ON A CLASS OF NON-NOETHERIAN V-RINGS

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Abstract. Right V-rings $R$ with infinitely generated right socle $\text{Soc}(R_R)$ such that $R/\text{Soc}(R_R)$ is a division ring are characterized as those non-noetherian rings over which a cyclic right module is either non-singular or injective. Furthermore, it is shown that a non-noetherian, right V-ring $S$ is Morita-equivalent to a ring of this type iff all singular simple right $S$-modules are isomorphic and every direct sum of uniform modules with an injective module over $S$ is extending.

1. Introduction. A ring $R$ is said to be of type (*) if $R$ is a right V-ring with infinitely generated $\text{Soc}(R_R)$ such that $R/\text{Soc}(R_R)$ is a division ring. Rings of this type have been considered to show, among others, that:

a) there are right SI-rings which are not left SI ([7, Example 3.2]),

b) there are right V-rings which are not left V ([8, Example 6.19]), and

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c) there exist rings, not necessarily right noetherian, over which direct
sums of uniform right modules and an injective right module are extending
([9]).

In view of the importance of these rings it would be interesting to inves-
tigate the behavior of modules over them, especially that of cyclic modules.

We will show that rings of type (*) are exactly rings which are not right
noetherian and each of their cyclic right modules is either non-singular or
injective. Moreover, a ring \( R \) is Morita-equivalent to a ring of type (*) if and
only if \( R \) is a non-noetherian right V-ring such that \( R \) satisfies the module
theoretic condition stated in c) and all singular simple right \( R \)-modules are
isomorphic.

2. Preliminaries. Throughout this paper, all rings have identity and
all modules are unitary right modules, unless otherwise stated. For a module
\( M \) we denote by \( Soc(M) \), \( Z(M) \) and \( J(M) \) the socle, the singular submodule
and the Jacobson radical of \( M \), respectively. If \( M = Soc(M) \), we say that
\( M \) is a semisimple module. A ring \( R \) is called a semisimple ring if \( RR \) is
semisimple. A module \( M \) is called a singular (resp., non-singular) module
if \( Z(M) = M \) (resp., \( Z(M) = 0 \)). A ring \( R \) is called right non-singular if
\( Z(RR) = 0 \).

For general background and terminology we refer to Anderson-Fuller
[1] and Faith [5].

A ring \( R \) is called a right (left) SI-ring if every singular right (left)
\( R \)-module is injective. SI-rings have been introduced and investigated by
Goodearl [7]. The following structure theorem is useful in our investigation.

**Lemma 1** ([7, 3.11]). A ring \( R \) is right SI if and only if \( R \) is right non-
singular and \( R = K \oplus R_1 \oplus \cdots \oplus R_n \), a ring direct sum, where \( K/Soc(KK) \) is
semisimple and each \( R_i \) is Morita-equivalent to a right SI-domain \( D_i \) which
is not a division ring.
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It is known that if $D$ is a right SI-domain, then $D$ is right noetherian, right hereditary and for each non-zero right ideal $C$ of $D$, $D/C$ is semisimple. By [10, Corollary 5], a ring $R$ is right SI if and only if every cyclic singular right $R$-module is injective. This shows that right SI-domains are precisely the same as right PCI-domains introduced by Faith [4].

We record some other well-known results which will be referred to throughout the paper.

**Lemma 2** ([10]). If every cyclic singular $R$-module is injective, then $R$ is a right SI-ring.

**Lemma 3** ([6], [11]). A ring $R$ is the direct sum of a semisimple ring and a right PCI-domain (equivalently, a right SI-domain) if and only if every cyclic $R$-module is projective or injective.

We notice that the statement "$R$ is a direct sum of a semisimple ring and a right SI-domain" means that $R$ is either a semisimple ring, or a right SI-domain, or the direct sum of two such rings.

**Lemma 4** ([2]). Let $M$ be a finitely generated quasi-injective module and $\{N_i\}$ be a system of infinitely many independent submodules of $M$ whose sum is $N$. Then $M/N$ has infinite uniform dimension.

3. **The results.** For convenience we say that a ring $R$ satisfies condition (E) if every cyclic $R$-module is non-singular or injective. Notice that homomorphic images of rings satisfying (E) also satisfy (E).

