Modules which are invariant under monomorphisms of their injective hulls

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Abstract

In this paper certain injectivity conditions in terms of extensions of monomorphisms are considered. In particular, it is proved that a ring $R$ is a quasi-Frobenius ring if and only if every monomorphism from any essential right ideal of $R_R$ into $R_R^{(R)}$ can be extended to $R_R$. Also, known results on pseudo-injective modules are extended. Dinh raised the question if a pseudo-injective CS module is quasi-injective. The following results are obtained: $M$ is quasi-injective if and only if $M$ is pseudo-injective and $M^2$ is CS. Furthermore, a uniform pseudo-injective is quasi-injective. As a consequence of this it is shown that over a right Noetherian ring $R$, quasi-injective modules are precisely pseudo-injective CS modules.

1 Introduction

Throughout the paper rings are associative with identity and modules are unitary (right) modules. Let $M$ and $N$ be two right $R$-modules over a ring $R$. $M$ is called (pseudo-) $N$-injective if, for any submodule $A$ of $N$, every homomorphism (resp. monomorphism) in $\text{Hom}_R(A,M)$ can be extended to an element of $\text{Hom}_R(N,M)$. $M$ is called quasi-injective (pseudo-injective) if it is (pseudo-) $M$-injective. $M$ and $N$ are called relatively injective if $M$ is $N$-injective and $N$ is $M$-injective. A submodule $K$ of $M$ is said to be a
complement in \( M \) of a submodule \( B \) if \( K \) is a maximal submodule among those which have zero intersection with \( B \). Complement submodules of \( M \) coincide with the submodules of \( M \) which do not have any proper essential extension in \( M \). Also, if \( A \) is a complement in \( M \) and \( B \) is a complement in \( A \), then \( B \) is a complement in \( M \). A CS module is one in which complement submodules are direct summands. \( M \) is called a continuous module if it is a CS module and submodules of \( M \) isomorphic to direct summands of \( M \) are again direct summands. If \( M \) is continuous and \( A \) and \( B \) are two direct summands of \( M \) with \( A \cap B = 0 \), then \( A \oplus B \) is also a direct summand of \( M \).

For other properties of complements and CS/continuous modules and the proofs of the above mentioned properties, the reader is referred to [5] and [13].

In this paper a weaker form of pseudo-\( N \)-injectivity is considered and it is proved, in particular, that a ring \( R \) is quasi-Frobenius if and only if monomorphisms from essential right ideals of \( R \) into \( R^{(n)} \) can be extended to \( R_R \). Also it is shown that a module \( M \) is invariant under monomorphisms of its injective hull if and only if every monomorphism from any essential submodule of \( M \) can be extended to \( M \). This extension property is used to characterize when semi-prime/right nonsingular rings are SI (see [9]).

Pseudo-injectivity has been studied by several authors such as Dinh, Jain, Singh, Teply, Tuganbaev and others (see [3], [12], [11], [17], [18], [19]). It was first introduced by Jain and Singh [12]. Teply [18] constructed examples of pseudo-injective modules which are not quasi-injective. In [3] Dinh raised the question if a pseudo-injective CS module is quasi-injective. He stated in [4] that the answer is affirmative if we assume further that \( M \) is nonsingular. In this paper we prove the following: \( M \) is quasi-injective if and only if \( M \) is pseudo-injective and \( M^2 \) is CS. Every uniform pseudo-injective module is quasi-injective. Consequently, over a right Noetherian ring \( R \), quasi-injective modules are precisely pseudo-injective CS modules.

2 Essentially pseudo-\( N \)-injectivity

In this section we consider a weaker form of pseudo-\( N \)-injectivity.

**Definition 1** Let \( M \) and \( N \) be two modules. \( M \) is said to be essentially pseudo-\( N \)-injective if for any essential submodule \( A \) of \( N \), any monomorphism \( f : A \to M \) can be extended to some \( \text{gH}(N, M) \). \( M \) is called essentially pseudo-injective if \( M \) is essentially pseudo-\( M \)-injective.
Obviously any pseudo-$N$-injective module is essentially pseudo-$N$-injective, but the converse is not true in general.

