Nonnegative $m$th Roots of Nonnegative 0-Symmetric Idempotent Matrices

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ABSTRACT

Nonnegative $m$th roots of nonnegative 0-symmetric idempotent matrices have been characterized. As an application, a characterization of nonnegative matrices $A$ whose Moore-Penrose generalized inverse $A^+$ is some power of $A$ is obtained, thus yielding some well-known theorems.

1. INTRODUCTION

Let $A$ be an $m \times n$ real matrix. Consider the Penrose [8] equations

$$AXA = A,$$

$$XAX = X,$$

$$(AX)^T = AX,$$

$$(XA)^T = XA,$$

where $X$ is an $n \times m$ real matrix and $T$ denotes the transpose. Consider also the equations

$$A^kXA = A^k,$$

$$AX =XA,$$

where $k$ is some positive integer.
For a rectangular matrix $A$ and for a nonempty subset $\lambda$ of $\{1, 2, 3, 4\}$, $X$ is called a $\lambda$-inverse of $A$ if $X$ satisfies Eq. (i) for each $i \in \lambda$. In particular, the $(1, 2, 3, 4)$-inverse of $A$ is the unique Moore-Penrose generalized inverse. The unique solution $X$ of (2), (1$^k$), and (5) is a square matrix called the Drazin inverse of $A$, where $k$ is the smallest positive integer such that $\text{rank} A^k = \text{rank} A^{k+1}$.

A matrix $A = (a_{ij})$ is called 0-symmetric if $a_{ij} = 0$ implies $a_{ji} = 0$. Thus every symmetric matrix and every positive matrix is 0-symmetric. If a matrix $A$ is a direct sum of matrices $A_i$, then $A_i$ will be called summands of $A$.

The problem of finding the $m$th roots of any matrix $A$ is an important classical problem (see Gantmacher [4], Chapter 8). In this paper our aim is to study the nonnegative $m$th roots of nonnegative 0-symmetric idempotent matrices. Theorem 1 of this paper reduces the study of the nonnegative $m$th roots of any nonnegative 0-symmetric idempotent matrix to the nonnegative $k$th roots of matrices of the form $xy^T$ ($x$, $y$ positive vectors with $y^T x = 1$), and to the nonnegative solution of a system of simultaneous equations of the type $X_1X_2\ldots X_d = x_1y_1^T, \ldots, X_dX_1\ldots X_{d-1} = x_d y_d^T$ ($x_i$, $y_i$ positive vectors with $y_i^T x_i = 1$). Clearly, $xy^T$ is the only nonnegative $k$th root of rank 1 of the positive idempotent matrix $xy^T$. However, the nonnegative $k$th roots of ranks greater than 1 are not considered, and it remains open to determine such roots. In Sec. 4, we use the reduction obtained in Theorem 1 to characterize the nonnegative matrices $A$ such that $A^k$ is 0-symmetric and $A^{k+1} = A$ for some positive integer $k$. This, in particular, determines all nonnegative matrices $A$ whose generalized inverse $A^+$ is some power of $A$. This result generalizes the recent results of Harary and Minc [5] for nonnegative matrices $A$ with $A^{-1} = A$ and that of Berman [1] for nonnegative matrices $A$ with $A^+ = A$.

1.1. Notation and Conventions

- $S_n$: the group of permutations on $\{1, 2, \ldots, n\}$.
- $A^+$: Moore-Penrose generalized inverse.
- $A_i$: Drazin inverse.
- $A > 0$: a matrix with nonnegative entries.
- $A > 0$: a matrix with positive entries.
- $\mathcal{E}$: a set of nonnegative matrices.
- $\nabla_{\mathcal{E}}$: $\{X | X^m \in \mathcal{E}\}$.
- $\nabla_{\mathcal{E}}$: $\{X > 0 | X^m \in \mathcal{E}\}$.
- $C^{(i)}_{pq}$: the $(p, q)$th block of the $i$th power of a partitioned matrix $C$.