**Lemma 5.** Let $R$ be a semiprimary ring. If $R$ satisfies (E), then $R$ is semisimple.

**Proof.** Let $R$ be a semiprimary ring satisfying (E) and let $S = Soc(J(R))$. Then $S$ is an $R/J(R)$-module. Write $R_R = e_1R \oplus \cdots \oplus e_nR$
where each $e_i$ is a primitive idempotent of $R$. Now, if $J(R)$ is nonzero, then there is an $e_i$ with $e_iJ(R) \neq 0$. Since $e_iR$ is a local module with Jacobson radical $e_iJ(R)$, it is clear that $e_iR/e_iJ(R)$ is a singular simple $R$-module. Hence $R/J(R)$ can not be non-singular as an $R$-module. Thus, by (E), $R/J(R)$ must be an injective $R$-module, and so $J(R)$ contains an injective simple submodule, a contradiction. Hence $J(R) = 0$. □

Next we characterize rings satisfying condition (E).

**Theorem 6.** For a ring $R$ the following conditions are equivalent:

(i) Every cyclic $R$-module is non-singular or injective;

(ii) $R$ is either a ring direct sum of a semisimple ring and a right SI-domain or $R$ is a ring of type (*)

Consequently, a non-noetherian ring satisfies (E) if and only if it is a ring of type (*)

**Proof.** (i) \(\Rightarrow\) (ii). By (i), every cyclic singular $R$-module is injective. Hence byLemma 2, $R$ is right SI. Therefore by Lemma 1, $R = A \oplus B$, where $A/Soc(A_A)$ is semisimple and $B$ is a semiprime right noetherian ring with zero right and (left) socle.

**Case 1.** $B \neq 0$.

If $B_B$ is not uniform, there are finitely many uniform right ideals $U_1, ..., U_m$ of $B$ ($m \geq 2$) such that $U_1 \oplus \cdots \oplus U_m$ is essential in $B_B$. Let $V_i$ be a non-zero proper submodule of $U_i$. Then $B/V_i$ is not non-singular, and so injective by (E). Hence the injective hull of $U_1 \oplus \cdots \oplus U_{i-1} \oplus U_{i+1} \oplus \cdots \oplus U_m$ is cyclic for every $i = 1, 2, ..., m$. Therefore, the injective hull of the $B$-module $B$ is finitely generated. Since $B$ is right noetherian and semiprime, $B$ is semisimple by [5, Theorem 20.12], a contradiction to $Soc(B_B) = 0$. Hence $B_B$ is uniform, and so $B$ is a right SI-domain.

Since $Soc(A_A)$ is essential in $A$, the $R$-module $A/Soc(A_A)$ is singular. Hence if $A \neq Soc(A_A)$, then the factor module $R/Soc(A_A)$ cannot be non-
singular, and hence it is injective by (E). But in this case $B$ is a right
self-injective domain, consequently a division ring, a contradiction. Thus,
we must have $A = Soc(A_A)$, proving that $R$ is a direct sum of a semisimple
ring and a right SI-domain.

Case 2. $B = 0$.

In this case $R$ is a right SI-ring such that $R/Soc(R_R)$ is a semisimple
ring. If $Soc(R_R)$ is finitely generated, then $R$ is a right artinian ring, hence
semisimple by Lemma 5, and so we are done.

It remains to consider the case when $Soc(R_R)$ is infinitely generated.
Let $S = eR$ be a minimal right ideal of $R$ where $e$ is an idempotent, and
$K = (1 - e)R$. It follows that $K/Soc(K_R)$ is singular and nonzero. Hence,
by (E), $R/Soc(K_R)$ must be injective, and so $S_R$ is injective. Consequently,
every idempotent minimal right ideal of $R$ is injective.

Therefore, the sum $I$ of all idempotent minimal right ideals of $R$ is an
(two-sided) ideal of $R$, and so $R/I$ satisfies (E). Moreover, for any nilpotent
right ideal $N$ of $R$, $Soc(R_R).N = 0$. Hence $Soc(R_R)^2 \subseteq I$. This together
with the fact that $R/Soc(R_R)$ is a semisimple ring shows that $R/I$ is a
semiprimary ring. Hence, by Lemma 5, $R/I$ is semisimple. This shows that
$J(R) = 0$. Therefore, every simple $R$-module is injective, i.e. $R$ is a right
V-ring. In particular, every finitely generated right ideal of $R$ in $Soc(R_R)$
is a direct summand of $R_R$. This implies that $Soc(R_R)$ is a von Neumann
regular ideal of $R$. Since $R/Soc(R_R)$ is semisimple, we conclude that $R$ is a
von Neumann regular ring (cf. [8, Lemma 1.3]).

