When is a Semilocal Group Algebra Continuous?

S.K.Jain, Pramod Kanwar, and J.B.Srivastava

Department of Mathematics, Ohio University, Athens, OH 45701, USA
Email: jain@ohiou.edu
Department of Mathematics Ohio University-Zanesville, Zanesville, OH 43701, USA
Email: pkanwar@math.ohiou.edu
Department of Mathematics Indian Institute of Technology, Delhi 110016, India
Email: jbsrivas@maths.iitd.ernet.in

Abstract
It is shown that (i) a semilocal group algebra $KG$ of an infinite nilpotent group $G$ over a field $K$ of characteristic $p > 0$ is CS (equivalently continuous) if and only if $G = P \times H$, where $P$ is a locally finite, infinite p-group and $H$ is a finite abelian group whose order is not divisible by $p$, (ii) if $K$ is a field of characteristic $p > 0$ and $G = P \times H$ where $P$ is an infinite locally finite p-group (not necessarily nilpotent) and $H$ is a finite group whose order is not divisible by $p$ then $KG$ is CS if and only if $H$ is abelian. Furthermore, commutative semilocal group algebra is always continuous and for $PI$ group algebras this holds for local group algebras; however this result is not true, in general.

1 Introduction

Semilocal finitely $\sum -CS$ group algebras of a solvable or linear group were characterized earlier as precisely self-injective group algebras (See Theorem 0.1 in [3] and related results in [2]). The purpose of this paper is to continue our investigation as to when a semilocal group algebra $KG$ is continuous? We consider the cases when (i) $G$ is nilpotent, and (ii) $G = P \times H$ where $P$ is an infinite locally finite $p-$group and $H$ is a finite group whose order is not divisible by $p$ (= char $K$). It is known in general that if $KG$ is continuous then $G$ is locally finite [1]. Theorem 4.3 shows that a semilocal group algebra $KG$ of an infinite nilpotent group $G$ over a field $K$ of characteristic $p > 0$ is CS (equivalently continuous) if and only if $G = P \times H$
where $P$ is an infinite locally finite $p$–group and $H$ is finite abelian group whose order is not divisible by $p$. Theorem 4.1 shows that every commutative semilocal group algebra is continuous. This raises a natural question as to whether every $PI$–semilocal group algebra is also continuous? Example 5.1 shows that this is not true, in general. However, the result holds for any local $PI$ group algebra, that is, such a group algebra is continuous.

2 Definitions and Notation

Unless otherwise stated throughout $K$ will denote a field of characteristic $p > 0$ and $G$, a group. $KG$ or $K[G]$ will denote the group algebra of a group $G$ over a field $K$. $R$ will denote a ring with identity and $J(R)$ its Jacobson radical. $R$ is called local if it has a unique maximal right ideal. If $R = KG$ is local over a field $K$ then $\text{char } K = p > 0$ and $G$ is a $p$–group. A ring $R$ is called semilocal if $R/J(R)$ is semisimple artinian. If $R = KG$ is semilocal then $G$ is a torsion group. A ring $R$ is called semiperfect if $R/J(R)$ is semisimple artinian and idempotents modulo $J(R)$ can be lifted. For a locally finite group $G$, $KG$ is semiperfect if and only if $G/O_p(G)$ is finite, where $O_p(G)$ denotes the maximal normal $p$–subgroup of $G$.

A ring $R$ is called a right $CS$–ring if it satisfies any one of the following equivalent conditions referred to as $(C_1)$–condition : (i) Each right ideal is essential in $eR$, $e = e^2$ ; (ii) Each complement right ideal is of the form $eR$, $e = e^2$. A ring $R$ is called right finitely $\sum-CS$ ring if for each positive integer $n$, the $n \times n$ matrix ring is also a right $CS$–ring. A ring $R$ is called right quasi-continuous (also known as right $\pi$–injective) if it satisfies any one of the following equivalent statements:

(i) For all right ideals $A_1$, $A_2$ with $A_1 \cap A_2 = (0)$, each projection $\pi_i : A_1 \oplus A_2 \longrightarrow A_i$, $i = 1, 2$ can be lifted to an endomorphism of $R_R$.