**Example 1** Let $p$ be a prime. The $\mathbb{Z}$-module $\mathbb{Z}/p^2\mathbb{Z}$ is not pseudo-$(\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z})$-injective since the obvious isomorphism $\iota : p\mathbb{Z}/p^3\mathbb{Z} \rightarrow \mathbb{Z}/p^2\mathbb{Z}$ can not be extended to any element of $Hom(\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}, \mathbb{Z}/p^2\mathbb{Z})$, but it is essentially pseudo-$(\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z})$-injective.

The following proposition provides a characterization of essentially pseudo-$N$-injectivity.

**Proposition 1** Let $M$ and $N$ be two modules and $X = M \oplus N$. The following conditions are equivalent:

(i) $M$ is essentially pseudo-$N$-injective;

(ii) For any complement $K$ in $X$ of $M$ with $K \cap N = 0$, $M \oplus K = X$.

**Proof.** (i) $\Rightarrow$ (ii) Let $K$ be a complement in $X$ of $M$ with $K \cap N = 0$, and $\pi_M : M \oplus N \rightarrow M$ and $\pi_N : M \oplus N \rightarrow N$ be the obvious projections. Note that $M \oplus K = M \oplus \pi_N(K)$ so that $\pi_N(K)$ is essential in $N$.

Now define $\theta : \pi_N(K) \rightarrow \pi_M(K)$ as follows: For $k \in K$ with $k = m + n$ $(m \in M, n \in N)$, $\theta(n) = m$. Then $\theta$ is a monomorphism by the $K \cap N = 0$ assumption. Hence $\theta$ can be extended to some $g : N \rightarrow M$, since $M$ is essentially pseudo-$N$-injective. Now let $T = \{n + g(n) : n \in N\}$. It is easy to see that $M \oplus T = X$. Also, $T$ contains $K$ essentially by modularity. Since $K$ is a complement, this implies $T = K$. Now the conclusion follows.

(ii) $\Rightarrow$ (i) Assume (ii). Let $A$ be an essential submodule of $N$ and $f : A \rightarrow M$ be a monomorphism. Let $H = \{a - f(a) : a \in A\}$. Obviously, $H \cap N = 0$. Also note that $M \oplus H = M \oplus \pi_N(H) = M \oplus A$, which is essential in $X$. Let $K$ be a complement in $X$ of $M$ containing $H$. By the previous argument and modularity $H$ is essential in $K$, so that $K \cap N = 0$. By assumption we have $M \oplus K = X$. Now let $\phi : M \oplus K \rightarrow M$ be the obvious projection. Then the restriction $\phi|_N$ is the desired extension of $f$. The proof is now complete.

**Proposition 2** If $M$ is essentially pseudo-$N$-injective, every direct summand of $M$ is essentially pseudo-$N$-injective.
Proof. Let \( X = M \oplus N \) and assume \( M = M_0 \oplus A \). Let \( K \) be a complement in \( M_0 \oplus N \) of \( M_0 \) with \( K \cap N = 0 \). Then \( M \oplus K \) is essential in \( X \). Since \( K \) is a complement submodule, the preceding argument implies that \( K \) is also a complement in \( X \) of \( M \). Now by Proposition 1 \( M \oplus K = X \). Then \( M_0 \oplus K = M_0 \oplus N \), which yields the conclusion again by Proposition 1.

The next example shows that essentially pseudo-N-injectivity is not inherited by direct sums.

Example 2 Let \( F \) be a field and \( R = \begin{pmatrix} F & F \oplus F \\ 0 & F \end{pmatrix} \). Consider the \( R \)-modules \( N = \begin{pmatrix} F & F \oplus F \\ 0 & 0 \end{pmatrix} \), \( S_1 = \begin{pmatrix} 0 & 0 \oplus F \\ 0 & 0 \end{pmatrix} \), \( S_2 = \begin{pmatrix} 0 & F \oplus 0 \\ 0 & 0 \end{pmatrix} \). Then \( S_1 \) and \( S_2 \) are both essentially pseudo-N-injective. But since the identity map of \( S_1 \oplus S_2 \) obviously can not be extended to an element of \( \text{Hom}(N, S_1 \oplus S_2) \), \( S_1 \oplus S_2 \) is not essentially pseudo-N-injective.

