Let us briefly describe Resco’s result (Cf.[R]) and indicate how it can be imitated in order to give another proof of the equivalence (i) $\Leftrightarrow$ (vii) in Theorem 5.4 above.

Let $K$ be a division ring with center $k$ and $T = K[t](k[t])^{-1} = K \otimes k(t)$ be the central localization of $K[t]$. A square matrix $A \in M_n(K)$ determines a structure of left $K[t]$-module on the space of rows $^nK$ via $t(v_1, \ldots, v_n) = (v_1, \ldots, v_n)A$ and $\alpha(v_1, \ldots, v_n) = (\alpha v_1, \ldots, \alpha v_n)$. This action can be extended into a left $T$-module structure for $^nK$ if and only if for any $f(t) \in k[t]$, $f(A)$ is invertible. Such a matrix is called totally transcendental. Resco (Cf.[R] or P.M.Cohn [Co2] Theorem 8.4.9 p. 391) showed that $T$ is a left (and right) $V$-domain if and only if any two totally transcendental square matrices of the same size are similar (with Cohn’s terminology $K$ is said to be matrix homogeneous). Resco’s result was used by Faith and Menal [FM] to construct an example of a left noetherian left annihilator ring which is not left artinian. In fact we can use similar methods as the ones used by Resco to give another proof of a part of the above theorem. In our case if $R = K[t; \sigma, \delta]$ is simple we don’t need localization and for any matrix $A \in M_n(K)$, we can define a left $R$-module structure on the space $^nK$ via:

$$ t.(v_1, \ldots, v_n) = (\sigma(v_1), \ldots, \sigma(v_n))A + (\delta(v_1), \ldots, \delta(v_n)). $$

Let us denote this left $R$-module by $(^nK, A)$. As in Resco’s paper one can easily show that every left $R$-module of finite length is induced by a square matrix (the proof is even simpler since we don’t have to check that the matrix is totally transcendental). Adapting the Resco’s arguments we then find back the equivalence (i) $\Leftrightarrow$ (vii) in Theorem 5.4 above. Of course, in this circumstances, the unique simple left $R = K[t; \sigma, \delta]$-module is $K$ with the $R$-module structure given by $f(t).x = f(\delta)(x)$, for $x \in K$.

6. $K[t; \sigma, \delta]$ as Both Left and Right $V$-domain

We will now examine the right structure of the Ore extension $R = K[t; \sigma, \delta]$. If $\sigma$ is an automorphism then $R$ is a right and left principal ideal domain. Hence, in this case, $R$ is a right $V$-domain if and only if it is a left $V$-domain. So we will assume from now on that $\sigma$ is not onto and study conditions that make $R$ a left and right $V$-domain. Let us recall that $R$ is a 2-fir. This implies, in particular, that for any atom $p \in R$, the ring $End_p(R/pR)$ is a division ring and hence $R/pR$ is indecomposable. Recall that, for a 2-fir $R$, if $fR \cap gR \neq 0$, $f, g \in R$ then there
Let us fix a basis for every \( a \) monic, \( R/fR \) is indecomposable. If we assume that \( f \) is an atom we know, by the remark made at the beginning of this section, that (2). We proceed by induction on the numbers of factors in a factorization of the previous case.

Since \( b \) \( fR \) is an atomic left factor of \( aR \) of that there exists \( aR \) modules injective itself. This contradiction shows that \( aR \) not uniform. If \( f \) is not an atom and that \( R/fR \) is injective we conclude that it is not uniform (Cf.\( [R/fR] \) is indecomposable. If we assume that \( R/fR \) is not injective.

\[ \begin{align*}
R/fR &= aR/(agR) \\
&\cong R/gR
\end{align*} \]

This contradicts (1).