The diagonal of any square matrix shall mean the main diagonal. By a vector we shall mean a column vector.
2. MAIN RESULTS

THEOREM 1. Let $\mathcal{E}$ be the set of all nonnegative 0-symmetric idempotent matrices. Then $A \in + \mathcal{E}$ if and only if there exists a permutation matrix $P$ such that $PAP^T$ is a direct sum of square matrices of the following (not necessarily all) three types:

(I) $C_{11}$, where $C_{11}^m = xy^T$, for some positive vectors $x$ and $y$ such that $y^Tx = 1$.

(II) 
\[
\begin{pmatrix}
0 & C_{12} & 0 & \cdots & 0 \\
0 & 0 & C_{23} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & C_{d-1d} \\
C_{d1} & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

where $(C_{12}C_{23} \cdots C_{d1})^{m/d} = x_1 y_1^T, \ldots, (C_{d1}C_{12} \cdots C_{d-1d})^{m/d} = x_d y_d^T$, $x_i$, $y_i$ are positive vectors of the same order with $y_i^Tx_i = 1$; $x_i$ and $x_j$, $i \neq j$, are not necessarily of the same order; $d|m$, $d \neq 1$, and the zeros on the diagonal are square matrices of appropriate orders.

(III) 
\[
\begin{pmatrix}
0 & C_{13} & \cdots & C_{1l} \\
0 & 0 & C_{23} & \cdots & C_{2l} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & C_{l-1l} \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

where $l < m$, the $C_{ij}$'s are nonnegative matrices of appropriate orders, and the zeros on the diagonal are square matrices.

THEOREM 2. Let $\mathcal{E}$ be the set of all nonnegative symmetric idempotent matrices. Then $A \in + \mathcal{E}$ if and only if there exists a permutation matrix $P$ such that $PAP^T$ is a direct sum of square matrices of the following (not necessarily all) three types:

(I) $C_{11}$, where $C_{11}^m = xx^T$ and $x$ is a positive unit vector.
(II)

\[
\begin{bmatrix}
0 & C_{12} & 0 & 0 & \cdots & 0 \\
0 & 0 & C_{23} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & C_{d-1d} \\
C_{d1} & 0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix},
\]

where \((C_{12}C_{23}\cdots C_{d1})^{m/d} = x_1x_1^T, \ldots, (C_{d1}C_{12}\cdots C_{d-1d})^{m/d} = x_dx_d^T\); the \(x_i\)'s are positive unit vectors (not necessarily of the same order); \(d|m, d \neq 1\); and the zeros on the diagonal stand for the square matrices of appropriate orders.

(III)

\[
\begin{bmatrix}
0 & C_{12} & C_{13} & \cdots & \cdots & \cdots & C_{1l} \\
0 & 0 & C_{23} & \cdots & \cdots & \cdots & C_{2l} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & C_{l-1l} \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0
\end{bmatrix},
\]

where \(l \leq m\), the \(C_{ij}\)'s are nonnegative matrices of appropriate orders, and the zeros on the diagonal stand for the square matrices.

3. PRELIMINARY RESULTS AND PROOFS OF THEOREMS 1 AND 2

In order to prove Theorems 1 and 2 we shall prove a few lemmas. We first recall that if \(A, B\) are nonnegative matrices of orders \(m \times n, n \times k\), respectively, such that \(AB = 0\), then for any \(i, 1 \leq i \leq n\), the \(i\)th column of \(A\) and the \(i\)th row of \(B\) cannot both be nonzero. We now prove

**Lemma 1.** Let \(A, C\) be nonnegative (not necessarily square) matrices such that \(AC = 0\), and \(XA + CY > 0\) for some matrices \(X\) and \(Y\) (not necessarily nonnegative). Then \(A = 0\) or \(C = 0\).