Now, let $R/Soc(R_R) = S_1 \oplus \cdots \oplus S_t$, where each $S_i$ is a minimal right
ideal of $R/Soc(R_R)$. Take an element $x \in R$ such that $x + Soc(R_R)$ generates
$S_1$. Then $Soc(xR)$ is infinitely generated and $xR/Soc(xR)$ is singular. On
the other hand, since $R$ is regular,

$$R_R = xR \oplus C$$
for some right ideal $C$ of $R$. It follows that $R / \text{Soc}(xR) \cong (xR / \text{Soc}(xR)) \oplus C$, and so $R / \text{Soc}(xR)$ is not non-singular, hence it is injective by (E). In particular, $C$ is an injective cyclic $R$-module. Clearly, $C / \text{Soc}(C)$ is semisimple and hence of finite composition length. By Lemma 4, $\text{Soc}(C)$ must be finitely generated. It yields $C = \text{Soc}(C)$, i.e. $C$ is contained in $\text{Soc}(R_R)$, and so we have

$$R / \text{Soc}(R_R) \cong xR / \text{Soc}(xR).$$

Thus $R / \text{Soc}(R_R)$ is a division ring, proving the fact that $R$ is a ring of type (*)

$(ii) \Rightarrow (i)$. If $R$ is a direct sum of a semisimple ring and a right SI-domain, then $R$ satisfies (E) by Lemma 3. Hence we need only consider the case when $R$ is of type (*), i.e. $R$ is a right V-ring with infinitely generated $\text{Soc}(R_R)$ such that $R / \text{Soc}(R_R)$ is a division ring. Obviously, $R$ is right non-singular and hence a right SI-ring by Lemma 1.

Let $N$ be a non-zero right ideal of $R$. If $N \not\subseteq \text{Soc}(R_R)$, then $N + \text{Soc}(R_R) = R$ and $(N + \text{Soc}(R_R)) / \text{Soc}(R_R)$ is simple. Hence there is an element $y \in N$ such that $yR + \text{Soc}(R_R) = R$. From this, $R / N$ is either zero or non-singular, as desired.

If $N \subseteq \text{Soc}(R_R)$, we consider the factor module $\overline{R} = R / N$. Since $R$ is a right SI-ring,

$$\overline{R} = \overline{U} \oplus \overline{V},$$

where $\overline{U}_R$ is non-singular and $\overline{V}_R$ is singular and injective. Put $\text{Soc}(R_R) = N \oplus M$, and denote by $\overline{M}$ the image of $M$ in $\overline{R}$. By assumption, $\overline{R} / \overline{M}$ is a simple module over $R$. It follows that

$$(\overline{V} \oplus \overline{M}) / \overline{M}$$

is either zero or simple. (It is clear that $\overline{V} \cap \overline{M} = 0$.) If it is zero, then $\overline{V} = 0$ and so $\overline{R} = \overline{U}$ is non-singular. If it is simple, then $\overline{V}$ is simple and so
\[ \overline{V} \oplus \overline{M} = \overline{R}. \] In particular \( \overline{M}_R \) is finitely generated, i.e. \( \overline{M}_R \) is a direct sum of finitely many (injective) simple \( R \)-modules. Thus \( (R/N)_R \) is injective. \( \square \)

A module \( M \) is called an extending (or CS-) module if every submodule of \( M \) is essential in a direct summand of \( M \) (or equivalently, if every complement submodule of \( M \) is a direct summand). Extending modules have been extensively studied by many authors. We refer to [3] for references on this subject.

Rings, for which direct sums of certain type of extending modules are extending, have been recently considered in [9]. It was shown there that, for a ring \( R \), every uniform \( R \)-module has composition length at most 2 if and only if every direct sum of uniform \( R \)-modules is extending. In general, such a ring need not be right semi-artinian.

