q-HYPERCYCLIC RINGS

S. K. JAIN AND D. S. MALIK

0. Introduction. A ring $R$ is called $q$-hypercyclic (hypercyclic) if each cyclic ring $R$-module has a cyclic quasi-injective (injective) hull. A ring $R$ is called a $qc$-ring if each cyclic right $R$-module is quasi-injective. Hypercyclic rings have been studied by Caldwell [4], and by Ososky [12]. A characterization of $qc$-rings has been given by Koehler [10]. The object of this paper is to study $q$-hypercyclic rings. For a commutative ring $R$, $R$ can be shown to be $q$-hypercyclic ($= qc$-ring) if $R$ is hypercyclic. (Theorems 4.2 and 4.3). Whether a hypercyclic ring (not necessarily commutative) is $q$-hypercyclic is considered in Theorem 3.11 by showing that a local hypercyclic ring $R$ is $q$-hypercyclic if and only if the Jacobson radical of $R$ is nil. However, we do not know if there exists a local hypercyclic ring with nonnil radical [12]. Example 3.10 shows that a $q$-hypercyclic ring need not be hypercyclic. A characterization of local $q$-hypercyclic rings is given in Theorem 3.9 by showing that local $q$-hypercyclic rings are precisely $qc$-rings. The structure of a semi-perfect $q$-hypercyclic ring is given in Theorem 5.7 whence it follows as a consequence that if $R$ is a semi-perfect $q$-hypercyclic ring then each cyclic right $R$-module is a finite direct sum of indecomposable quasi-injective modules. Finally, a characterization of right or left perfect $q$-hypercyclic (hypercyclic) rings is given in Section 6. Our results depend upon a number of lemmas. Lemma 5.1 regarding the quasi-injective hull of $A \oplus B$, where $B$ contains a copy of the injective hull $E(A)$ of $A$, though straightforward, is also perhaps of interest by itself, besides being a key lemma in the proof of our Theorem 5.5. We also make use of Koehler's characterization of $qc$-rings as those which are direct sum of rings each of which is semisimple artinian, or a rank $0$ duo maximal valuation ring.

1. Notation and definitions. All rings considered have unity and unless otherwise stated all modules are unital right modules. If $M$ is a module, then $E(M)$ (q.i.h. $(M)$ ) will denote the injective hull (quasi-injective hull) of $M$. An idempotent $e$ of a ring $R$ is called primitive if the module $eR$ is indecomposable. $J$ will denote the Jacobson radical of the ring $R$. $S(R)$ ($S(R_R)$) will denote the right (left) socle of $R$. Let $X \subseteq R$, then $r_R(X)$ ($l_R(X)$) will denote the right (left) annihilator of $X$ in $R$.

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$N \subset M$ will denote that $N$ is a large submodule of $M$.

$R$ is called right (left) duo if every right (left) ideal of $R$ is a twosided ideal of $R$. $R$ is a right (left) valuation ring if right (left) ideals of $R$ are linearly ordered. $R$ is called a right (left) bounded ring if every non-zero right (left) ideal of $R$ contains a non-zero twosided ideal of $R$. $R$ is called a duo (valuation, bounded) ring if $R$ is both right and left duo (valuation, bounded).

A module $M$ is called local if $M$ has a unique submodule. A ring $R$ is called semi-perfect if $R/J$ is artinian and idempotents modulo $J$ can be lifted, or equivalently every finitely generated module has a projective cover. $R$ is called right (left) perfect if every right (left) $R$-module has a projective cover, or equivalently, $R/J$ is artinian and every non-zero right (left) $R$-module has a maximal submodule. $R$ is called uniserial if $R$ is an artinian principal ideal ring. An $R$-module $M$ is said to have finite Azumaya diagram (A.D) [5] if

$$M = \oplus \sum_{i=1}^{k} M_i,$$

where each $R$-submodule $M_i$ has a local endomorphism ring.

2. Preliminary results.

**Lemma 2.1.** Let $M$ be quasi-injective. If $E(M) = \oplus \sum_{i=1}^{n} K_i$ is a direct sum of submodules $K_i$, then

$$M = \oplus \sum_{i=1}^{n} (M \cap K_i).$$

**Proof.** See ([7], Theorem 1.1).

The following is a well known equivalence between mod-$R$, the category of right $R$-modules and mod-$R_n$, the category of right $R_n$-modules, where $R_n$ is the $n \times n$ matrix ring over $R$.