(ii) $R$ satisfies the condition $(C_1)$ given above and the condition $(C_3)$: If $eR \cap fR = (0)$, $e = e^2$, $f = f^2$ then $eR \oplus fR = gR$ where $g = g^2$.

A ring $R$ is known as right continuous (as defined by von Neumann) if it satisfies the condition $(C_1)$ and the condition $(C_2)$: If $aR \simeq eR$, $e = e^2$ then $aR = fR$, $f = f^2$. It is known $(C_2) \Rightarrow (C_3)$ and so every right continuous
ring is right quasi-continuous ring (also known as right $\pi-$injective ring). A ring $R$ is called principally right self-injective if each $R-$homomorphism from $aR \rightarrow R$, $a \in R$, can be lifted to an $R-$endomorphism of $R_R$. The concepts of CS, quasi-continuous, continuous, principally injective and injective are right-left symmetric for group algebras. Thus we will omit the prefix right or left when dealing with these concepts for group algebras.

A group $G$ is called locally finite if each finite subset generates a finite subgroup.

3 Preliminaries

In this section, we give the results that are used often in the proofs of main results. We begin with a lemma that if $KG$ is $\pi-$injective (= quasi-continuous) then the torsion elements form a subgroup of $G$.

**Lemma 3.1.** If $KG$ is quasi-continuous then the torsion elements of $G$ form a locally finite normal subgroup of $G$. In particular, if $KG$ is continuous then $G$ is locally finite, ([1], Theorem 4.3).

**Proof.** Let $T$ be the set of torsion elements of $G$. Let $a, b \in T$. Let $H = \langle a, b \rangle$. Then $\omega(H) = KG(a-1)+KG(b-1)$ and $\text{r.ann } \omega(H) = \text{r.ann } KG(a-1) \cap \text{r.ann } KG(b-1)$. Suppose $\text{r.ann } KG(a-1) \cap \text{r.ann } KG(b-1) = (0)$. Then the projection $\pi_1 : \text{r.ann } KG(a-1) \oplus \text{r.ann } KG(b-1) \rightarrow \text{r.ann } KG(a-1)$ can be extended to $\pi_1^* : KG_{KG} \rightarrow KG_{KG}$. Let $\pi_1^*(1) = x \in KG$. Now $\pi_1^*(y) = \pi_1(y) = y$, for all $y \in \text{r.ann } KG(a-1)$. Also, $\pi_1^*(y) = xy$. This implies $(x - 1) \in \text{l.ann}(\text{r.ann } KG(a-1)) = KG(a-1)$. Furthermore, $\pi_1^*(r.ann KG(b-1)) = 0$ and so $x(r.ann KG(b-1)) = 0$. This implies $x \in \text{l.ann}(\text{r.ann } KG(b-1)) = KG(b-1)$. Therefore, $1 = x + (1 - x) \in KG(b-1) + KG(a-1) \subseteq \omega(KG)$, a contradiction. Thus $\text{r.ann } \omega(H) \neq 0$, which yields that $H = \langle a, b \rangle$ is finite. This gives $H \subseteq T$ and so $T$ is a subgroup of $G$. The above argument, using ([7], 3.1.2, p.68), can be extended to show by induction that any finite subset of $T$ generates a finite subgroup.

Let $KG$ be continuous. If $g \in G$ has infinite order then $1 - g$ is regular.
and hence invertible, a contradiction. Hence $G$ is torsion. That $G$ is locally finite follows from the first part. This completes the proof. 

The following lemma is stated in ([5], Theorem 4.1) for a group algebra over a field. However, the proof carries over to group algebra over a division ring.

**Lemma 3.2.** Let $D$ be a division algebra over a field $K$ with characteristic $p > 0$. Let $P$ be a locally finite $p$–group. Then $DP$ is a continuous local ring.

**Proof.** We sketch its proof briefly just for the convenience of the reader. Firstly, $\omega(DP)$ is nil and hence $J(DP) = \omega(DP)$ ([4], Corollary, p.682). Thus $DP$ is a local algebra. Next $\alpha, \beta \in DP \Rightarrow \alpha, \beta \in DH$, where $H = \langle \text{Supp}(\alpha) \cup \text{Supp}(\beta) \rangle$ is a finite $p$–group. Thus $DH$ is local selfinjective and therefore uniform. This implies $DP$ is local uniform with nil radical. Hence $DP$ is continuous. 