**Proof.** Assume \(A \neq 0\), \(C \neq 0\). Then \(AC = 0\) implies that there exists a zero column of \(A\) (hence of \(XA\)) and a zero row of \(C\) (hence of \(CY\)). But then \(XA + CY\) cannot be positive, a contradiction.
Lemma 2. Let $A, C_1, \ldots, C_n$ be nonnegative matrices such that $AC_i = 0$ ($C_i A = 0$), $i = 1, \ldots, n$, and $XA + \sum_{i=1}^{n} C_i Y_i > 0$ ($AX + \sum_{i=1}^{n} Y_i C_i > 0$) for some nonnegative matrices $X, Y_i$, $1 \leq i \leq n$. Then $A = 0$ or all $C_i$'s are zero.

Proof. Observe $A(\sum_{i=1}^{n} C_i) = 0$ and $XA + (\sum_{i=1}^{n} C_i)(\sum_{i=1}^{n} Y_i) > 0$, and apply Lemma 1.

Lemma 3. Let $A, B, C,$ and $D$ be nonnegative matrices of orders $m \times n$, $n \times m$, $n \times m$, and $m \times n$, respectively, such that $AC = 0 = DB$ and each entry on the diagonal of $BA + CD$ is nonzero. Then the $j$th column of $A$ is zero if and only if the $j$th row of $B$ is zero.

If in addition, $AB = 0$, then $A = 0 = B$.

Proof. If $A, B, C,$ or $D$ is zero, then the proof is trivial. So assume each of the matrices $A, B, C,$ and $D$ is not zero. Let the $j$th column of $A$ be zero. Then the $j$th column of $BA$ is zero. Since the diagonal of $BA + CD$ is nonzero, this implies that the $j$th column of $CD$ cannot be zero. Hence the $j$th column of $D$ cannot be zero. But then $DB = 0$ implies that the $j$th row of $B$ is zero. The converse can be proved similarly.

The last statement follows trivially.

Lemma 4. Let $\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$ be a nonnegative matrix such that the diagonal blocks are square matrices and each entry on the diagonal of $D$ is nonzero. Then

$$+ \sqrt{m \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} + \sqrt{m} D & 0 \\ 0 & + \sqrt{0} \end{pmatrix}}.$$ 

Proof. Let $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \in + \sqrt{m \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}}$.

Then

$$CC^{-1} = C^{m-1}C = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$
implies

\[ C_{11}^{(m-1)} + C_{12}^{(m-1)} C_{21}^{-1} = D, \quad C_{11}^{(m-1)} C_{12} = C_{21}^{(m-1)} C_{11} = C_{21}^{(m-1)} C_{12} = 0, \]

and

\[ C_{11}^{(m-1)} C_{11} + C_{12}^{(m-1)} C_{21} = D, \quad C_{11} C_{12}^{(m-1)} = C_{21} C_{11}^{(m-1)} = C_{21} C_{12}^{(m-1)} = 0. \]

Thus, by Lemma 3, \( C_{12} = 0 = C_{21} \). Then \( C_{11}^{m} = D \) and \( C_{22}^{m} = 0 \). Hence

\[
\begin{pmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{pmatrix}
\in
\begin{pmatrix}
\sqrt{D} & 0 \\
0 & \sqrt{0}
\end{pmatrix},
\]

completing the proof.

**Lemma 5.** Let

\[
C = \begin{bmatrix}
C_{11} & \cdots & C_{1n} \\
\vdots & \ddots & \vdots \\
C_{n1} & \cdots & C_{nn}
\end{bmatrix} \in +^m \sqrt{\begin{bmatrix}
\alpha_1 & 0 \\
\vdots & \ddots \\
0 & \alpha_n
\end{bmatrix}},
\]

where \( C_{ij} \) is a nonnegative matrix of order \( l_i \times l_j \), and \( A_i \) is a positive square matrix of order \( l_i \), \( 1 \leq i \leq n \). Then there exists an \( \sigma \in S_n \) such that