**Lemma 2.2.** Let

$$F = \sum_{i=1}^{n} x_i R$$

be a free $R$-module with free basis $\{x_i | 1 \leq i \leq n\}$. Then $M_R \rightarrow \text{Hom}_R(F, M)$ is a category isomorphism between mod-$R$ and mod-$R_n$ with inverse

$$N_{R_n} \rightarrow N \otimes_{R_n} F.$$
**Lemma 2.3.** Let \( R/J \) be artinian, \( I \) a right ideal of \( R \),

\[
R/I = \bigoplus_{i=1}^{k} M_i.
\]

Then \( k \) is composition length of \( R/J \).

**Proof.** See ([12], Lemma 1.8).

**Lemma 2.4.** Let \( I \) be a two-sided ideal of \( R \) and let \( E \) be an injective \( R \)-module. Then

\[
0:_E I = \{ x \in E | xI = 0 \}
\]

is injective as an \( R/I \)-module.

**Proof.** See ([13], Proposition 2.27).

**Lemma 2.5.** Let \( R \) be semiperfect and \( q \)-hypercyclic. Then \( R_R \) is self-injective.

**Proof.** Let \( I \) be a right ideal of \( R \) such that \( R/I \) is the quasi-injective hull of \( R \). Let \( \phi: R \to R/I \) be the embedding. Since \( R/I \) contains a copy of \( R \), \( R/I \) is injective. Let \( \phi(R) = B/I \). Then \( B/I \subset R/I \). Hence \( B \subset R \). Since \( R \cong B/I \), \( B/I \) is projective. Thus \( B = I \oplus K \) for some \( K_R \subset B_R \). Now

\[
R \cong \frac{B}{I} = \frac{I \oplus K}{I} \cong K.
\]

Therefore \( E(R) \cong E(K) \). But then \( I \oplus K \subset R \) implies

\[
E(R) = E(I) \oplus E(K) \cong E(I) \oplus E(R).
\]

Since \( E(R) \cong R/I \), \( E(R) \) is a finite direct sum of indecomposable modules, by Lemma 2.3. Thus \( E(R) \) has finite Azumaya-Diagram [5]. Therefore, \( E(R) \cong E(R) \oplus E(I) \) implies \( E(I) = 0 \). Hence \( I = 0 \). Thus \( R \) is self-injective.

**Lemma 2.6.** Let \( R \) be \( q \)-hypercyclic. Then every homomorphic image of \( R \) is also \( q \)-hypercyclic.

**Proof.** Let \( A \) be a twosided ideal of \( R \). Let \( \overline{R} = R/A \). Let \( \overline{R}/I \) be a cyclic \( \overline{R} \)-module, where \( \overline{I} = I/A \). But \( \overline{R}/I \cong R/I \). Since \( A \subset I \), \( A \) annihilates \( R/I \). Let \( R/K \) be the quasi-injective hull of \( R/I \) as an \( R \)-module. Then

\[
\frac{R}{K} \cong \operatorname{End}_R \left( \frac{E(R)}{I} \right) \frac{R}{I}.
\]

Then it follows that \( A \) annihilates \( R/K \). Thus \( R/K \) may be regarded as an \( \overline{R} \)-module. Since \( R/K \) is quasi-injective as an \( R \)-module, \( R/K \) is quasi-injective as an \( \overline{R} \)-module. Since \( A \) is a twosided ideal and annihilates
R/K, A ⊆ K. Hence

\[ \frac{R}{K} \cong \frac{R}{A} \frac{K}{A}. \]

Clearly \( \bar{R}/\bar{K} \) is the quasi-injective hull of \( \bar{R}/\bar{T} \) as an \( \bar{R} \)-module. Hence \( \bar{R} \) is q-hypercyclic.

**Lemma 2.7.** Let \( R \) be a finite direct sum of rings, \( \{ R_i \mid 1 \leq i \leq n \} \). Then \( R \) is q-hypercyclic if and only if each \( R_i \) is q-hypercyclic for all \( i, 1 \leq i \leq n \).

