MONOTONICITY OF NONNEGATIVE MATRICES

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Abstract. We present a nonnegative rank factorization of a nonnegative matrix $A$ for the case in which one or both of $A^{(1)}A$ and $AA^{(1)}$ are nonnegative. This gives, in particular, a known result for characterizing nonnegative matrices when $A^\dagger$ or $AA^\dagger$ is nonnegative. We applied this characterization to the derivation of known results based on the characterization of nonnegative monotone matrices.

A matrix $A = (a_{ij})$ is nonnegative if $a_{ij} \geq 0$ for all $i, j$, and the nonnegativity is expressed as $A \succeq 0$. If there exists a matrix $X$ such that $X$ satisfies the following equations, for $\lambda \subseteq \{1, 2, 3, 4, 5\}$: (1) $AXA = A$, (2) $XAX = X$, (3) $AX = (AX)^T$, (4) $XA = (XA)^T$, and (5) $AX =XA$, then $X$ is called a $\lambda$-inverse of $A$, also known as a generalized inverse of $A$. A $\lambda$-inverse of $A$ is denoted $A^{(\lambda)}$. If $A^{(\lambda)} \succeq 0$, then $A$ is referred to as $\lambda$-monotone. For $\lambda = \{1, 2, 3, 4\}$, $X$ is the Moore–Penrose inverse of $A$. For $\lambda = \{1, 2, 5\}$, then $X$ indicates the group inverse of $A$. Whereas the Moore–Penrose inverse always exists and is unique, the group inverse exists if and only if the index of $A$ is 1 and unique. The Moore–Penrose and group inverses of $A$ are denoted by $A^\dagger$ and $A^\#$, respectively. For $\lambda = 1$, the matrix $X = A^{(1)}$ is known as the 1-inverse of $A$. For an example of the applications of 1-inverses in interval linear programming, see Ben–Israel and Greville [1]. Related work has been motivated by the utility of characterizing a nonnegative matrix $A$ such that a linear system $Ax = b$ has a nonnegative solution or a best approximate nonnegative solution when the output matrix $B$ is also nonnegative. Several sufficiency conditions have been demonstrated under a variety of hypotheses. For a linear system $Ax = b$, $x = A^{(1,3)}b$ is a best approximate solution to the minimum norm, or $x = A^{(1)}b$ is a solution provided that the system $Ax = b$ is consistent. Along these lines, some authors have studied the conditions under which $A^{(\lambda)}$ is nonnegative. For example, for $\lambda = \{1, 2, 3, 4\}$, see ([1], Theorem 5.2), for $\lambda = \{1, 5\}$, see ([5], Theorem 1), and for $\lambda = 1$, see ([6], Theorem 2). Under a weaker hypotheses, Jain–Tynan [4] considered nonnegative matrices $A$ such that $A^{(1,3)}A$ is nonnegative or $A^{(1,4)}A$ is nonnegative. An $n \times n$ nonnegative matrix is monomial if each row and each column has exactly one nonzero entry. Unless otherwise stated, by ”vector” we mean a ”column vector”.

The purpose of this paper is to improve the known results presented in [4]. This work characterizes nonnegative matrices $A$ such that $A^{(1)}A$ or $AA^{(1)}$ is nonnegative. As a consequence, some known results are obtained for the cases in which $A^{(1)}$ is

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1S. K. Jain would like to dedicate this paper to the honor of Professor P. Lee, National University of Taiwan, upon his retirement.
nonnegative or $A^{(1,3)}$ is nonnegative. A new characterization is presented for the case in which the matrix $A$ has a monotone group inverse.

The reader is referred to [1], [2], and [7] for definitions and results relating to generalized inverses.

1. Preliminaries

We first state the following key result due to Flor [3], which characterizes nonnegative idempotent matrices.

**Lemma 1.** If $E$ is any nonnegative idempotent matrix of rank $d$, then there exists a permutation matrix $P$ such that

$$PEP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where the zeros in the matrices are zero blocks of appropriate size, $C,D \geq 0$, $J = XY^T$,

$$X = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_d \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 & 0 & \cdots & 0 \\ 0 & y_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y_d \end{bmatrix},$$

$x_i$ and $y_i$ are positive vectors with $y_i^Tx_i = 1$, and $y_i^T$ is the transpose of $y_i$.