A ring \( R \) is said to satisfy condition \((E')\) if every direct sum of uniform \( R \)-modules and an injective \( R \)-module is extending. Condition \((E')\) has been considered first in [9] where the following result was obtained:

**Lemma 7 ([9]).** A ring \( R \) satisfying \((E')\) is right semi-artinian, i.e. every non-zero \( R \)-module has a non-zero socle. If \( R \) is right non-singular, then \( R \) satisfies \((E')\) if and only if \( R \) is right SI and each non-singular uniform \( R \)-module has composition length at most 2.

Now we consider some semiprime rings satisfying condition \((E')\).

**Theorem 8.** For a non-noetherian semiprime ring \( R \) consider the following conditions:

(i) \( R \) satisfies \((E')\) and all singular simple \( R \)-modules are isomorphic;
(ii) \( R/\text{Soc}(R_R) \) is a simple artinian ring;
(iii) \( R \) is Morita-equivalent to a ring \( S \) such that \( S/\text{Soc}(S_S) \) is a division ring.

Then \((i) \Rightarrow (ii) \Leftrightarrow (iii)\).
Proof. (i) $\Rightarrow$ (ii). By Lemmas 1 and 7, $R/Soc(R_R)$ is a semisimple ring. In particular, $R/Soc(R_R)$ is a direct sum of finitely many singular simple $R$-modules, which are isomorphic to each other by (i). Hence $R/Soc(R_R)$ is a simple artinian ring, proving (ii).

(ii) $\Rightarrow$ (iii). Since $R$ is semi-prime, (ii) gives that $R$ is von Neumann regular. In particular, $R$ is right non-singular. Hence, by Lemma 1, $R$ is a right SI-ring. Using the same argument as that in the first part of the proof of Theorem 6 (the last part of Case 2) we can show that $R_R$ has the following direct decomposition:

\begin{equation}
R_R = R_1 \oplus \cdots \oplus R_n,
\end{equation}

where each $R_i/Soc(R_i)$ is simple and $R_i/Soc(R_i) \cong R_j/Soc(R_j)$ for each $i, j = 1, 2, \ldots, n$. Note that each $Soc(R_i)$ is infinitely generated, and since $R$ is right SI, $R$ is right hereditary (cf. [7]).

For each $i$, with $i = 2, 3, \ldots, n$, there exists a homomorphism $\varphi_i'$ of $R_1$ onto $R_i/Soc(R_i)$. Let $\varphi_i''$ be the canonical homomorphism of $R_i$ onto $R_i/Soc(R_i)$. By the projectivity of $R_1$, there is a homomorphism $\varphi_i$ from $R_1$ to $R_i$ such that $\varphi_i'' \cdot \varphi_i = \varphi_i'$. Hence $\varphi_i(R_1)$ is not semisimple, and so $R_i = \varphi_i(R_1) + Soc(R_i)$. Therefore there is a submodule $C_i$ of $R_i$ with $C_i \subset Soc(R_i)$ such that

\[ R_i = \varphi_i(R_1) \oplus C_i. \]

Hence $C_i$ is a direct sum of finitely many minimal submodules of $R_i$.

Moreover, since $\varphi_i(R_1)$ is projective, we have

\[ R_1 \cong \varphi_i(R_1) \oplus Ker(\varphi_i). \]

Hence each $R_i$ is isomorphic to a direct summand of $R_1 \oplus C_i$. Put

\[ A = R_1 \oplus C_2 \oplus \cdots \oplus C_n. \]
Then $A$ is a finitely generated right ideal of the regular ring $R$ and $A/Soc(A)$ is a simple $R$-module. Write $A = eR$ where $e$ is an idempotent. Moreover, if we denote by $M$ the external direct sum of $n$ copies of $eR$, then

$$M \cong R_R \oplus L$$

for some submodule $L$ of $M$. This shows that $eR$ is a projective generator of the category Mod-$R$, and hence $R$ is Morita-equivalent to $\text{End}_R(eR) \cong eRe$. It is clear that $eRe$ is a semiprime ring and $eRe/Soc(eRe)$ is a division ring. This proves $(iii)$.