**Proof.** This is straightforward.

3. **Local q-hypercyclic rings.** In this section we study local q-hypercyclic rings and show that over such rings every cyclic module is quasi-injective. Throughout this section unless otherwise stated \( R \) will denote a local q-hypercyclic ring.

**Lemma 3.1.** If \( I \) is a right ideal of \( R \), then \( E(R/I) \) is indecomposable.

**Proof.** Let q.i.h. \( (R/I) = R/A \). Since \( R/A \) is indecomposable, \( E(R/I) \) is indecomposable.

**Lemma 3.2.** Right ideals of \( R \) are linearly ordered.

**Proof.** Let \( A \) and \( B \) be right ideals of \( R \). Suppose

\[ \frac{A}{A \cap B} \neq 0, \quad \frac{B}{A \cap B} \neq 0. \]

Then

\[ \frac{A}{A \cap B} \oplus \frac{B}{A \cap B} \subseteq \frac{R}{A \cap B}. \]

Hence

\[ E\left( \frac{R}{A \cap B} \right) = E\left( \frac{A}{A \cap B} \right) \oplus E\left( \frac{B}{A \cap B} \right) \oplus K. \]

By Lemma 3.1, \( E\left( \frac{R}{A \cap B} \right) \) is indecomposable. Hence either

\[ \frac{A}{A \cap B} = 0 \quad \text{or} \quad \frac{B}{A \cap B} = 0. \]

Thus either \( A \subseteq B \) or \( B \subseteq A \).

**Lemma 3.3.** Left ideals of \( R \) are linearly ordered.

**Proof.** This follows by ([8], Theorem 1) and Lemma 3.2.
**Lemma 3.4.** Let $I$ be a non-zero right ideal of $R$. If q.i.h. $(R/I) \cong R$, then $R/I$ is injective.

**Proof.** Let $\phi: R/I \to R$ be the embedding. Let $\phi(1 + I) = x$. Then $R/I \cong xR$. Let $A = xR$. Then $R$ is quasi-injective hull of $A$. Thus

$$R = \text{End}_R(R)A = RA = RxR.$$ 

Therefore, $x \notin J$, and hence $x$ is a unit. Thus $A = R$. Hence $R/I$ is injective.

**Lemma 3.5.** Let $I$ be a non-zero right ideal of $R$ such that $R/I$ is quasi-injective. Suppose $S(RR) = 0$. Then $I$ contains a non-zero twosided ideal of $R$.

**Proof.** Since $R$ is local, $r_R(J) = S(RR) = 0$. We may assume that $I \neq J$. Let $x \in J$ and $x \notin I$. Then $I \subseteq xR$. By linear ordering on right ideals either $x^{-1}I \subseteq I$ or $I \subseteq x^{-1}I$. Suppose $x^{-1}I \subseteq I$. Define

$$\phi: \frac{xR}{I} \to \frac{R}{I}$$ 

by $\phi(xa + I) = a + I$. Since $x^{-1}I \subseteq I$, $\phi$ is well defined. Then $\phi$ can be extended to $f: R/I \to R/I$. Let $f(1 + I) = t + I$. Then

$$1 + I = \phi(x + I) = f(x + I) = tx + I.$$ 

Therefore $1 - tx \in I$. Since $tx \in J$, $1 - tx$ is a unit. Thus $I = R$. Hence $I \subseteq x^{-1}I$. Let

$$y = xa \in xI, a \in I.$$ 

Since $a \in I \subseteq x^{-1}I$, $xa \in I$. Thus $xI \subseteq I$. Hence for all $x \in J$, $x \notin I$, $xI \subseteq I$. Thus $JI \subseteq I$. If $JI = 0$ then

$$I \subseteq r_R(J) = 0.$$ 

Since $I$ is non-zero, $JI \neq 0$. Therefore $JI$ is a non-zero twosided ideal of $R$ contained in $I$.

**Lemma 3.6.** $R$ is left bounded or $R$ is right bounded.

**Proof.** Case 1. If $\text{Soc}(R) \neq 0$, then by linear ordering on left ideals, $\text{Soc}(RR)$ is a non-zero twosided ideal contained in each left ideal and hence $R$ is left bounded.