(a) \( C_{\sigma(j),\sigma(k)} \neq 0, C_{jk} = 0 \ \forall k \neq \sigma(j), j = 1, \ldots, n. \)

(b) \( C_{\sigma(j),\sigma(k)} \cdots C_{\sigma(n),\sigma(1)} = A_1 \) \ ([Equivalently, if \( d_1 \) is the smallest positive integer such that \( \sigma^{d_1}(\cdot) = \cdot \), then \( (C_{\sigma(j),\sigma(k)} \cdots C_{\sigma(n),\sigma(1)})^{m/d_1} = A_1 \).]

(c) \( \sigma^m = I \), the identity permutation.

(d) There exists a permutation matrix \( P \) such that \( PCP^T \) is a direct sum of square matrices of the types (I) or (II) described in Theorem 1.

**Proof.** Since

\[
C^m = \begin{bmatrix}
A_1 & 0 \\
\vdots & \ddots \\
0 & A_n
\end{bmatrix},
\]
we get
\[ C_{ik}^{(m-1)}C_{kj} = 0 = C_{ik}C_{kj}^{(m-1)} \text{ for all } i \neq j \]  
(6)

and
\[ A_j = I_{j_1}C_{ii}^{(m-1)} + \cdots + I_{j_r}C_{ii}^{(m-1)} + \cdots + I_{j_n}C_{ii}^{(m-1)}. \]  
(7)

Assume \( C_{ij}^{(m-1)} \neq 0 \). Then \( C_{ij}^{(m-1)} \neq 0 \), and thus by (6), (7), and Lemma 2, \( C_{ij} = 0 \) \( \forall i \neq j \). Note that \( C_{ij} \neq 0 \) and \( A_i = C_{ii}C_{ij}^{(m-1)}. \) Hence each row of \( C \) has one and only one nonzero block. Since the matrix \( C^m \) has no zero column, the same is true for the matrix \( C \). Therefore, there is one and only one nonzero block in each row and in each column of \( C \). This determines a permutation \( \sigma \in S_n \) such that
\[ C_{\sigma(i)j} = 0, \quad C_{ij} = 0 \quad \forall k \neq \sigma(i), \quad j = 1, \ldots, n. \]  
(8)

Then from (7) and (8), \( A_j = C_{i_1}C_{ii}^{(m-1)} \cdots C_{i_r}C_{ii}^{(m-1)} \). But then \( C_{ij} \neq 0 \), \( C_{j_1}C_{ii}^{(m-1)} \neq 0 \), \( \ldots \), \( C_{j_r}C_{ii}^{(m-1)} \neq 0 \) imply \( p_1 = \sigma(j_1), \quad p_2 = \sigma(j_2), \quad \ldots, \quad p_{m-1} = \sigma^{m-1}(j) \), and \( j = \sigma^m(j) \). Hence \( \sigma^m = I \), the identity permutation, proving (b) and (c).

Since any permutation \( \sigma \) can be expressed as a product of disjoint cycles, (d) follows by straightforward computations.

Lemma 6. Let \( 0 \neq C \in \mathbb{M}_n \), where \( 0 \) is a square matrix of order \( n \). Then there exists a permutation matrix \( P \) such that
\[
P C P^T = \begin{bmatrix}
0 & C_{12} & \cdots & C_{1i} \\
0 & 0 & \cdots & C_{ii} \\
& \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & C_{i-1i} \end{bmatrix},
\]
where \( l < m \), the 0's on the diagonal stand for square matrices, and the \( C_{ij} \)'s are nonnegative matrices of appropriate orders.

