ROOTS OF SYMMETRIC IDEMPOTENT BOOLEAN MATRICES *

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Abstract

Boolean matrices $A$ such that $A^m$ is symmetric and idempotent are characterized. As two of the applications, characterizations of Boolean matrices $A$ such that for some positive integer $m$, (i) $A^t = A^m$, or (ii) $A^m = A^{m+1}$, where $A^m$ is symmetric, are derived.

1. INTRODUCTION

We consider the Boolean algebra $\{0, 1\}$ equipped with the operations of Boolean addition and multiplication, which are the same as the usual operations, except that $1 + 1 = 1$. In this paper, we deal exclusively with Boolean matrices, i.e., matrices over the Boolean algebra $\{0, 1\}$. Matrix addition and multiplication are defined in the usual way. For a comprehensive survey of Boolean matrices, we refer to [3].

A characterization of (entrywise) nonnegative matrices $A$ such that $A^m$ is symmetric and idempotent was obtained in [2]. In the present paper, we consider a similar problem for Boolean matrices. We first introduce some notation. As usual the group of permutations of $\{1, 2, \cdots, n\}$ will be denoted by $S_n$. $J_\ell$ will denote the $\ell \times \ell$ matrix of all ones and if the order of the matrix is not relevant, then we will simply denote it by $J$. For a matrix $A$, $A^T$ and $A^t$ will denote the transpose of $A$ and the Moore-Penrose inverse of $A$, respectively. If $A_1, \cdots, A_k$ are matrices, then $A_1 \oplus \cdots \oplus A_k$ will denote their direct sum

$$
\begin{bmatrix}
A_1 & 0 & \cdots & 0 \\
0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_k
\end{bmatrix}
$$

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* Research partially supported by Baker Fund Award, Ohio University to S.K. Jain while he was visiting professor at Indian Statistical Institute in November-December 1993.
The main result gives a complete characterization of matrices $A$ such that $A^m$ is symmetric and idempotent.

2. THE MAIN RESULT

THEOREM 1: Let $A$ be an $n \times n$ matrix. Then $A^m$ is symmetric and idempotent if and only if there exists a permutation matrix $Q$ such that $QAQ^T$ is a direct sum of square matrices of the following (not necessarily all) three types.

(I) $C_{11}$, where $C_{11}^m = J$

\[
\begin{bmatrix}
0 & C_{12} & 0 & \cdots & 0 \\
0 & 0 & C_{23} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & C_{d-1,d} \\
C_{d1} & 0 & \cdots & 0 & 0
\end{bmatrix}
\]

where $d \mid m$, $d \neq 1$, the zeros on the diagonal are square matrices of appropriate order and

\[
(C_{j+1,j+1} \cdots C_{j+d-1,j})^{m/d} = J, \quad j = 0, 1, \cdots, d
\]

where the subscripts are to be interpreted modulo $d$.

(II) A block matrix $[C_{i,j}]$, $i, j = 1, 2, \cdots, \ell$ where $\ell \leq m$, $C_{i,j} = 0$ if $i \geq j$ and $C_{i,i}$ is square.

Before proving Theorem 1, we obtain some preliminary results. The proofs of the following lemmas are similar to those of Lemmas 4-6 in [2] and hence are omitted.

LEMMA 1: Suppose $C^m = D$ where $C, D$ are $n \times n$ matrices conformally partitioned as

\[
C = \begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{bmatrix}, \quad D = \begin{bmatrix}
D_{11} & 0 \\
0 & 0
\end{bmatrix}
\]

where the diagonal blocks are square and each diagonal entry of $D_{11}$ is 1. Then $C_{12}, C_{21}$ are zero matrices, $C_{11}^m = D_{11}$ and $C_{22}^m = 0$.

LEMMA 2: Let $C$ be an $n \times n$ matrix such that $C^m = 0$. Then there exists a permutation matrix $Q$ such that $QCQ^T$ is the block matrix $[C_{i,j}]$, $i, j = 1, 2, \cdots, \ell$ where $\ell \leq m$, $C_{i,j} = 0$ if $i \leq j$ and $C_{ii}$ is square.