Assume now that \( f \) is not an atom and that \( R/fR \) is injective. If \( R/fR \) is indecomposable it will be uniform which again contradicts (1). Thus we can write \( R/fR = \frac{x_1R+x_2R}{fR} \oplus \frac{x_2R+x_1R}{fR} \) for some \( x_1, x_2 \in R \) such that each direct summand is not zero. If either \( x_1R \cap fR = 0 \) or \( x_2R \cap fR = 0 \), then the corresponding summand is isomorphic to \( R/fR \) and, as a direct summand of the injective module \( R/fR \), is injective itself. This contradiction shows that \( x_iR \cap fR \neq 0, \ i = 1, 2 \).

By the property of 2-firs recalled at the beginning of this section, we get \( x_iR + fR = y_iR \), for some \( y_1, y_2 \in R \). Let us write \( f = y_iy_i \) for \( i = 1, 2 \). we obtain \( R/fR = y_1R/fR \oplus y_2R/fR \cong R/z_1R \oplus R/z_2R \). Since the direct summands \( y_1R/fR \) and \( y_2R/fR \) are both nonzero, \( y_1 \) and \( y_2 \) are both nonunits, so \( z_1 \) and \( z_2 \) have shorter factorizations than \( f \). As direct summands of the injective module \( R/fR \), the modules \( R/z_iR \) are injective. Our induction hypothesis then gives the desired contradiction.

(3) If \( aR \) is not closed then \( aR \) is essential in some right ideal \( I \neq aR \). We claim that there exists \( b \in R \) such that \( aR \) is essential in \( bR \). For \( i \in I \setminus aR \) the essentiality of \( aR \) in \( I \) shows that \( iR \cap aR \neq 0 \). By the 2-fir property of \( R \), \( iR + aR = bR \), for some \( b \in R \). Then \( aR \subseteq bR \subseteq I \), proving our claim.

Let us write \( a = bc \), where \( c \) not a unit. Then \( bcR \subseteq bR \) and so \( cR \subseteq R \). If \( p \) is an atomic left factor of \( c \) we have \( cR \subseteq pR \subseteq R \). So, \( pR \subseteq R \). Choose \( q \notin pR \).

By essentiality of \( pR \) in \( R \), \( pR \cap qR \neq 0 \) and then, by the property of 2-fir, there exists \( h \in R \) such that \( pR + qR = hR \). So, \( pR \subseteq hR \); write \( p = hr \), for some \( r \in R \). Since \( q \notin pR \), \( pR \) is strictly contained in \( hR \), i.e. \( r \) is not a unit. Thus \( h \) is a unit and we have \( pR + qR = R \), for any \( q \notin pR \). This shows that \( pR \) is maximal. Thus
$R/pR$ is simple and hence uniform. Part (1) then yields that $\sigma$ must be onto. This contradiction shows that $aR$ is closed. □

**Theorem 6.2.** Let $R = K[t; \sigma, \delta]$ be a left $V$-domain. Then the following statements are equivalent:

(i) There exists an element $f \in R$ such that $R/fR$ is injective.
(ii) $\sigma$ is onto.
(iii) $R$ is a right principal Ore $V$-domain.
(iv) for every nonzero $f \in R$, $R/fR$ is injective.
(v) There exists an irreducible polynomial $f \in R$ such that $R/fR$ is uniform.
(vi) There exists an irreducible polynomial $f \in R$ such that $R/fR$ is CS.
(vii) $R$ is a right PCI domain.
(viii) $Q_{\max}^r(R)$ is directly finite.

**Proof.** (i)$\Rightarrow$(ii) follows by Proposition 6.1.

(iii)$\Rightarrow$(iv). Since $R$ is both a right and left principal Ore domain when $\sigma$ is onto, the implication (ii)$\Rightarrow$(iii) is due to the symmetry of some of the statements in Theorem 3.2.

(iv)$\Rightarrow$(v). This is clear since injective indecomposable modules are uniform.

The equivalence (v)$\Leftrightarrow$(vi) is true because an indecomposable module is CS if and only if it is uniform.