**Lemma 2.** ([2], p.68) Let $A$ be a nonnegative $r \times n \ (n \times r)$ matrix of rank $r$. Then $A$ has a nonnegative right (left) inverse if and only if it has a monomial submatrix of rank $r$.

2. Main Results

**Theorem 3.** Let $A$ be a nonnegative $n \times n$ matrix of rank $d$. Then the following are equivalent:

(i) There exists an $A^{(1)}$ such that $A^{(1)}A \geq 0 \ (AA^{(1)} \geq 0)$.

(ii) There exists a permutation matrix $P \ (Q)$ such that $PAP^T = FG, \ (QAQ^T = F_1G_1)$,

where $F = \begin{bmatrix} (a^{11})_{ij} \\ (a^{21})_{ij} \\ (a^{31})_{ij} \\ (a^{41})_{ij} \end{bmatrix}$ is an $n \times d$ full column rank nonnegative matrix,

$(a^{11})_{ij}$ are nonnegative $d \times d$ block matrices, where blocks are column vectors,
\[ G = \begin{bmatrix} Y^T & Y^T D & 0 & 0 \end{bmatrix}, \quad Y^T = \begin{bmatrix} y_1^T & 0 & \cdots & 0 \\ 0 & y_2^T & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & y_d^T \end{bmatrix}, \quad y_i^T \text{ are positive vectors}, \]

and \( D \) is a nonnegative matrix.

\[ \begin{bmatrix} X \\ 0 \end{bmatrix}, \quad G_1 = \begin{bmatrix} (b_{11}^{(1)})_{ij} & (b_{12}^{(1)})_{ij} & (b_{13}^{(1)})_{ij} & (b_{14}^{(1)})_{ij} \end{bmatrix}, \]

where \( (b_{ik}^{(1)})_{ij} \) are nonnegative \( d \times d \) block (row vector) matrices,

\[ x_i \text{ are positive vectors, and } C \text{ is a nonnegative matrix.} \]

\( A^{(1,2)} A \geq 0 \) (\( AA^{(1,2)} \geq 0 \)).

Proof. Let \( A \) be an \( n \times n \) nonnegative matrix of rank \( d \) such that \( A^{(1)} A \geq 0 \), for some \( A^{(1)} \). Since \( A^{(1)} A \) is idempotent, by Flor (Lemma 1) there exists a permutation matrix \( P \) such that

\[ PA^{(1)} A^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \]

where \( J, X, Y, B, \) and \( C \) are as defined in the Lemma 1. Note that rank \( A^{(1)} A = \rank A = \rank J = r \). We next partition \( PAP^T \) in conformity with the partition of \( PA^{(1)} A^T \) and let

\[ PAP^T = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}. \]

Since \( PAP^T PA^{(1)} A^T = PAP^T \),

\[ \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}. \]

This implies that the third and fourth columns of \( PAP^T \) are zero columns, the second column is a right \( D \) multiple of the first column, and \( A_{i1} \), \( i = 1, 2, 3, 4 \) satisfies the equation \( UJ = U \) in the variable \( U \). Thus, the rank of \( A \) is the rank of the first column of the above block partitioned matrix \( A \). To solve the equation
$UJ = U$, we partition $U$, which is in conformity with the partitioning of $J$, as in Lemma 1, and write $U = [U']_J$ accordingly as a $d \times d$ block matrix. By multiplying $U$ by $J$ and comparing its entries with the corresponding entries of $J$, we obtain the result that each block submatrix $U_{ij}$ is of rank $\leq 1$ and, indeed, it is of the form $U_{ij} = u_{ij}y^T$, where $u_{ij}$ is a nonnegative vector of length $d$. This yields $U = [u_{ij}]Y^T$.