Case 2. $\text{Soc}(R) = 0$.

Let $I$ be a nonzero right ideal of $R$. If $R$ is the quasi-injective hull of $R/I$, then $R/I$ is quasi-injective by Lemma 3.4. Hence $I$ contains a non-zero twosided ideal (Lemma 3.5).

Let $R/A$ be the quasi-injective hull of $R/I$, for some non-zero right ideal $A$ of $R$. Then by Lemma 3.5 $A$ contains a non-zero twosided ideal,
say $B$. Let $\phi: R/I \to R/A$ be the embedding and let $\phi(1 + I) = x + A$. Let $a \in B$. Then

$$\phi(a + I) = xa + A = A.$$ 

Therefore $a \in I$. Thus $B \subseteq I$. Therefore $I$ contains a non-zero ideal $B$. Hence $R$ is right bounded.

**Lemma 3.7.** $J$ is nil.

**Proof.** Let $a \in J$. Suppose $a^n \neq 0$ for any positive integer $n$. Let

$$S = \{a^n | n > 0\}.$$ 

By Zorn's lemma there exists an ideal $P$ of $R$ maximal with respect to the property that $P \cap S = \phi$. Then $P$ is prime. Hence $R/P$ is a prime local $\theta$-hypercyclic ring. Thus $R/P$ is either left bounded or right bounded. Then it follows that $R/P$ is a domain. Since $R/P$ is also local and $\theta$-hypercyclic ring, $R/P$ is self-injective and hence a division ring. Therefore $P$ is a maximal ideal of $R$. Thus $P = J$, a contradiction. Hence $J$ is nil.

**Lemma 3.8.** $R$ is duo.

**Proof.** It suffices to show that for $0 \neq y \in R, yR = Ry$. Let $0 \neq y \in R$. Suppose $yr \notin Ry$. By linear ordering on left ideals $Ry \subseteq Ry$. Therefore

$$y = xyr \quad \text{for some } x \in R.$$ 

If $x \in J$ then $x^n = 0$ for some $n$. Then

$$y = xyr = x^2 yr^2 = \ldots = x^nyr^n = 0,$$

which is a contradiction. Thus $x$ is a unit. Hence

$$xyr = y \Rightarrow yr = x^{-1}y \in Ry,$$

which is again a contradiction. Thus $yR \subseteq Ry$. By symmetry $Ry \subseteq yR$. Hence $Ry = yR$.

We now prove the main result of this section.

**Theorem 3.9.** Let $R$ be a local ring. Then $R$ is $\theta$-hypercyclic if and only if $R$ is a $qc$-ring.

**Proof.** Let $R$ be $\theta$-hypercyclic and let $A$ be a non-zero right ideal of $R$. Then by Lemma 3.8, $A$ is a twosided ideal of $R$. But then by Lemma 2.6, $R/A$ is a self-injective ring. Thus $R/A$ is a quasi-injective $R$-module, proving that $R$ is a $qc$-ring. The converse is obvious.

The following example shows that a $\theta$-hypercyclic ring need not be hypercyclic.
Example 3.10. Let $F$ be a field, $x$ an indeterminant over $F$. Let
\[ W = \{ \{ \alpha_i \} \mid \{ \alpha_i \} \text{ is a well ordered sequence of nonnegative real numbers} \}. \]
Let
\[ T = \left\{ \sum_{i=0}^{\infty} a_i x^{\alpha_i} \mid a_i \in F, \{ \alpha_i \} \in W \right\}. \]
Then $T$ is a local, commutative ring and
\[ J(T) = \left\{ \sum_{i=0}^{\infty} a_i x^{\alpha_i} \in T \mid a_0 > 0 \right\}. \]
Let
\[ R = \frac{T}{xJ(T)}. \]
Then as shown in [4], $R$ is a commutative local hypercyclic ring. Then $R$ is $q$-hypercyclic (Theorem 4.3). But $R/S$, where $S$ is the socle of $R$, is not hypercyclic. Since $R/S$ is a homomorphic ring of $R$, $R/S$ is $q$-hypercyclic by Lemma 2.6. Note that $R/S$ is a commutative local ring with zero socle.