Proof. If \( C = 0 \) then the proof is trivial. So assume \( C \neq 0 \). Then \( m > 1 \). We shall prove this result by induction on \( m \). So suppose \( m = 2 \). Then \( C^2 = 0 \) implies that there exists a \( \sigma \in S_n \) and \( 1 < r < n \) such that \( \sigma(1)\text{th}, \ldots, \sigma(r)\text{th} \)
rows and $\sigma(r+1)th, \ldots, \sigma(n)$th columns of $C$ are zero. This gives a permutation matrix $P$ such that $PCP^T$ is of the required form. We now assume that the result is true for $m=k-1$ and prove the result for $m=k$. Since $C^k=0$ we have $(C^{k-1})^2=0$. By induction there exists a permutation matrix $P_1$ such that

$$P_1C^{k-1}P_1^T = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}.$$

Without loss of generality, we can assume that each row of the matrix block $D$ is nonzero. Let

$$P_1CP_1^T = \begin{pmatrix} A & E \\ B & F \end{pmatrix}.$$

Then $P_1C^kP_1^T = 0$ gives

$$\begin{pmatrix} A & E \\ B & F \end{pmatrix} \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} = 0.$$

This implies $AD=0 = BD$. But since no row of $D$ is zero, we get $A=0 = B$. Thus

$$P_1CP_1^T = \begin{pmatrix} 0 & E \\ 0 & F \end{pmatrix}.$$

Then

$$\begin{pmatrix} 0 & E \\ 0 & F \end{pmatrix}^{k-1} = (P_1CP_1^T)^{k-1} = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}.$$

implies $F^{k-1}=0$. Again by the induction assumption, there exists a permutation matrix $P_2$ such that

$$P_2F^{p-1}P_2^T = \begin{pmatrix} 0 & F_{12} & F_{13} & \cdots & F_{1q} \\ 0 & 0 & F_{23} & \cdots & F_{2q} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & F_{q-1q} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$q < k-1$. 
Then

\[ P = \begin{pmatrix} I & 0 \\ 0 & P_2 \end{pmatrix} P_1 \]

is a desired permutation matrix.

Proof of Theorem 1.

"Only if" part.

Let \( A \in \sqrt{B} \). Then there exists a matrix \( B \in \mathbb{R} \) such that \( A^m = B \). Since \( B \) is a 0-symmetric idempotent matrix, there exists, by Flor [3], a permutation matrix \( P \) such that

\[
PBP^T = \begin{pmatrix} A_1 & 0 \\ & \ddots & 0 \\ & & A_s \end{pmatrix},
\]

where \( A_i = x_i y_i^T \), \( x_i, y_i \) are positive vectors with \( y_i^T x_i = 1 \), and \( s \) is the rank of \( B \). The proof now follows by Lemmas 4, 5, and 6.

The converse is clear.

Proof of Theorem 2.

In the proof of Theorem 1, we observe that if \( B \) is symmetric, then \( A_i = x_i x_i^T \), where \( x_i \) is a positive vector with \( x_i^T x_i = 1 \). This completes the proof.

4. APPLICATIONS OF MAIN RESULTS

In this section we use our main results to obtain characterizations of nonnegative matrices \( A \) such that \( A^k \) is 0-symmetric and \( A^{k+1} = A \) for some positive integer \( k \). This gives, in particular, characterization of matrices \( A \) whose generalized inverses are some power of \( A \) (cf. [1], [5]).

Theorem 3. Let \( A \) be a nonnegative matrix. Then \( A^m \) is 0-symmetric and \( A^{m+1} = A \) if and only if there exists a permutation matrix \( P \) such that \( PAP^T \) is a direct sum of matrices of the following (not necessarily all) three
types:

(i) \( xy^T \), where \( x \) and \( y \) are positive vectors with \( y^T x = 1 \).