LEMMA 3: Let $C = [C_{i,j}]$, $i, j = 1, 2, \cdots, n$ be a block matrix where $C_{i,j}$ is $\ell_i \times \ell_j$ and suppose $C^m = J_{\ell_1} \oplus \cdots \oplus J_{\ell_\ell}$. Then there exists $\sigma \in S_n$ such that
(a) \( C_{j} = \text{the zero matrix except when } \ell = \sigma(j) \)

(b) \( (C_{j} \sigma(j) C_{\sigma(j)} \sigma(j) \cdots C_{\sigma(j)^{d_j - 1} \sigma(j)})^{m/d_j} = J_{d_j} \)

where \( d_j \) is the least positive integer such that \( \sigma^{d_j}(j) = j \).

Consequently, there exists a permutation matrix \( Q \) such that \( QCQ^T \) is a direct sum of square matrices of types (I) and (II) (not necessarily both types) given in Theorem 1.

**Proof of Theorem 1.** The "if" part is easy. To prove the "only if" part, note that since \( A^m \) is symmetric, idempotent, there exists a permutation matrix \( Q \) such that \( QAQ^T = J_{\ell_1} \oplus \cdots \oplus J_{\ell_m} \oplus 0 \). This is seen by first reducing \( A^m \) to the Frobenius Normal Form and then using the well-known fact (see [1], for example) that an irreducible symmetric idempotent matrix must be equal to \( J \). The proof now follows from Lemmas 1, 2 and 3.

3. APPLICATIONS

**THEOREM 2:** Let \( A \) be an \( n \times n \) matrix. Then \( A^m \) is symmetric and \( A^{m+1} = A \) if and only if there exists a permutation matrix \( Q \) such that \( QAQ^T \) is a direct sum of the following (not necessarily both) two types.

(i) A square matrix of all ones or all zeros

\[
\begin{bmatrix}
  0 & J_{\ell_1} & 0 & \cdots & 0 \\
  0 & 0 & J_{\ell_2} & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & \cdots & \cdots & \cdots & J_{\ell_{d-1}} \\
  J_{\ell_d} & \cdots & \cdots & \cdots & 0
\end{bmatrix}
\]

where \( d \mid m \).

**Proof:** Note that \( A^m \) is idempotent. Therefore, by Theorem 1 there exists a permutation matrix \( Q \) such that \( QAQ^T \) is a direct sum of square matrices of type (I), (II) or (III). Since \( A^{m+1} = A \), each summand \( S \) of \( QAQ^T \) satisfies \( S^{m+1} = S \).

If \( S \) is of type (I), then \( S^m = J \). Thus \( S \) cannot have a zero row and, therefore, \( S = S^{m+1} = SJ = J \). If \( S \) is of type (II), then using a similar argument we can show that \( S \) must be of type (II). Finally, if \( S \) is of type (III), then \( S^m = 0 \). Thus \( S = 0 \).

**REMARK:** As an immediate consequence of Theorem 2, we see that if \( A \) is an \( n \times n \) matrix such that \( A^t \) exists (in which case it, in fact, is \( A^T \), see for example [4]) and if \( A^t = A^m \), then there exists a permutation matrix \( Q \) such that \( QAQ^T \) is a direct sum of square matrices of types (i) or (ii).
As another application, we can obtain the characterization of Boolean matrices $A$ such that $A^m$ is symmetric and $A^{m+1} = A^m$ in the following theorem.

THEOREM 3: Suppose $A$ is a Boolean matrix such that $A^m = A^{m+1}$ and $A^m$ is symmetric. Then there exists a permutation matrix $Q$ such that $QAQ^T$ is a direct sum of matrices of the following types:

(i) A square matrix of all ones or all zeros.

(ii) A nilpotent matrix partitioned as $\ell \times \ell$ block matrix $[C_{i,j}]$ where $C_{i,j} = 0$ if $i \geq j$ and $C_{ii}$ is square, $\ell \leq m$.

Proof. Follows from Theorem 1.

REFERENCES