Restricting $U$ to the submatrix $A_{ii}$, we may write $A_{ii} = ((a^{(k)})_{ij})Y^T$, $k = 1, 2, 3, 4$, where $(a^{(k)})_{ij}$ are nonnegative column vectors of length $d$. Therefore,

$$PAP^T = \begin{bmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
A_{31} & A_{32} & A_{33} & A_{34} \\
A_{41} & A_{42} & A_{43} & A_{44}
\end{bmatrix}$$

$$= \begin{bmatrix}
((a^{(1)})_{ij})Y^T & ((a^{(1)})_{ij})Y^TD & 0 & 0 \\
((a^{(2)})_{ij})Y^T & ((a^{(2)})_{ij})Y^TD & 0 & 0 \\
((a^{(3)})_{ij})Y^T & ((a^{(3)})_{ij})Y^TD & 0 & 0 \\
((a^{(4)})_{ij})Y^T & ((a^{(4)})_{ij})Y^TD & 0 & 0
\end{bmatrix}$$

$$= \begin{bmatrix}
((a^{(1)})_{ij}) \\
((a^{(2)})_{ij}) \\
((a^{(3)})_{ij}) \\
((a^{(4)})_{ij})
\end{bmatrix}
\begin{bmatrix}
Y^T & Y^TD & 0 & 0
\end{bmatrix}$$

$$= FG,$$

a nonnegative full rank factorization of $PAP^T$, as desired. Given the condition $AA^{(1)} \geq 0$, we can obtain a similar factorization. This proves that $(i)$ implies $(ii)$.

Note that simply by interchanging the columns of $F$ and the rows of $G$, $A = (P^TF)(GP) = F'G'$ (say) is a nonnegative rank factorization of $A$.

Next, we show that $(ii) \implies (iii)$. By $(ii)$, $A = F'G'$ is a full rank factorization of $A$. Recall that a full row rank matrix possesses a right inverse, and a full column rank matrix possesses a left inverse.

$$P^{-1}G = \begin{bmatrix} Y \\
0 \\
0 \\
0
\end{bmatrix}$$

is a right inverse $G'$ of $G'$ (Note that $Y$ is a $d \times d$ diagonal block matrix). Choosing $A^{(1,2)} = G'F'$, where $F'$ is some left inverse of $F'$, we have $A^{(1,2)}A = G'F'F'G' = G'G' \geq 0$, as desired.

$(iii) \implies (i)$ is obvious. $

The following result, which is an immediate consequence of the above theorem, is well known.

**Corollary 4.** The class of nonnegative $\{1\}$-monotone matrices is the same as the class of nonnegative $\{1,2\}$-monotone matrices.

As in ([4], Example 3), $AA^{(1)}$ may be nonnegative, but $AA^{(1)}$ need not be nonnegative. The proof of the theorem for the case in which both $AA^T$ and $A^TA$ are nonnegative in ([4], Theorem 7) is quite technical. Below is provided a very short argument and proof of a more general result.

**Theorem 5.** Let $A$ be a nonnegative $n \times n$ matrix of rank $d$. Then the following are equivalent:
There exists an $A^{(1)}$ such that $A^{(1)} A \geq 0$ and $A A^{(1)} \geq 0$.

(ii) $A$ has full rank nonnegative rank factorizations of the type $A = F' G'$ and $A = F'' G''$, where $G'$ has a nonnegative right inverse $G'_r$ and $F''$ has a nonnegative left inverse $F'_l$. Furthermore, $G'' = U G'$, where $U$ is a nonnegative invertible matrix, $G' = \begin{bmatrix} Y^T & Y^T D & 0 & 0 \end{bmatrix} P$ is as in Theorem 3 above, and $F' = F' V$, where $V$ is a nonnegative invertible matrix and $F' = P^T \begin{bmatrix} X & 0 & 0 \end{bmatrix}$. In other words, for two factorizations of the type stated, $F'$ and $F''$ are conjugate, and the same property holds for $G'$ and $G''$.

Proof. (i) $\Rightarrow$ (ii). The first part of the statement follows from Theorems 3 and 4. The last part of the statement is addressed by first considering $A = F' G' = F'' G$. Then $G'' = F'' F' G' = U G'$, where $U = F'' F'$ is a nonnegative $d \times d$ matrix of rank $d$ and, hence, is invertible. Similarly, $F' = F'' V$, where $V$ is invertible.