(ii) \[
\begin{pmatrix}
0 & \omega_{12} x_1 y_2^T & 0 & 0 & \cdots & 0 \\
0 & 0 & \omega_{23} x_2 y_3^T & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \omega_{d-1 \cdot d-1} x_{d-1} y_{d}^T \\
\omega_{d1} x_d y_1^T & 0 & 0 & \cdots & 0 & 0
\end{pmatrix},
\]

where \( x, y \), are positive vectors of the same order with \( y^T x = 1 \); \( x_i \) and \( x_j \), \( i \neq j \), are not necessarily of the same order; \( d \mid m \); and \( \omega_{12}, \ldots, \omega_{d1} \) are positive numbers with \( \omega_{12} \omega_{23} \cdots \omega_{d1} = 1 \).

(iii) A zero matrix.

**Proof.** "Only if" part: Clearly \( A^m \) is idempotent. Hence by Theorem 1, there exists a permutation matrix \( P \) such that \( PAP^T \) is a direct sum of the square matrices of the types (I), (II), or (III). Since \( A^{m+1} = A \), each summand \( S \) of \( PAP^T \) satisfies \( S^{m+1} = S \). If \( S \) is of type (I), then \( S = C_{11} \), where \( C_{11} = xy^T \) for some positive vectors \( x \) and \( y \) such that \( y^T x = 1 \). Since \( xy^T \) is idempotent of rank 1, there exists an invertible matrix \( U \) such that

\[
xy^T = U \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U^{-1},
\]

where zero block on the diagonal stands for a square matrix. It follows then that the first column of \( U \) is \( x \), and the first row of \( U^{-1} \) is \( y^T \). From Gantmacher [4, p. 235] we have

\[
\frac{\sqrt{m}}{\sqrt{xy^T}} = U \begin{pmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{m} \end{pmatrix} U^{-1}.
\]

Further, if \( R^{m+1} = R \) for some \( R \in \mathbb{R}^n \), then

\[
R \in U \begin{pmatrix} \sqrt{1} & 0 \\ 0 & 0 \end{pmatrix} U^{-1}.
\]
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Since $S \in +\sqrt[n]{xy^T}$, we obtain

$$S = U \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} U^{-1} = xy^T.$$ 

If $S$ is of type (II), then

$$S = \begin{pmatrix} 0 & C_{12} & 0 & \cdots & 0 \\ 0 & 0 & C_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ C_{d1} & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $(C_{12}C_{23}\cdots C_{d1})^{m/d} = x_1y_1^T$, $\ldots$, $(C_{d1}C_{12}\cdots C_{d-1,d})^{m/d} = x_dy_d^T$, $x_i$ and $y_i$ are positive vectors with $y_i/x_i = 1$, $d|m$, and the zeros on the diagonal stand for the square matrices of appropriate orders. Therefore,

$$S^m = \begin{pmatrix} x_1y_1^T & 0 & 0 & \cdots & 0 \\ 0 & x_2y_2^T & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ x_1y_1^T & 0 & 0 & \cdots & 0 \end{pmatrix}.$$ 

Since $S^m$ is an idempotent matrix of rank $d$, there exists an invertible matrix $U$ such that

$$S^m = U \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} U^{-1}.$$ 

This implies that the first $d$ columns of $U$ are

$$\begin{pmatrix} u_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_2 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \ddots \\ \vdots \\ u_d \end{pmatrix}.$$
and the first $d$ rows of $U^{-1}$ are $(v_i^T, 0, \ldots, 0), \ldots, (0, \ldots, 0, v_d^T)$ in this order, and $I_d$ is the $d \times d$ identity matrix. Let

$$u_i = \begin{bmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{id} \end{bmatrix} \text{ and } v_i = \begin{bmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{id} \end{bmatrix}, \quad 1 \leq i \leq d.$$ 

From Gantmacher [4, p. 235]

$$S \in U \begin{bmatrix} \sqrt{m}I_d & 0 \\ 0 & \sqrt{n}I_0 \end{bmatrix} U^{-1}.$$ 