Proposition 3 Let \( M \) and \( N \) be two modules. Then the following conditions are equivalent:

(i) \( M \) is \( N \)-injective ;

(ii) \( M \) is essentially pseudo-\( N/L \)-injective for every submodule \( L \) of \( N \).

Proof. (i) \( \Rightarrow \) (ii) follows from [13, Proposition 1.3].

(ii) \( \Rightarrow \) (i) Assume \( M \) is essentially pseudo-\( N/L \)-injective for every submodule \( L \) of \( N \). Let \( X = M \oplus N \), \( A \subseteq X \) with \( A \cap M = 0 \) and \( K \) be a complement in \( X \) of \( M \) containing \( A \). Also let \( T = K \cap N \). Since \( (M \oplus K)/K \) is essential in \( X/K \), then \( (M \oplus K)/K \) is essential in \( X/T \), and \( K/T \cap N/T = 0 \). Thus it is easy to see that \( K/T \) is a complement in \( X/T \) of \( (M \oplus T)/T \). Now by assumption and Proposition 1 we have \( (M \oplus T)/T \oplus K/T = X/T \). Hence \( M \oplus K = X \). Then by [5, Lemma 7.5] \( M \) is \( N \)-injective.

Corollary 1 \( M \) is injective if and only if \( M \) is essentially pseudo-\( N \)-injective for any cyclic module \( N \).

Corollary 2 A nonsingular module \( M \) is injective if and only if it is essentially pseudo-\( N \)-injective for any nonsingular cyclic module \( N \).
The following result generalizes [3, Theorem 2.2] and [11, Theorem 1].

**Theorem 1** If \( M \oplus N \) is essentially pseudo-\( N \)-injective then \( M \) is \( N \)-injective.

**Proof.** Call \( X = M \oplus N \). Let \( A \) and \( K \) be as in the proof of Proposition 3. Let \( \pi : M \oplus N \to N \) be the obvious projection. Then \( M \oplus K = M \oplus \pi(K) \) and thus \( \pi(K) \) essential in \( N \). Note that \( K \cong \pi(K) \). Pick any isomorphism \( f : \pi(K) \to K \). By assumption \( f \) can be extended to some monomorphism \( g : N \to X \). Then \( g(\pi(K)) = K \) is essential in \( g(N) \). But since \( K \) is a complement in \( X \), we must have \( K = g(N) \), whence \( \pi(K) = N \). Thus \( M \oplus K = X \). Now the result follows by [5, Lemma 7.5].

**Corollary 3** \( M \) is quasi-injective if and only if \( M^2 \) is essentially pseudo-\( M \)-injective.

Ososfky proved in [15] that a ring \( R \) is semisimple Artinian if and only if every cyclic right (left) \( R \)-module is injective.

**Corollary 4** A ring \( R \) is semisimple Artinian if and only if every countably generated right \( R \)-module is essentially pseudo-injective.

**Proof.** Let \( M \) be a cyclic right \( R \)-module. Then \((M \oplus R)^{(N)} \cong (M \oplus R)^{(N)} \oplus (M \oplus R)^{(N)} \), which is countably generated, whence essentially pseudo-injective. Thus \((M \oplus R)^{(N)} \)^2 is essentially pseudo-\( (M \oplus R)^{(N)} \)-injective. Then by Theorem 1, \((M \oplus R)^{(N)} \) is quasi-injective, whence \( R_{R_{R}} \)-injective. Therefore \( M \) is injective. Now the conclusion follows by Ososfky’s theorem.

**Corollary 5** ([3, Theorem 2.2]) If \( M \oplus N \) is pseudo-injective then \( M \) and \( N \) are relatively injective.

In what follows \( E(M) \) stands for the injective hull of \( M \) and we will consider \( M \) as a submodule of \( E(M) \). Also we will use the notation \( E_N(M) \) for the submodule of \( E(M) \) generated by all the isomorphic copies of \( N \). Note that \( E_N(M) \) is invariant under monomorphisms of \( End(E(M)) \) and that \( E_{R_{R}}(M) \) contains all elements of \( M \) with zero right annihilator in \( R \).

