NONNEGATIVE MINIMUM NORM LEAST SQUARES SOLUTIONS OF $AX = B$

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1. Notation and definitions. Let $A$ denote an $m \times n$ matrix and $X$ denote an $n \times m$ real matrix. Consider the equations: (1) $AXA = A$, (2) $XAX = X$, (3) $(AX)^T = AX$, (4) $(XA)^T =XA$, and (5) $AX = XA$, where $T$ denotes the transpose. Let $\lambda$ denote a nonempty subset of \{1, 2, 3, 4, 5\}. Then $X$ is called a $\lambda$-inverse of $A$ if $X$ satisfies equation (i) for each $i \in \lambda$. A $\lambda$-inverse of a matrix $A$ is denoted by $A^{(\lambda)}$. A \{1, 2, 3, 4\}-inverse of $A$ is the unique Moore-Penrose inverse of $A$ and is denoted by $A^\dagger$. A \{1, 2, 5\}-inverse of $A$ exists if and only if $m = n$ and $\text{rank } A = \text{rank } A^T$. A \{1, 2, 5\}-inverse is called a group inverse and is denoted by $A^#$. The group inverse $A^#$ is a polynomial in $A$. $A \geq 0$ means that all entries of $A$ are nonnegative, and $R(A)$ denotes the range of $A$.

2. Example. In this section we give an example for which the minimum norm least squares solution for the system $AX = I$ fails to be nonnegative, but for which the system $AX = B$ does have a nonnegative minimum norm least squares solution for some nonnegative idempotent matrix $B$.

Let

$$A = B = \begin{bmatrix}
UUT & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
CUT & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad \text{where } U = \begin{bmatrix}
1/\sqrt{2} & 0 \\
1/\sqrt{2} & 0 \\
0 & 1/\sqrt{2} \\
0 & 1/\sqrt{2}
\end{bmatrix}$$

$C = (1 \ 2 \ 3 \ 4 )$. It can be shown that $A^\dagger$, which is the minimum norm least squares solution of $AX = I$, is not nonnegative but that $A^\dagger A$, which is the minimum norm least squares solution of $AX = A$, is nonnegative. It is also of interest to note that $AA^\dagger \geq 0$ for this example.
3. Results

**Lemma 1.** Let \( A \) be any square matrix and \( E \) be an idempotent matrix,

(i) If \( R(A) \subset R(E) \), then \( EA = A \).

(ii) If \( AE = EA \) and rank \( A = \text{rank } (AE) \), then \( R(A) \subset R(E) \) and so \( A = EA \).

**Proof.** The proof is straightforward.

**Lemma 2.** If \( A, B \) are square matrices such that \( \text{rank } (AB) = \text{rank } A \), \( BA = AB \), and \( B^\# \) exists, then

(i) \( (A^tBA)^\# = (A^tBA)^t = A^tB^#A \) and

(ii) \( (AA^tB)^\# = B^#A^tA^tB \).

**Proof.** It is a consequence of lemma 1 that \( BB^#A = A \). The verification of (i) and (ii) follows from direct computation using this fact together with \( AB = BA \).

The theorem proved below gives the form of a nonnegative minimum norm least squares solution of \( AX = B \), where \( A, B \) are nonnegative matrices satisfying certain conditions including \( B^\# \geq 0 \). The characterization of nonnegative matrices \( B \) for which \( B^\# \geq 0 \) is given in [4]. The theorem below generalizes the known result when \( B = I \), [7].

Our theorem gives necessary conditions in order that a solution \( X_o \) to the minimization problem:

\[
\min \| AX - B \| ,
\]

is also a solution of the constrained minimization problem.

\[
\min \| AX - B \| , \quad X \geq 0.
\]

**Theorem 1.** Let \( A, B \) be nonnegative matrices such that \( B^\# \geq 0 \), \( AB = BA \), and rank \( AB = \text{rank } A \). If \( AX = B \) has a nonnegative minimum norm least squares solution, then there are permutation matrices \( P, Q \) such that

\[
PAQ^T = \begin{bmatrix}
A_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
ZA_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
SOLUTIONS OF $AX = B$

and

$$QX^T = \begin{bmatrix}
X_{11} & X_{11}^T & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

where some zero blocks may not appear and $A_{11}X_{11}$ are direct sums of matrices of the following two types (not necessarily both):

(I) $\beta xy^T, \beta \geq 0, x$ and $y$ positive unit vectors

(II)

$$\begin{bmatrix}
0 & \beta_{12}x_1y_2^T & 0 & 0 & \cdots & 0 \\
0 & 0 & \beta_{23}x_2y_3^T & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\beta_{d1}x_dy_1^T & 0 & 0 & \cdots & 0
\end{bmatrix}$$

$\beta_{ij} \geq 0, x_i, y_i$ positive unit vectors not necessarily of the same size.

Equivalently $A_{11} \succeq 0, X_{11} \succeq 0$.

PROOF. It is known that the minimum norm least squares solution of $AX = B$ is given by $X_0 = A^TB$, ([1], p. 119). By lemma 2, $(AX_0)^\# \succeq 0$ and $(X_0A)^\# = (X_0A)^T \succeq 0$. Then by [3] and [4] there exist permutation matrices $P_1, Q_1$ such that

$$P_1AX_0P_1^T = \begin{bmatrix}
J & JD & 0 & 0 \\
0 & 0 & 0 & 0 \\
CJ & CJD & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

and

$$Q_1X_0AQ_1^T = \begin{bmatrix}
J_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

where $J$ and $J_1$ are direct sums of matrices of types (I) and (II) (not necessarily both) and $C, D \succeq 0$.

Let $L = P_1AQ_1^T, M = Q_1X_0P_1^T$
Then

\[
LM = \begin{bmatrix}
J & JD & 0 & 0 \\
0 & 0 & 0 & 0 \\
CJ & CJD & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

and

\[
ML = \begin{bmatrix}
J_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Using the same argument as in Lemma 1 of [5], we obtain

\[
L = \begin{bmatrix}
L_{11} & L_{11}U & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

and

\[
M = \begin{bmatrix}
M_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
VM_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

where \(L_{11}M_{11} = J\) and \(M_{11}L_{11} = J_1\).

Arguments similar to the proof of Lemma 2 in [6] can be used to obtain permutation matrices \(P_2, Q_2\) such that

\[
P_2L_{11}Q_2^T \quad \text{and} \quad Q_2M_{11}P_2^T
\]

are direct sums of matrices of types (I) or (II). The proof depends on the fact that each row of the block partitioned matrix has one and only one nonzero block, likewise for the columns. The details are rather technical although straightforward.

Finally for

\[
P = \begin{bmatrix}
P_2 & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

and

\[
Q = \begin{bmatrix}
Q_2 & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
SOLUTIONS OF AX = B

PAQᵀ and QXᵀPT have the desired form. This completes the proof.

If the matrix B satisfies a somewhat weaker hypothesis than commuting with A, then the following theorem gives the form of the matrix A. However in this case we are unable to yield the form of the solution.

**Theorem 2.** Let A, B be nonnegative matrices such that B# ≥ 0, R(A) ⊂ R(B), and rank A = rank AB. If the system AX = B has a nonnegative minimum norm least squares solution, then there are permutation matrices P, Q such that

\[
    PAQ = \begin{bmatrix}
        J & 0 & 0 \\
        0 & 0 & 0 \\
        CJ & 0 & 0 \\
        0 & 0 & 0
    \end{bmatrix}
\]

where J is a direct sum of matrices of the types (I) and (II) as in Theorem 1.

**References**


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