Next, we record below a wellknown fact.

**Lemma 3.3.** If $G$ is a locally finite group and $K$ is a field with $\text{char } K = p > 0$ then $KG$ is semilocal if and only if $KG$ is semiperfect.

The lemmas which follow are stated for easy reference.

**Lemma 3.4** ([7], Lemma 10.1.6). If $KG$ is semilocal then $G$ is torsion.

**Lemma 3.5** ([8], Theorem 7.4.10, p.230). Let $K$ be a field of characteristic $p > 0$, and let $G$ be a finite group. Then $KG$ has no nonzero nilpotent elements if and only if $G$ is an abelian $p'$–group.

The following theorem is due to Farkas.

**Theorem 3.1** ([7], Exercise 10(ii), p.107). $KG$ is principally selfinjective if and only if $G$ is locally finite.

Next we state a fact which is a consequence of the above theorem and the result that if $KG$ is principally self-injective then $KG$ satisfies the condition $(C_2)$ (Lemma [6], page 119, Ex.46).

**Lemma 3.6.** Let $G$ be a locally finite group. Then $KG$ is continuous if and only if $KG$ is CS.
4 Main Results

The following theorem provides plenty of examples of continuous rings.

**Theorem 4.1.** Let $KG$ be a semilocal group algebra of an abelian group $G$ over a field $K$ of characteristic $p > 0$. Then $KG$ is continuous.

Proof. Since $KG$ is semilocal, $G$ is torsion by Lemma 3.4. But then $G$ is locally finite because $G$ is abelian. Thus $J(KG) = N^*(KG)$ and $KG$ is semiperfect. Therefore, $G/O_p(G)$ is finite ([7], 10.1.3, p.409). This yields, $G \simeq O_p(G) \times A$, where $A$ is a finite abelian group such that $p \nmid |A|$. Write $P = O_p(G)$. Now, $KG \simeq KP \otimes_K KA \simeq \bigoplus_{i=1}^n K_iP$, where $KA \cong \bigoplus_{i=1}^n K_i$ and $K_i$s are field extensions over $K$. Since $P$ is a locally finite $p$–group, $K_iP$ is a continuous ring (Lemma 3.2). This yields $KG$ is a continuous ring.

The question arises whether the above theorem holds for $PI$ group algebras. The answer is, in general, negative (See Example 5.1). However, the result is true for local group algebras.

**Theorem 4.2.** Let $KG$ be a local $PI$ group algebra of a group $G$ over a field $K$ of char $K = p > 0$. Then $KG$ is continuous.

Proof. Since $KG$ is local, $G$ is a $p$–group. Furthermore, since $KG$ has $PI$, $G$ contains a $p$–abelian subgroup of finite index and hence $G$ is solvable-by-finite. Because torsion solvable-by-finite groups are locally finite, $G$ is locally finite $p$–group. Therefore, by Lemma 3.2, $KG$ is continuous.

Next, we give a complete characterization of a semilocal continuous group algebra of a nilpotent group.

**Theorem 4.3.** Let $G$ be an infinite nilpotent group and let $K$ be a field of characteristic $p > 0$. Then the following statements are equivalent:

1. $KG$ is semiperfect continuous.
2. $KG$ is semilocal continuous.
3. $KG$ is semiperfect CS.
4. $KG$ is semilocal CS.

5. $G = P \times A$, where $P$ is infinite locally finite $p$–group and $A$ is a finite abelian group such that $p \nmid |A|$.

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) : Since $KG$ is continuous, $G$ is locally finite (Lemma 3.1). Thus $J(KG)$ is nil and so $KG$ is semiperfect.