**Proposition 4** \( M \) is essentially pseudo-\( N \)-injective if and only if \( E_N(M) \subseteq M \).
Proof. Assume \( E_N(M) \subseteq M \) and let \( B \) be an essential submodule of \( N \), and \( f : B \to M \) be a monomorphism. There exists some monomorphism \( g : N \to E(M) \) such that \( g|_B = f \). By assumption \( g(N) \subseteq M \). Thus \( g \) is the desired extension of \( f \), whence \( M \) is essentially pseudo-\( N \)-injective.

Conversely assume that \( M \) is essentially pseudo-\( N \)-injective. We will use the same argument as in [13, Lemma 1.13]: Let \( h : N \to E(M) \) be a monomorphism. Let \( A = h^{-1}(M) \). Then \( A \) is essential in \( N \). Thus, by assumption, the restriction \( h|_A \) extends to some \( \theta : N \to M \). Now assume \( h(n) \neq \theta(n) \) for some \( n \in N \). Then \( x = h(n) - \theta(n) \neq 0 \). Since \( M \) is essential in \( E(M) \), there exists some \( r \in R \) such that \( 0 \neq xr = h(nr) - \theta(nr) \in M \). But then \( h(nr) \in M \) so that \( nr \in A \). This is a contradiction since \( \theta|_A = h|_A \). Now the conclusion follows.

Corollary 6 \( M \) is essentially pseudo-injective if and only if it is invariant under monomorphisms in \( \text{End}(E(M)) \).

Corollary 7 Let \( \{A_i\} \) be a family of submodules of a module \( N \), \( B = \Sigma A_i \) and assume \( M \) is essentially pseudo-\( A_i \)-injective for each \( i \). Then \( M \) is essentially pseudo-\( B \)-injective.

Proof. Let \( f : B \to E(M) \) be a monomorphism. Then \( f(B) = \Sigma f(A_i) \). By assumption and Proposition 4, \( f(B) \) is contained in \( M \). Now the conclusion follows again by Proposition 4.

The converse of the Corollary 7 does not hold in general.

Example 3 Let \( p \) be a prime. It is easy to see that the \( \mathbb{Z} \)-module \( \mathbb{Z}/p^2\mathbb{Z} \) is not essentially pseudo-\( \mathbb{Z}/p^3\mathbb{Z} \)-injective, but it is trivially essentially pseudo-(\( \mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z} \))-injective.

Corollary 8 Let \( E \) be an injective module and \( A \) be any submodule of \( E \). Then \( X = \Sigma \{C | C \leq E, C \cong A\} \) is essentially pseudo-injective.

Proof. First note that \( E(X) \) is a summand of \( E \). As in the proof of Corollary 7, for any monomorphism \( f : X \to E(X) \), \( f(X) \) is contained in \( X \). The conclusion follows by Proposition 4.

Goodearl defined a right SI-ring to be one over which every singular right module is injective ([9]). Such rings are precisely right nonsingular rings over which singular right modules are semi-simple (see [5]).
Theorem 2 Let \( R \) be a ring which is either right nonsingular or semi-prime. The following conditions are equivalent:

(i) \( R \) is a right SI-ring;

(ii) Any two cyclic singular right \( R \)-modules are relatively essentially pseudo-injective;

(iii) For any two cyclic singular right \( R \)-modules \( B \) and \( C \), \( E_B(C) \subseteq C \).

Proof. (i) \( \Rightarrow \) (ii) Trivial.

(ii) \( \Leftrightarrow \) (iii) By Proposition 4.

(ii) \( \Rightarrow \) (i) Assume (ii). Then cyclic singular right \( R \)-modules are relatively injective by Proposition 3. So if \( C \) and \( M \) are singular right \( R \)-modules and \( C \) is cyclic, then \( C \) is \( M \)-injective by the above argument and [13, Proposition 1.4]. This implies, by [5, Corollary 7.14], that all singular right \( R \)-modules are semi-simple.

