Semilocal CS matrix rings of order > 1 over group algebras of solvable groups are selfinjective

K.I. Beidar,\textsuperscript{a} S.K. Jain,\textsuperscript{b,*} Pramod Kanwar,\textsuperscript{c} and J.B. Srivastava \textsuperscript{d}

\textsuperscript{a} Department of Mathematics, National Cheng-Kung University, Tainan, Taiwan
\textsuperscript{b} Department of Mathematics, Ohio University, Athens, OH 45701, USA
\textsuperscript{c} Department of Mathematics, Ohio University–Zanesville, Zanesville, OH 43701, USA
\textsuperscript{d} Department of Mathematics, Indian Institute of Technology, Delhi 110016, India

Received 30 April 2003
Communicated by Kent R. Fuller

Keywords: CS-rings, finitely $\Sigma$-CS modules, Semilocal rings, Locally finite groups, Solvable groups, Linear groups.

The following theorem is a sort of an addendum or a sequel to our earlier paper [1]. This result generalizes Theorem 4.3 in [1] to group algebras of solvable groups. For definitions and terminology the reader is referred to [1].

\textbf{Theorem 0.1.} Let $K$ be a field and $G$ be a group. Suppose one of the following conditions is satisfied.

(a) $G$ is a locally finite group.
(b) The group algebra $K G$ is semilocal and $G$ is either a solvable group or a linear group.

Then the following are equivalent.

(i) $M_n(KG)$, $n > 1$, is a right CS-ring.
(ii) $M_1(KG)$ is a right CS-ring.
(iii) $KG$ is right selfinjective.
(4) $G$ is finite.

**Proof.** (1) $\Rightarrow$ (2) is well-known [2, Lemmas 7.1 and 17.8].

(2) $\Rightarrow$ (3). First assume that $G$ is a locally finite group. Write $R = M_2(KG)$. By [4, Corollary 8.4], it is enough to prove that $R$ is right continuous, that is, any right ideal of $R$ which is isomorphic to a direct summand of $R$ is itself a direct summand of $R$. Observe that for any two elements $a$ and $b$ of $R = M_2(KG)$ there exists a right selfinjective subring $T$ containing $a$, $b$, and the unity of $R$. To see this, let $a$ and $b$ be any two elements of $R$ and let $H$ be the subgroup generated by the supports of the entries of $a$ and $b$. Since $G$ is locally finite, $H$ must be finite. Hence $KH$ is right selfinjective [3, Theorem 2.8, p. 79].

So $T = M_2(KH)$ is the desired right selfinjective subring of $R$ containing $a$, $b$, and the unity of $R$. To prove $R$ is right continuous, let $I$ be a right ideal of $R$ and let $I \trianglelefteq eR$ for some idempotent $e \in R$. Suppose $\varphi : I_R \rightarrow eR_R$ is an isomorphism. Let $a = a^{-1}(v)$. Then, as shown above, there exists a right selfinjective subring, say $S$, containing unity of $R$ such that $a, e \in S$. Obviously $\varphi$ induces an isomorphism $aS \rightarrow eS$. Since $S$ is right selfinjective, $eS$ is an injective right $S$-module. Therefore $aS = eS$ for some idempotent $v \in S$. Since $S$ contains the unity of $R$, it follows that $a^2 = eR$. Consequently $R$ is right continuous.

Now let $KG$ be semisimple and $G$ be either solvable or linear. By [3, Section 3, p. 322]

$$J(KG) = N^* (KG)$$

$$= \{ a \in KG : aS \text{ is nilpotent for every finitely generated subring } S \text{ of } KG \}.$$ In particular, $J(KG)$ is nil. By [3, Theorem 1.5, p. 409], $G$ is locally finite. But then by what we have proved above, $KG$ is right selfinjective.

(3) $\Rightarrow$ (1) follows from the fact that the matrix ring over a right selfinjective ring is again right selfinjective and right selfinjective rings are right CS-rings.

The equivalence of (3) and (4) is well-known [3, Theorem 2.8, p. 79].

**Remark 1.** The above proof shows that if $R$ is a right CS-ring with unity 1 such that any two elements $a, b \in R$ are contained in a right selfinjective subring $S$ having the same unity 1 then $R$ is right continuous. In particular, if $R = M_2(T)$ for some ring $T$ then $R$ (and hence $T$) is right selfinjective.

**Remark 2.** It follows from Remark 1 that a group algebra of a locally finite group is right CS if and only if it is right continuous.

**Remark 3.** Theorem 0.1 is not true if the order of the matrix ring is not greater than 1. For example, the group ring of an infinite locally finite $\mu$-group over a field of characteristic $\mu$ is a local right CS-ring but is not right selfinjective.

**Acknowledgment**

The authors thank the referee for his helpful suggestions.
References