(ii) $\Rightarrow$ (i) is clear.

Remark 6. The above theorem can be invoked to yield the known characterizations of nonnegative $\lambda$-monotone matrices for the subsets $\lambda$ of $\{1, 2, 3, 4, 5\}$. Jain-Snyder [6] provided a description of $\lambda$-monotone matrices for $\lambda = \{1\}$ and for the case in which $A^{(1)}$ is a polynomial in $A$. The above theorems provide, as a consequence, an explicit characterization of nonnegative matrices $A$ such that $A^{(1, 3)} \geq 0$ (see Berman-Plemmons [2], Theorem 6.2, p. 123).

Theorem 7. Let $A$ be a nonnegative matrix. Then the following are equivalent.

(i) $A^{(1, 3)} \geq 0$.

(ii) $A = P^T \begin{bmatrix} X & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} W \begin{bmatrix} Y^T & 0 & 0 \end{bmatrix} P$, where $X$ and $Y$ are $n \times d$ matrices, as in Lemma 1, $W$ is a nonsingular $d \times d$ monomial matrix, and $P$ is a permutation matrix.

(iii) $A^\dagger \geq 0$.

(iv) $A^{(1, 4)} \geq 0$.

Proof. (i) $\implies$ (ii). Let $A^{(1, 3)} \geq 0$. Then by choosing $A^{(1)} = A^{(1, 3)}$, we have $A^{(1)} A \geq 0$ and $A A^{(1)} \geq 0$. Both $A^{(1)} A$ and $A A^{(1)}$ are symmetric, which implies that $C = 0$ and $D = 0$ in part (ii) of the statement of Theorem 3. Thus, by invoking Theorem 3, $A$ has full rank nonnegative rank factorizations of the types $A = F' G'$ and $A = F'' G''$, where $G'$ has a nonnegative right inverse $G'_r$ and $F''$ has a nonnegative left inverse $F''_l$.

Furthermore, $G'' = U G'$, where $U$ is a nonnegative invertible matrix and $G' = \begin{bmatrix} Y^T & 0 & 0 & 0 \end{bmatrix} P$. Also, $F' = F'' V$, where $V$ is a nonnegative invertible matrix.
and \( F'' = P^T \begin{bmatrix} X & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \). Then

\[
A = F''G'' = P^T \begin{bmatrix} X \\ 0 \\ 0 \\ 0 \end{bmatrix} U G' = P^T \begin{bmatrix} X \\ 0 \\ 0 \end{bmatrix} U \begin{bmatrix} Y^T & 0 & 0 \end{bmatrix} P.
\]

Also,

\[
A = F'G' = F''V \begin{bmatrix} Y^T & 0 & 0 \end{bmatrix} P = P^T \begin{bmatrix} X \\ 0 \\ 0 \end{bmatrix} U \begin{bmatrix} Y^T & 0 & 0 \end{bmatrix} P.
\]

By equating the two expressions for \( A \) and using the properties of \( X \) and \( Y \), we obtain \( U = V \). Furthermore,

\[
A^{(1,3)} = P^T \begin{bmatrix} Y \\ 0 \\ 0 \\ 0 \end{bmatrix} U^{-1} \begin{bmatrix} X^T & 0 & 0 \end{bmatrix} P \geq 0,
\]

which implies that \( U^{-1} \geq 0 \). This shows, by Lemma 2, that \( U \) is monomial. This proves \((i) \implies (ii)\).

\((ii) \implies (iii)\). In the proof of \((i) \implies (ii)\), the formula given for \( A^{(1,3)} \) also holds for \( A^1 \). Hence, \( A^1 \geq 0 \).

\((iii) \implies (iv)\) is obvious. The statement \((iv)\) yields the statement \((ii)\) exactly in the same manner as the proof of the statement \((i) \implies (ii)\). Since we have already shown \((ii) \implies (iii) \implies (i)\), it follows that \((iv) \implies (i)\). This completes the proof.