Now, if \( R \) is right nonsingular, the conclusion immediately follows by the preceding remark and the above argument. Else, assume that \( R \) is semi-prime. Since singular modules are semi-simple, \( Z(RR)^2 = 0 \), whence \( Z(RR) = 0 \). Now the conclusion follows by the above argument.

3 Pseudo-injectivity

Proposition 5 Let \( M \) and \( N \) be two modules and \( X = M \oplus N \). The following conditions are equivalent:

(i) \( M \) is pseudo-\( N \)-injective;

(ii) For any submodule \( A \) of \( X \) with \( A \cap M = A \cap N = 0 \), there exists a submodule \( T \) of \( X \) containing \( A \) with \( M \oplus T = X \).

Proof. (i) \( \Rightarrow \) (ii) Assume (i) and let \( A \) satisfy the assumptions of (ii). Also let \( \pi_M \) and \( \pi_N \) be as in the Proposition 1, and define \( \theta : \pi_N(A) \to \pi_M(A) \) as follows: \( \theta(\pi_N(a)) = \pi_M(a) \), for \( a \in A \). Then, by assumption, \( \theta \) extends to some \( g \in Hom(N,M) \). Let \( T = \{ n + \theta(n) | n \in N \} \). Then we have \( M \oplus T = X \) and \( A \subseteq T \), as required.

(ii) \( \Rightarrow \) (i) Assume (ii). Let \( B \) be a submodule of \( N \) and \( f : B \to M \) be a monomorphism. Call \( A = \{ b - f(b) | b \in B \} \). Then \( A \cap M = A \cap N = 0 \).
Now, by assumption, there exists a submodule $T$ of $X$ containing $A$ with $M \oplus T = X$. Let $\pi : M \oplus T \to M$ be the obvious projection. Then the restriction $\pi|_{N}$ is the desired extension of $f$.

Jain and Singh proved in [12, Theorem 3.7] that for a nonsingular module $M$ with finite uniform dimension, the following conditions are equivalent: (i) $M$ is pseudo-injective; (ii) $M$ is invariant under any monomorphism (“isomorphism” in the terminology of [12]) of $\text{End}(E(M))$ (i.e. $M$ is essentially pseudo-injective by Corollary 6). The following result extends it to any module with finite uniform dimension.

**Theorem 3** A module $M$ with finite uniform dimension is pseudo-injective if and only if it is essentially pseudo-injective.

**Proof.** Let $M$ be essentially pseudo-injective and $A$ be a submodule of $M$ with a monomorphism $f : A \to M$. Call $B = f(A)$. Pick, by Zorn’s Lemma, two submodules $A'$ and $B'$ of $M$ such that $A \oplus A'$ and $B \oplus B'$ are essential in $M$. Now, $E(M) = E(A) \oplus E(A') = E(B) \oplus E(B')$ and $E(A) \cong E(B)$. Then by [13, Theorem 1.29] and since $M$ has finite uniform dimension, we have $E(A') \cong E(B')$. Thus $A'$ and $B'$ have isomorphic essential submodules $U \subseteq A'$ and $V \subseteq B'$. Then $A \oplus U$ and $B \oplus V$ are essential submodules of $M$. Let $\phi : U \to V$ be any isomorphism. Then there exists an isomorphism $\theta : A \oplus U \to B \oplus V$ such that $\theta|_{A} = f$. By assumption $\theta$ extends to some $g \in \text{End}(M)$. Obviously, $g|_{A} = f$. Therefore the conclusion follows.

Note that, in [1, Theorem 2.1] Alamelu gives a proof of the equivalence in Theorem 3 without the finite dimension assumption. However the proof is incorrect. In summary, the proof states that, for a module $M$ which is invariant under monomorphisms of its injective hull, and for any monomorphism $f : N \to M$ where $N$ is a submodule of $M$, $f$ can be extended to a monomorphism $f'' : E(M) \to E(M)$. This is not correct as the following example shows: Let $M$ be any directly infinite injective module with $M = N \oplus B$ where $M \cong N$ and $B$ is nonzero. Also let $f : N \to M$ be any isomorphism. Obviously $f$ can not be extended to a monomorphism in $\text{End}(E(M))$.