$(iii) \Rightarrow (ii)$ is clear. □

The following result is a consequence of Theorems 6 and 8, which also shows how the conditions (E) and (E') are related to each other.

**Corollary 9.** For a non-noetherian ring $R$, the following conditions are equivalent:

(i) $R$ is a right V-ring satisfying (E') and all singular simple right $R$-modules are isomorphic;

(ii) $R$ is a right V-ring and $R/Soc(R_R)$ is simple artinian;

(iii) $R$ is Morita-equivalent to a ring satisfying (E), i.e. a ring of type (*).

Because we are concerned with the question of characterizing rings, Morita-equivalent to rings of type (*), we restricted ourself in Theorem 8 on those rings which are simple artinian modulo socle. In fact, Theorem 8 can be extended to a general form as discussed below.

Let $R$ be a non-noetherian semiprime ring and assume that $R/Soc(R_R)$ is semisimple. Hence $R$ is von Neumann regular. Let $R_1$ be a principal right ideal of $R$ such that $(R_1 + Soc(R_R))/Soc(R_R)$ is simple. Then $R_R = R_1 \oplus R_1'$ and $Soc(R_1)$ is infinitely generated. Therefore, by an easy induction proof
we obtain the following direct decomposition for $R_R$:

$$R_R = R_1 \oplus \cdots \oplus R_n,$$

where each $R_i/Soc(R_i)$ is simple and each $Soc(R_i)$ is infinitely generated. (Notice that in this case some factor modules $R_i/Soc(R_i)$ may not be isomorphic to each other.)

Moreover, by Lemma 1, $R$ is right SI and hence right hereditary. Therefore, we may use the same argument as that in the proof of Theorem 8, to find finitely many independent principal right ideals of $R$, say $A_1, \ldots, A_t$, with the following properties:

1. Each $A_j/Soc(A_j)$ is simple,
2. For $j \neq k$, $A_j/Soc(A_j) \not\cong A_k/Soc(A_k)$,
3. Each $R_i$ is isomorphic to a direct summand of exactly one $A_j$.

Let $A = A_1 \oplus \cdots \oplus A_t$ and as earlier let $M$ be the external direct sum of $n$ copies of $A_R$. Then we get $M \cong R_R \oplus C$ for some submodule $C$ of $M$. Hence $A$ is a generator of Mod-$R$. Since $A$ is a finitely generated right ideal of $R$, $A = fR$ for some idempotent $f \in R$. Therefore, $R$ is Morita-equivalent to the ring $End_R(fR) \cong fRf$. It is also easy to see that $fRf/Soc(fRf)$ is a direct sum of finitely many division rings.

If we assume moreover that all idempotents of $R$ are central, then (2) is a ring-direct decomposition of $R$. Furthermore, in this case, $R$ is a right V-ring. Then we conclude easily that each $R_i$ in (2) is a ring of type (*).

Thus we have proved the following result.

**Theorem 10.** For a non-noetherian semiprime ring $R$, consider the following conditions:

(i) $R$ satisfies $(E')$;

(ii) $R/Soc(R_R)$ is a semisimple ring;

(iii) $R$ is Morita-equivalent to a ring $T$ such that $T/Soc(T_T)$ is a direct sum of finitely many division rings.
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Then \((i) \Rightarrow (ii) \Leftrightarrow (iii)\). In addition, if all idempotents of \(R\) are central, then \((i)\) and \((ii)\) are equivalent to the condition:

\((iii')\) \(R\) is a ring-direct sum of finitely many rings satisfying \((E)\), i.e. rings of type \((^*)\).

In connection with condition \((E')\) we would like to mention a result obtained in [9]:

\textit{For a right non-singular ring \(R\), the following conditions are equivalent:}

\((a)\) Every direct sum of injective \(R\)-modules and uniform \(R\)-modules is extending;

\((b)\) Every direct sum of extending \(R\)-modules is extending;

\((c)\) \(R\) is right artinian, and every uniform \(R\)-module has length at most 2.

In a revised version, the authors of [9] have shown furthermore that, under this condition (that \(R\) is right non-singular), \((a)\) – \((c)\) are equivalent to:

\((d)\) Every \(R\)-module is extending.

It has also been shown in [9], that in general, a ring satisfying condition \((b)\) is semiprimary with Jacobson radical square zero.

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