The characterization of nonnegative matrices having nonnegative group inverses was considered by Jain–Kwak–Goel [5]. Although some authors have provided equivalent conditions for the monotonicities of various generalized inverses, the conditions required for the monotonicity of a monotone group have not been studied except in [5]. Stochastic matrices having nonnegative group inverses are considered in [5]. It is interesting that an application of Theorem 3 provides a new equivalent statement for the monotonicity of the group inverse.

**Theorem 8.** Let \( A \) be a nonnegative matrix of index 1. Then the following statements are equivalent:

\((i)\) \( A^# \geq 0 \).

\((ii)\) There exists a permutation matrix \( P \) such that

\[
PAP^T = FG = \begin{bmatrix} ((a^{11})_{ij}) & 0 & 0 \\ 0 & ((a^{31})_{ij}) & 0 \end{bmatrix} \begin{bmatrix} Y^T & Y^T D & 0 & 0 \end{bmatrix}
\]

is a full rank nonnegative factorization, where \( ((a^{11})_{ij}) \) is a nonnegative monomial \( d \times d \) block matrix (the block entries of which are the columns \( (a^{11})_{ij} \)), the block
submatrix \(((a^{31})_{ij})\) is a constant multiple of the block submatrix \(((a^{11})_{ij})\), and

\[
Y^T = \begin{bmatrix}
y_1^T & 0 & \cdots & 0 \\
0 & y_2^T & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & y_d^T
\end{bmatrix}, \ y_i \text{ are positive vectors.}
\]

(iii) There exists a permutation matrix \(P\) such that

\[
PAP^T = F_1G_1 = \begin{bmatrix}
X \\
0 \\
CX \\
0
\end{bmatrix} \begin{bmatrix}
((b^{11})_{ij}) & ((b^{12})_{ij}) & 0 & 0 \\
\end{bmatrix}
\]
is a full rank nonnegative factorization, where \(((b^{11})_{ij})\) is a nonnegative monomial \(d \times d\) block matrix, the block entries of which are row vectors \(((b^{11})_{ij})\), the submatrix \(((b^{12})_{ij})\) is a constant multiple \(C\) of the block matrix \(((b^{11})_{ij})\), and

\[
X = \begin{bmatrix}
x_1 & 0 & \cdots & 0 \\
0 & x_2 & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & x_d
\end{bmatrix}, \ x_i \text{ are positive vectors.}
\]

Proof. We will prove \((i) \iff (ii)\). Assume \((i)\). Since \(A^\#A = AA^\# \succeq 0\), by Theorem 3, there exists a permutation matrix \(P\) such that

\[
PAP^T = FG = (F_1G_1)
\]

where

\[
F = \begin{bmatrix}
((a^{11})_{ij}) \\
((a^{21})_{ij}) \\
((a^{31})_{ij}) \\
((a^{41})_{ij})
\end{bmatrix}
\]
is an \(n \times d\) full column rank nonnegative matrix, \(((a^{11})_{ij})\) are nonnegative \(d \times d\) block matrices (the \(i - j\)th block entry of the block matrix \(((a^{11})_{ij})\) is column \((a^{11})_{ij})\), and

\[
G = \begin{bmatrix}
Y^T & Y^TD & 0 & 0
\end{bmatrix}, \ Y^T = \begin{bmatrix}
y_1^T & 0 & \cdots & 0 \\
0 & y_2^T & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & y_d^T
\end{bmatrix}.
\]

Note that

\[
F_1 = \begin{bmatrix}
X \\
0 \\
CX \\
0
\end{bmatrix}
\]

and \(G_1 = \begin{bmatrix}
((b^{11})_{ij}) & ((b^{12})_{ij}) & ((b^{13})_{ij}) & ((b^{14})_{ij})
\end{bmatrix}\), where \(((b^{1k})_{ij})\) are nonnegative \(d \times d\) block matrices (the \(i - j\)th block entry of the block matrix \(((b^{1k})_{ij})\) is
the row vector \(((b^{1k})_{ij})\) and \(X = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x_d \end{bmatrix}\). By comparing the two factorizations, we obtain \(((a^{21})_{ij}) = 0, ((a^{41})_{ij}) = 0, ((b^{13})_{ij}) = ((b^{14})_{ij}) = 0\). Thus,