(v)$\Rightarrow$(vii). By Proposition 6.1, $\sigma$ is onto. Theorem 3.2 shows that for all $f, g \in R \setminus \{0\}$, $Rf + gR = R$ i.e. $R/gR$ is divisible for all nonzero $g \in R$. Since $\sigma$ is onto, $R$ is a right principal ideal domain. Thus, $R/gR$ is injective for all $0 \neq g \in R$, as required.

(vii)$\Rightarrow$(viii). If $R$ is a right PCI-domain, $Q_{\max}^r(R)$ is a division ring.

(viii)$\Rightarrow$(i). Let us first show that nonzero elements of $R$ are left invertible in $Q := Q_{\max}^r(R)$. This is part of folklore; If $0 \neq a \in R$, there exists $q \in Q$ such that $a = aqa$. If $qa - 1 = 0$, we are done. If not then $0 \neq qa - 1 \in \text{rann}Qa := \{x \in Q \setminus \{0\} \mid ax = 0\}$. Since $R \subset_c Q$, we get that $\text{rann}Ra \neq 0$. This is impossible since $R$ is a domain.

Now, if $x \in Q$, there exists an essential right ideal $E$ such that $0 \neq xE \subseteq R$. In particular there exists an element $e \in R$ such that $0 \neq xe \in R$. By the above $xe$ is left invertible in $Q$, i.e. there exists $q \in Q$ such that $qxe = 1$. Direct finiteness of $Q$ implies that $xeq = 1$. This shows that every nonzero element of $Q$ is right invertible and hence $Q$ is a division ring. This shows that $R$ is left and right Ore. Hence $\sigma$ is onto. □

**Remark 6.3.** Damiano showed that a right PCI-domain $R$ which is left coherent is a left PCI-domain ([D]), However his proof contains an error. He claims, by invoking a result in Stenstrom’s book ([S] XI, Corollary 3.2 ) that the right maximal quotient ring of $R$ is flat as right $R$-module. In fact the right maximal quotient ring is flat as a left $R$-module. This leads Damiano to the wrong conclusion that the right and left maximal quotient ring of $R$ are isomorphic and $R$ must be a left Ore domain.
Since a polynomial ring $R = K[t; \sigma, \delta]$ is always a semifir, it is both left and right coherent. If correct, Damiano’s theorem would imply that if $R = K[t; \sigma, \delta]$ is a left Ore $V$-domain then it is also a right Ore domain. Consequently $\sigma$ would be onto. We do not have any example when $K[t; \sigma, \delta]$ is a left $V$-domain with $\sigma$ not onto.

References


[Co2] P. M. Cohn: Skew Fields, Theory of General Division Rings, Encyclopedia in Math., Vol. 57,

11-14.


[Ja2] N. Jacobson: The equation $x' \equiv xd - dx = b$, Bull. A.M.S. 50(1944), 902-905.

[Lam1] T. Y. Lam: A First Course in Noncommutative Rings, Graduate Texts in Math., Vol. 131,

[Lam2] T. Y. Lam: Exercises in Classical Ring Theory, Problem Books in Mathematics, Springer-


[LL1] T. Y. Lam and A. Leroy: Principal one-sided ideals in Ore polynomial rings, in Algebra and


appear in proceedings of a conference held in Chennai (India) in December 2006

[LQ] A. Leroy and A. Ozturk:F-independence and factorizations in 2-firs Communications in


107-114.


[S] B. Stenström: Rings of quotients, Die Grundlehren der mathematische Wissenschaften in
Einzeldarstellungen, Vol. 217, Springer Verlag, 1975

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ATHENS, OHIO, OH. 45701
E-mail address: Jain@math.ohiou.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720
E-mail address: lam@math.berkeley.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITÉ D’ARTOIS, 62307 LENS, FRANCE
E-mail address: leroy@euler.univ-artois.fr