A ring has rank 0 if every prime ideal is a maximal ideal. A valuation ring is called maximal if every family of pairwise solvable congruences of the form $x \equiv x_a(K_a)$ (each $x_a \in R$, each $K_a$ is an ideal of $R$) has a simultaneous solution [9].

We now give a necessary and sufficient condition for a local hypercyclic ring to be $q$-hypercyclic. In the next section we will show that a commutative hypercyclic ring is always $q$-hypercyclic.

**Theorem 3.11.** Let $R$ be local and hypercyclic. Then the following conditions are equivalent.

(i) $J$ is nil.

(ii) $R$ is $q$-hypercyclic.

*Proof.* (i) $\Rightarrow$ (ii). By [12], $R$ is duo, valuation, and self-injective. But then $R$ is maximal. Thus $R$ is a $qc$-ring [10], and hence $q$-hypercyclic. (ii) $\Rightarrow$ (i) follows from Lemma 3.7.

**Remark.** 3.12. It is not known whether there exists a semi-perfect (or equivalently local) hypercyclic ring with a non-nil radical ([12], p. 339).

4. **Commutative $q$-hypercyclic rings.** We begin with

**Lemma 4.1.** Let $R$ be commutative and $q$-hypercyclic. Then $R$ is self-injective.
Proof. This is obvious.

Theorem 4.2. Let $R$ be a commutative ring. Then the following are equivalent.

(i) $R$ is $q$-hypercyclic.
(ii) $R$ is a $qc$-ring.

Proof. This is similar to the proof of the Theorem 3.9.

Theorem 4.3. Let $R$ be a commutative hypercyclic ring. Then $R$ is $q$-hypercyclic.

Proof. Let $R$ be hypercyclic. Then by ([4], Theorem 2.5), $R$ is a finite direct sum of commutative local hypercyclic rings. So it suffices to show that a commutative local hypercyclic ring is $q$-hypercyclic. Let $R$ be commutative local and hypercyclic. Then by [4], $R$ is valuation and self-injective, and $J$ is nil. Then by ([9], Theorem 2.3), $R$ is maximal. Since $J$ is nil, $R$ has rank 0. Then $R$ is rank 0 maximal valuation ring. Thus $R$ is a $qc$-ring [10], proving the theorem.

5. Semi-perfect $q$-hypercyclic rings.

Lemma 5.1. Let $A$ and $B$ be right $R$-modules. Let $B$ be injective containing a copy of $E(A)$. Then

\[ q.i.h. \ (A \oplus B) = E(A) \oplus B. \]

Proof.

\[
q.i.h. \ (A \oplus B) = \text{End}_R(E(A) \oplus B)(A \oplus B) \\
= \left( \begin{array}{cc}
\text{Hom}_R(E(A), E(A)) & \text{Hom}_R(B, E(A)) \\
\text{Hom}_R(E(A), B) & \text{Hom}_R(B, B)
\end{array} \right) \left( \begin{array}{c}
A \\
B
\end{array} \right) \\
= \left( \begin{array}{c}
\text{Hom}_R(E(A), E(A))A + \text{Hom}_R(B, E(A))B \\
\text{Hom}_R(E(A), B)A + \text{Hom}_R(B, B)B
\end{array} \right) \\
= \left( \begin{array}{c}
E(A) \\
B
\end{array} \right) = E(A) \oplus B.
\]

The above lemma gives another proof of an interesting result of Koehler.

Corollary 5.2. ([11]). If the direct sum of any two quasi-injective modules is quasi-injective, then every quasi-injective module is injective.

Proof. Let $M$ be a quasi-injective right $R$-module. By Lemma 5.1

\[ q.i.h. \ (M \oplus E(M)) = E(M) \oplus E(M). \]

Then $M \oplus E(M) = E(M) \oplus E(M)$. Therefore $M \cong E(M)$, proving $M$ is injective.
The proof of the next lemma is exactly similar to Osofsky's ([12], Corollary 1.9).

**Lemma 5.3.** Let \( R \) be semiperfect and \( q \)-hypercyclic and let \( e \) be an idempotent in \( R \). Assume length of \( eR/eI = m \). Then any independent family of submodules of a quotient of \( eR \) has at most \( m \) elements.