In [6] and [7] Er studied the modules in which isomorphic copies of complements are again complements. These are called SICC-modules in [7]. The following result was proved in [12] for nonsingular modules, but the proof works for an arbitrary pseudo-injective module as well.
Lemma 1 (Jain and Singh [12, Lemma 3.1]) If $M$ is pseudo-injective then submodules of $M$ isomorphic to complements in $M$ are again complements.

Proof. Let $K$ be a complement in $M$ and $A$ be a submodule of $M$ with an isomorphism $f : A \to K$. Then $f$ extends to some $g \in \text{End}(M)$ by assumption. Pick, by Zorn’s Lemma, a complement $A'$ in $M$ essentially containing $A$. Then the restriction $g_{|A'}$ is obviously a monomorphism. Hence $K = g(A)$ is essential in $g(A')$. Since $K$ is a complement this implies $K = g(A')$, whence $A = A'$. The conclusion follows.

Remark Modules in which submodules isomorphic to complements are complements always decompose into relatively injective summands by [7, Lemma 4]. So Corollary 5 also follows from that result and Lemma 1. It is proved in [3, Corollary 2.8] that a pseudo-injective CS module is continuous. This result also follows from Lemma 1 and the definition of CS.

Dinh [3] raised the question whether a CS module $M$ which is pseudo-injective is quasi-injective, and stated in [4] that the answer is affirmative when $M$ is furthermore nonsingular. Now we present some partial answers to Dinh’s question.

Theorem 4 $M$ is quasi-injective if and only if $M$ is pseudo-injective and $M^2$ is CS.

Proof. Assume $M$ is pseudo-injective and $M^2$ is CS. Let $M_1$ and $M_2$ be two isomorphic copies of $M$ and $X = M_1 \oplus M_2$. Note that $M$ is continuous by the preceding remark.

First let $A$ be any complement in $X$ with $A \cap M_1 = 0$ and $A \cap M_2$ essential in $A$. There exist submodules $V$ and $V'$ of $M_2$ such that $V \oplus V' = M_2$ and $V$ contains $A \cap M_2$ essentially. Also since $M^2$ is CS by assumption, we have $A \oplus A' = X$ for some submodule $A'$ of $X$. Since $V$ is a direct summand of a continuous module, $V$ is continuous (see [13]), whence it has exchange property by [13, Theorem 3.4]. Since $V \cap A$ is essential in $A$ we have $V \cap A' = 0$. Thus we must have $V \oplus A' = X$. Hence $A$ is isomorphic to a summand, namely $V$ of $M_2$.

Now let $C$ be a submodule of $X$ such that $C \cap M_1 = 0$ and pick, by Zorn’s Lemma, a complement $K$ in $X$ of $M_1$ containing $C$. Again by Zorn’s Lemma, choose a complement $K_1$ in $K$ of $K \cap M_2$ and a complement $K_2$ in
$K$ of $K_1$ containing $K \cap M_2$. Note that $K \cap M_2$ is essential in $K_2$ and that $K_1$ and $K_2$ are complements in $X$ by [5, 1.10]. By Proposition 5 there exists some submodule $T$ of $X$ containing $K_1$ with $M_1 \oplus T = X$. Then $T \cong M$ and $K_1$ is a complement in $T$, whence $K_1$ is isomorphic to a complement in $M_2$. Also by the preceding paragraph $K_2$ is isomorphic to a complement of $M_2$ too. Now consider the usual projection $\pi : M_1 \oplus M_2 \to M_2$. We have $M_1 \oplus (K_1 \oplus K_2) = M_1 \oplus (\pi(K_1) \oplus \pi(K_2))$, where $\pi(K_1) \cong K_1$. Hence by continuity of $M_2$ and the above argument, $\pi(K_1) \oplus \pi(K_2)$ is a summand of $M_2$. Now, since $K$ is a complement of $M_1$, $M_1 \oplus K = M_1 \oplus \pi(K)$ is essential in $X$. Then $\pi(K)$ is essential in $M_2$. Also, by choice of $K_i$, $K_1 \oplus K_2$ is essential in $K$. Then $\pi(K_1) \oplus \pi(K_2)$ is essential in $\pi(K)$, hence in $M_2$. This implies that $M_2 = \pi(K_1) \oplus \pi(K_2) = \pi(K)$. Thus $M_1 \oplus K = X$. Now it follows by [5, Lemma 7.5] that $M_1$ is $M_2$-injective. The proof is now complete.