(3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (5) : Since $KG$ is semilocal, $G$ is torsion (Lemma 3.4). Since $G$ is nilpotent, $G$ is locally finite. Thus $KG$ is semiperfect and so $G/O_p(G)$ is finite ([7], Theorem 10.1.5, p. 409). Let $P = O_p(G)$. Since $G$ is infinite, $P$ must be infinite. Furthermore, $G$ being nilpotent and locally finite, $P = O_p(G)$ is the unique $p$–Sylow subgroup of $G$, where $p$ does not divide the order of $G/P$. Let $|G/P| = n = q_1^{e_1} \cdots q_s^{e_s}$ where $q_i$s are primes different from $p$. Then $G = P \times H$ where $H = Q_1Q_2\cdots Q_s$ and $Q_i = \{ x \in G \mid o(x) = \text{some power of } q_i \}$. It may be noted that each $Q_i$ is a normal subgroup of $G$, because $G$ is a locally finite nilpotent group.

By hypothesis $KG$ is CS and since $G$ is shown to be locally finite, by Lemma 3.6 $KG$ is continuous. Now

$$KG = K[P \times H] \simeq KP \bigotimes_K KH \simeq KP \bigotimes \mathbb{M}_{n_i}(D_i)$$

$$\simeq \bigoplus_{i=1}^{s} KP \bigotimes_K \mathbb{M}_{n_i}(D_i),$$

where $D_i$ is a finite dimensional division algebra over $K$. Then

$$KG \simeq \bigoplus_{i=1}^{s} \mathbb{M}_{n_i}(D_iP)$$

By Utumi [9], $KG$ is continuous if and only if either $D_iP$ is selfinjective or $n_i = 1$. Since $P$ is infinite, $D_iP$ is not selfinjective. Hence each $n_i = 1$. So, $KH \simeq \bigoplus_{i=1}^{s} D_i$. Because $KH$ has no nonzero nilpotent elements, it follows by ([8], Theorem 7.4.10, p. 230) that $H$ must be abelian.

6
(5) ⇒ (1) : Under the given hypothesis $KG$ is semiperfect. The rest of the argument is exactly similar to the argument in Theorem 4.1.

We close this section with the following result that provides a possible direction for further investigation of continuous group algebras.

**Proposition 4.1.** Let $KG$ be a group algebra of a group $G = P \times H$ over a field of characteristic $p > 0$, where $P$ is an infinite locally finite $p$–group and $H$ is a finite group with $p \nmid |H|$. Then $KG$ is continuous if and only if $H$ is abelian.

**Proof.** Under the given hypothesis $KG$ is semiperfect. If $KG$ is continuous then the fact that $H$ is abelian follows as in the proof of (4) ⇒ (5) of Theorem 4.3. For the converse, we argue as in Theorem 4.1.

## 5 Examples

Our first example is of a semiperfect $PI$ group algebra which is not continuous.

**Example 5.1** Let $F$ be a finite field of characteristic $p > 0$. Let $U_n(F)$ be the group of all $n \times n$ upper triangular matrices whose entries are in $F$ with diagonal entries all equal to 1. $U_n(F)$ is a finite $p$–group which is nilpotent of class $n - 1$ ([10], Exercise 1.3 (iv), p.16). Let $P = U_n(F) \times P_0$, $n \geq 3$, where $P_0 = \prod_{i=1}^{\infty} C_p^{(i)} = \{(x_i) | x_i \in C_p^{(i)} \text{ and } x_i = e_i \text{ for all but finitely many } i\}$ is the restricted direct product of $C_p^{(i)}$, $1 \leq i \leq \infty$ and for all $i$, $C_p^{(i)} \simeq C_p$, the cyclic group of order $p$. Let $H = U_n(K)$, where $K$ is a finite field of characteristic $q \neq p$. Let $G = P \times U_n(K)$, $n \geq 3$ which is a nilpotent group. $KG$ satisfies $PI$ because $P$ is a $p$–abelian subgroup of finite index. Since $U_n(K)$ is not abelian, by Theorem 4.3, $KG$ is not continuous. Here $KG$ is semiperfect ([7], Theorem 10.1.5, p.409) with $PI$.

The example which follows shows that $KG$ can be continuous without $G$ being nilpotent.

**Example 5.2** Under the notation in Example 5.1, let $G = P \times A$, where
\( P = \prod_{n=1}^{\infty} U_n(F) \) is the restricted direct product and \( A \) is any finite abelian group whose order is not divisible by \( p \). Then \( KG \) is semiperfect and by Proposition 4.1, \( KG \) is continuous. Here \( G \) is not nilpotent.

References