**Proof.** Let \( \{ M_i | 1 \leq i \leq k \} \) be an independent family of submodules of \( eR/eI \). Then

\[
\frac{R}{eI} \cong (1 - e)R \oplus \left( \bigoplus_{i=1}^{k} M_i \right).
\]

Therefore \( E(R/eI) \) is a direct sum of length \( R/J - m + s \) terms, where \( s \geq k \). Thus q.i.h. \( (R/eI) \) is a direct sum of length \( R/J - m + s \) terms, by Lemma 2.1. Then Lemma 2.3 gives \( s \leq m \). Hence \( k \leq m \).

**Corollary 5.4.** Let \( R \) be semi-perfect and \( q \)-hypercyclic, \( e^2 = e \in R \), \( eR/eJ \) is simple. Then submodules of \( eR \) are linearly ordered.

**Proof.** This follows from Lemma 5.3.

**Theorem 5.5.** Let \( R \) be a semi-perfect \( q \)-hypercyclic ring. Then \( R \) is a finite direct sum of \( q \)-hypercyclic matrix rings over local rings.

**Proof.** \( R = e_1 R \oplus \ldots \oplus e_n R \), where \( e_i, 1 \leq i \leq n \) are primitive idempotents.

We will show that for \( i \neq j \), either \( e_i R \cong e_j R \), or

\[
\text{Hom}_R(e_i R, e_j R) = 0.
\]

Suppose for some \( i \neq j \), \( \text{Hom}_R(e_i R, e_j R) \neq 0 \). By renumbering, if necessary, we may assume that \( i = 1, j = 2 \). Let \( \alpha : e_1 R \rightarrow e_2 R \) be a non-zero \( R \)-homomorphism. Then \( e_1 R/\text{Ker} \alpha \) embeds in \( e_2 R \). Since \( e_2 R \) is indecomposable,

\[
E(e_1 R/\text{Ker} \alpha) \cong e_2 R.
\]

Hence \( B = e_2 R \oplus \ldots \oplus e_n R \) contains a copy of \( E(e_1 R/\text{Ker} \alpha) \). Now

\[
R/\text{Ker} \alpha \cong (e_1 R)/\text{Ker} \alpha \times e_2 R \times \ldots \times e_n R.
\]

Let \( A = (e_1 R)/\text{Ker} \alpha \). Then \( B \) is injective and contains a copy of \( E(A) \). Hence

\[
\text{Hom}_R(B, E(A))B = E(A).
\]

Since \( R \) is \( q \)-hypercyclic, for some right ideal \( I \),

\[
R/I \cong \text{q.i.h.} (R/\text{Ker} \alpha) \cong \text{q.i.h.} (A \times B) \cong E(A) \times B.
\]

Thus \( R/I \cong e_2 R \times B \). Then \( R/I \) is projective. Hence \( R = I \oplus K \) for some
right ideal $K$. Then
\[ K \cong R/I \cong e_2R \times e_2R \times \ldots \times e_nR. \]
Thus
\[ R = I \oplus K \Rightarrow e_1R \times e_2R \times \ldots \times e_nR \]
\[ \cong I \times e_2R \times e_2R \times \ldots \times e_nR. \]
Hence by Azumaya Diagram [5],
\[ e_1R \cong I \times e_2R. \]
Since $e_1R$ is indecomposable, $I = 0$. Consequently, $R = K$. Then
\[ e_1R \times e_2R \times \ldots \times e_nR \cong e_2R \times e_2R \times \ldots \times e_nR. \]
Again by Azumaya Diagram, $e_1R \cong e_2R$. Thus for $i \neq j$, either
\[ e_iR \cong e_jR \quad \text{or} \quad \text{Hom}_R(e_iR, e_jR) = 0. \]
Set $[e_kR] = \sum e_iR$, $e_iR \cong e_kR$. Renumbering if necessary, we may write
\[ R = [e_1R] \oplus \ldots \oplus [e_tR], \quad t \leq n. \]
Then for all $1 \leq k \leq t$, $[e_kR]$ is an ideal. Since for any $k$, $1 \leq k \leq n$, $e_kR$ is indecomposable,
\[ \text{End}_R(e_kR) \cong e_kRe_k \]
is a local ring.
Thus $[e_kR] = \oplus \sum e_iR$ is the $n_k \times n_k$ matrix ring over the local ring $e_kRe_k$ where $n_k$ is the number of $e_iR$ appearing in $\oplus \sum e_iR$. That the matrix ring is $q$-hypercyclic follows from Lemma 2.7.