\[
A = \begin{bmatrix} ((a^{11})_{ij}) \\ 0 \\ ((a^{31})_{ij}) \\ 0 \end{bmatrix} \begin{bmatrix} Y^T & Y^T D & 0 & 0 \end{bmatrix} = \begin{bmatrix} X \\ 0 \\ CX \\ 0 \end{bmatrix} \begin{bmatrix} ((b^{11})_{ij}) & ((b^{12})_{ij}) & 0 & 0 \end{bmatrix}.
\]

This implies that \(((a^{11})_{ij})Y^T = X((b^{11})_{ij})\). Observe that after multiplying the factors of \(A\), it follows that the \((1, 1)\)-block entry determines the rank of \(A\), since other block entries are multiple of the \((1, 1)\)-block entry. Furthermore, since \(Y^TY = I\), \(\text{rank}((a^{11})_{ij})) = d\). Similarly, \(X^TX = I\) yields \(\text{rank}(X((b^{11})_{ij})) = \text{rank}((b^{11})_{ij})) = d\). Now, \(GF = Y^T((a^{11})_{ij})\) is an invertible \(d \times d\) matrix according to the Cline Theorem ([1], Theorem 2, p.163). Also, by the Cline formula, \(A^# = F(GF)^{-2}G \geq 0\).

Since \(G\) has a nonnegative right inverse \(\begin{bmatrix} Y \\ 0 \\ 0 \\ 0 \end{bmatrix}\), it follows that \(F(GF)^{-2} \geq 0\). This implies that \(GF(GF)^{-2} \geq 0\), and so \((GF)^{-1} \geq 0\). Therefore, by Lemma 2, \(GF\) is a monomial matrix. Now, \(GF = Y^T((a^{11})_{ij})\), where \(Y^T = \begin{bmatrix} y^T_1 & 0 & \cdots & 0 \\ 0 & y^T_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y^T_d \end{bmatrix}\).

This shows that \(((a^{11})_{ij})\) is a block monomial matrix because \(y_i\) are positive vectors. This proves (ii). Let us now assume (ii). We have \(GF = Y^T((a^{11})_{ij})\). Because

\(((a^{11})_{ij})\) is a block monomial matrix and \(Y^T = \begin{bmatrix} y^T_1 & 0 & \cdots & 0 \\ 0 & y^T_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & y^T_d \end{bmatrix}\), \(y_i\) are positive vectors, it follows that \(Y^T((a^{11})_{ij})\) is a \(d \times d\) monomial nonnegative matrix.

By applying the Cline formula, \(A^# = F(GF)^{-2}G\), we obtain \(A^# \geq 0\), proving (i). The proof of (i) \(\Leftrightarrow\) (iii) is similar. This completes the proof. \(\blacksquare\)

We conclude with an illustration of Theorem 8.

We know that \(A^#\) is a polynomial in \(A\). Let us choose \(A\) to be a \(4 \times 4\) matrix with rank \(A = 2\). Following the form of the full rank factorization provided in Theorem 8 above, let

\[
F = \begin{bmatrix} 0 & 3 \\ 0 & 4 \\ 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}.
\]
Then

\[ A = FG = \begin{bmatrix} 0 & 0 & 6 & 9 \\ 0 & 0 & 8 & 12 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}. \]

\[ GF = \begin{bmatrix} 0 & 7 \\ 3 & 0 \end{bmatrix}, (GF)^{-1} = \frac{1}{21} \begin{bmatrix} 0 & 7 \\ 3 & 0 \end{bmatrix}, (GF)^{-2} = \frac{1}{21} I. \]

\[ A^# = F(GF)^{-2}G = \frac{1}{21} FG = \frac{1}{21} A. \]

We note an interesting fact: \((GF)^{-1} = p(GF)\), where \(p(t)\) is a polynomial over a field, if and only if \(A^# = p(A)\). In this example, \(p(t) = \frac{1}{21} t\).

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