Since $S^{m+1} = S$, we get

$$S \in U \begin{bmatrix} \sqrt{m}I_d & 0 \\ 0 & 0 \end{bmatrix} U^{-1}.$$ 

Thus

$$S = U \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix} U^{-1},$$

where $W^m = I_d$. Also

$$S = \begin{bmatrix} 0 & C_{12} & 0 & 0 & \cdots & 0 \\ 0 & 0 & C_{23} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & C_{d-1,d} \\ C_{d1} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$ 

So if $W = (w_{ij})$, then simple computations give all $w_{ij} = 0$ except
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\[ S = U \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} U^{-1} \]

\[
\begin{bmatrix}
0 & w_{12}x_1 y_2^T & 0 & 0 & \cdots & 0 \\
0 & 0 & w_{23}x_2 y_3^T & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & w_{d-1d}x_{d-1} y_d^T \\
w_{d1}x_d y_1^T & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}.
\]

Finally, suppose \( S \) is of type (III). Then \( S^{m+1} = S \) gives \( S = 0 \), completing the proof.

The converse is clear.

**Theorem 4.** Let \( A \) be a nonnegative matrix. Then \( A' = A^{m-1} \) for some positive integer \( m \) if and only if there exists a permutation matrix \( P \) such that \( P A P^T \) is a direct sum of matrices of the following (not necessarily all) three types:

(i) \( xx^T \), where \( x \) is a positive unit vector.

(ii) \[
\begin{bmatrix}
0 & w_{12}x_1 x_2^T & 0 & 0 & \cdots & 0 \\
0 & 0 & w_{23}x_2 x_3^T & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & w_{d-1d}x_{d-1} x_d^T \\
w_{d1}x_d x_1^T & 0 & 0 & \cdots & 0 & 0
\end{bmatrix},
\]

where \( x_i \) are positive unit vectors; \( x_i \) and \( x_j \), \( i \neq j \), are not necessarily of the same order; \( d|m \); and \( w_{12}, \ldots, w_{d1} \) are positive numbers with \( w_{12}w_{23} \cdots w_{d1} = 1 \).

(iii) A zero matrix.

**Proof.** Follows from Theorems 2 and 3.
5. REMARKS

(1) As special cases of Theorem 4 we can obtain theorems of Harary and
Minc [5] and Berman [1], characterizing nonnegative matrices $A$ such that
$A^{-1} = A$ and $A^T = A$ respectively.

(2) We can also derive the nonnegative solutions of the matrix equation
$X^m = I$, where $m$ is a positive integer, from Theorem 4. The solutions are
square matrices $A$ such that for some permutation matrix $P$, $PAP^T$ is a direct
sum of matrices $A_i$, where $A_i$ is an identity matrix or a matrix of the form

\[
\begin{pmatrix}
0 & a_1 & 0 & \cdots & 0 \\
0 & 0 & a_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 0 \\
a_d & 0 & \cdots & \cdots & 0
\end{pmatrix}
\]

where $a_1 a_2 \cdots a_d \cdots a_{d-1} = 1$, $a_i > 0$, $1 \leq i \leq d$, and $d|m$.

The referee has informed us that M. Lewin [7] has also characterized the
nonnegative solutions of $X^m = I$.

(3) A special case of Theorem 3 answers a question of Berman [1] for
characterizing the nonnegative matrices which are equal to a $(1,2)$-inverse of
themselves (equivalently $A = A^T$) under the hypothesis that $A^2$ is 0-symmetric.
We note from Theorem 3 that if $A^3 = A$, (i.e., $A$ is equal to a $(1,2)$-inverse),
then $A$ is 0-symmetric if and only if $A^2$ is 0-symmetric.

(4) In another paper [6] we have characterized nonnegative matrices $A$
whose Moore-Penrose generalized inverse $A^+$ is nonnegative and is equal to
some polynomial in $A$ with scalar coefficients. This result generalizes Theo-
rem 4 of this paper.

REFERENCES

1. A. Berman, Nonnegative matrices which are equal to their generalized inverse,
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