We now proceed to study $q$-hypercyclic rings which are matrix rings over local rings.

**Theorem 5.6.** Let $S = T_n$ be the $n \times n$ $q$-hypercyclic matrix ring over a local ring $T$. Then $T$ is $q$-hypercyclic.

**Proof.** Let $e$ be a primitive idempotent of $S$ and let $eS/eI$ be a quotient of $eS$. Since $S$ is $q$-hypercyclic,
\[ \text{q.i.h.} \left( \frac{eS}{eI} \right) \cong \frac{S}{A} \]
for some right ideal $A$ of $S$. But since submodules of $eS$ are linearly ordered, $S/A$ is indecomposable. Thus $S/A \cong fS/fK$, ([2], Lemma 27.3), for some primitive idempotent $f$ of $S$, which may be chosen to be $e$ by
itself. Thus $S/A \cong eS/eB$ for some right ideal $B$ of $S$. Since category isomorphism (Lemma 2.2) takes $T$ to $eS$, every quotient of $T$ has quasi-injective hull a quotient of $T$, proving that $T$ is a $q$-hypercyclic ring.

**Theorem 5.7.** Let $R$ be a semi-perfect and $q$-hypercyclic ring. Then $R$ is a finite direct sum of matrix rings over local $qc$-rings.

**Proof.** Combine Theorems 5.5, 5.6 and 3.9.

We are unable to show if, in general, the $n \times n$ matrix ring $S$ over a local $q$-hypercyclic ring is again $q$-hypercyclic. However we will show in the next section that the result is true if $S$ is a perfect ring. In the following theorem we prove that each cyclic $S$-module is a finite direct sum of indecomposable quasi-injective modules and generalise this to the case when $S$ is any semi-perfect $q$-hypercyclic ring in Theorem 5.9.

**Theorem 5.8.** Let $S = T_n$ be the $n \times n$ matrix ring over a local $q$-hypercyclic ring. Then every cyclic $S$-module is a direct sum of indecomposable quasi-injective $S$-modules.

**Proof.** Let $I$ be a right ideal of $S$. Let $e \in S$ be a primitive idempotent of $S$. Since the category isomorphism (Lemma 2.2) takes $T$ to $eS$ every quotient of $eS$ is quasi-injective. Let

$$\frac{S}{I} = \bigoplus_{i=1}^{k} M_i,$$

where the $M_i$ are indecomposable $S$-modules. Since $S$ is semi-perfect and $M_i$ indecomposable,

$$M_i = (e_i S)/(e_i A),$$

where $e_i$ is a primitive idempotent of $S$. Thus $S/I$ is a direct sum of indecomposable quasi-injective $S$-modules.

**Theorem 5.9.** Let $R$ be a semi-perfect and $q$-hypercyclic ring. Then every cyclic $R$ module is a direct sum of indecomposable quasi-injective $R$-modules.

**Proof.** Combine Theorems 5.7 and 5.8.

6. Perfect $q$-hypercyclic rings. A ring $R$ is called right (left) perfect if every right (left) $R$-module has a projective cover. A theorem of Bass [3] states that the following conditions on a ring $R$ are equivalent.

(i) $R$ is right perfect.

(ii) $R$ satisfies minimum conditions on principal left ideals.

(iii) $R/J$ is artinian and every right $R$-module has a maximal submodule.
LEMMA 6.1. Let \( R \) be a local right perfect and \( q \)-hypercyclic ring. Then \( R \) is hypercyclic.

Proof. By Theorem 3.9, \( R \) is a \( qc \)-ring and hence \( R \) is duo. Let \( I \) be a nonzero right ideal of \( R \). Then \( R/I \) is quasi-injective and indecomposable, and hence \( E = E(R/I) \) is indecomposable.