The following is a key result.

Lemma 2 The following conditions hold:

(i) A uniform pseudo-injective module $M$ is quasi-injective.

(ii) Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of uniform modules $M_i$. $M$ is quasi-injective if and only if it is pseudo-injective.

Proof. (i) Let $A$ be a submodule of $M$ and $f : A \to M$ be a nonzero homomorphism. If $\text{Ker}(f) = 0$ then $f$ can be extended to an element of $\text{End}(M)$ by assumption. So assume $\text{Ker}(f) \neq 0$. Let $\delta = i_A - f$, where $i_A : A \to M$ is the inclusion map. Since $\text{Ker}(f) \neq 0$ and $M$ is uniform, $\text{Ker}(\delta) = 0$. Then by pseudo-injectivity assumption $\delta$ can be extended to some $g \in \text{End}(M)$. Now $1 - g$ is obviously an extension of $f$. The conclusion follows.

(ii) Let $M$ be pseudo-injective. Then, by Corollary 5, $M(I - i)$ is $M_i$-injective for all $i \in I$. Now by part (i) and since direct summands of pseudo-injectives are obviously pseudo-injective, each $M_i$ is quasi-injective. Therefore $M$ is quasi-injective.

Theorem 5 Over a right Noetherian ring $R$, a module $M$ is quasi-injective if and only if $M$ is a pseudo-injective CS-module.

Proof. Let $M$ be a pseudo-injective CS module. Then $M$ is a direct sum of uniform submodules by [14]. Now the result follows by Lemma 2.
Before proving the next result, note that $R$ is called a right countably
$\Sigma$-CS ring if $R^{(n)}_R$ is a CS module.

**Theorem 6** The following conditions are equivalent for a ring $R$:

(i) $R$ is a quasi-Frobenius ring;

(ii) Every projective right $R$-module is essentially pseudo-$R_R$-injective;

(iii) $R^{(n)}_R$ is essentially pseudo-$R_R$-injective;

(iv) $R$ is a right countably $\Sigma$-CS ring with finite uniform dimension and $R_R$
     is essentially pseudo-injective.

Proof. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious, and (i) $\Rightarrow$
(iv) follows from the fact that every injective module is CS, and (iii) $\Rightarrow$ (i)
follows by Theorem 1.

(iv) $\Rightarrow$ (i) Since $R_R$ has finite uniform dimension, then $R_R$ is pseudo-
injective by Theorem 3. By assumption $R_R = \bigoplus_{i=1}^n e_i R$ for some uniform
right ideals $e_i R$. By Corollary 5 $e_i R$ are relatively injective. Also by Lemma 2
each $e_i R$ is a quasi-injective right $R$-module. Thus $R$ is right self-injective
with finite uniform dimension. Hence $R$ is a semiperfect right countably $\Sigma$-
CS ring. This implies by [10] that $R$ is Artinian. Now the conclusion follows.

The following results were proved in [7, Theorem 2, Corollary 4, Theorem
3, Theorem 4] for modules in which submodules isomorphic to complements
are complements. Each pseudo-injective module satisfies this property by
Lemma 1, whence we have the following corollaries.

**Corollary 9** Any decomposition of a pseudo-injective module into indecom-
posable submodules complements summands.

**Corollary 10** A essentially pseudo-injective module with finite uniform di-
mension has the internal cancellation property.

Recall that every right $R$-module over a right Noetherian ring $R$ is locally
Noetherian.

**Corollary 11** If $M$ is a locally Noetherian pseudo-injective module, then
$M = A \oplus B$, where $A$ is a maximal quasi-injective summand, $B$ has no quasi-
injective summands, and $A$ and $B$ have no nonzero isomorphic submodules.

**Corollary 12** A locally Noetherian Dedekind-finite pseudo-injective module
has internal cancellation property.
References