First we show that the submodules of \( E \) are linearly ordered. Let \( aR \) and \( bR \) be submodules of \( E \). Let \( A = r_R(a) \). Since ideals of \( R \) are linearly ordered either

\[
r_R(a) \subseteq r_R(b) \text{ or } r_R(b) \subseteq r_R(a).
\]

To be specific let \( A = r_R(a) \subseteq r_R(b) \). Let \( E' = 0 : E^A \). Then \( aR, bR \subseteq E' \). By Lemma 2.4, \( E' \) is injective as an \( R/A \) module. Hence \( E' \) is quasi-injective as an \( R \)-module. Since \( E' \subseteq E \) and \( E \) is injective and indecomposable, \( E' \) is indecomposable as an \( R \)-module and hence \( E' \) is indecomposable as an \( R/A \)-module. Let \( \bar{R} = R/A \). Then \( E' \cong E_R(\bar{R}) \), the injective hull of \( \bar{R} \) as an \( \bar{R} \)-module. Hence \( E' \cong R/A \). Since submodules of \( R/A \) are linearly ordered, submodules of \( E' \) are linearly ordered. Thus \( aR \subseteq bR \) or \( bR \subseteq aR \). Hence submodules of \( E \) are linearly ordered. But then \( E \) must be local, since \( R \) is right perfect. Hence \( E \) is cyclic. Therefore \( R \) is hypercyclic, proving the theorem.

THEOREM 6.2. Let \( S = T_n \) be the \( n \times n \) matrix ring over a local ring \( T \). Let \( S \) be right perfect. Then \( S \) is \( q \)-hypercyclic if and only if \( T \) is \( q \)-hypercyclic.

Proof. Let \( T \) be \( q \)-hypercyclic. Then \( T \) is right perfect local and \( q \)-hypercyclic. Thus by Theorem 6.1, \( T \) is hypercyclic. Further, by Theorem 3.9, \( T \) is a \( qc \)-ring. Since \( T \) is hypercyclic, by ([12], Theorem 1.17), \( S \) is hypercyclic. Let \( e \in S \) be a primitive idempotent. Then as before, the category isomorphism (Lemma 2.2) takes \( T \) to \( eS \). Hence quotients of \( eS \) are quasi-injective and each quotient has injective hull a quotient of \( eS \). Let \( I \) be a right ideal of \( S \). By Theorem 5.8,

\[
\frac{S}{I} = \bigoplus_{i=1}^{k} M_i,
\]

where for all \( 1 \leq i \leq k \), \( M_i \) are indecomposable and quasi-injective. Then

\[
M_i \cong (e_iS)/(e_iA)
\]

for some primitive idempotent \( e_i \in S \). Hence

\[
E(M_i) \cong E[(e_iS)/(e_iA)] \cong (e_iS)/(e_iB) \text{ for all } 1 \leq i \leq k.
\]

Since \( S \) is right perfect and hypercyclic, submodules of \( (e_iS)/(e_iB) \) are linearly ordered. Hence for all \( 1 \leq i \leq k \), submodules of \( E(M_i) \) are linearly ordered. Let
\[ H = \text{q.i.h.} (S/I). \]

Then
\[ H = \bigoplus_{i=1}^{k} (H \cap E(M_i)). \]

Let \( K_i = H \cap E(M_i) \). Then submodules of \( K_i \) are linearly ordered for all \( 1 \leq i \leq k \). But then since \( S \) is right perfect, for all \( 1 \leq i \leq k \), \( K_i \) is cyclic. Therefore,
\[ K_i = (f_i S)/(f_i D) \]
where \( f_i \in S \) is a primitive idempotent, \( 1 \leq i \leq k \). Thus
\[ H \cong \bigoplus_{i=1}^{k} (f_i S)/(f_i D). \]

Then \( H \) is isomorphic to a quotient of \( S \), proving that \( S \) is \( q \)-hypercyclic.

The converse follows from Theorem 5.6.

**Theorem 6.3.** Let \( R \) be right perfect. Then \( R \) is \( q \)-hypercyclic if and only if \( R \) is a finite direct sum of matrix rings over local \( qc \)-rings.

**Proof.** Combine Theorems 5.5 and 6.2.

**Theorem 6.4.** Let \( R \) be right perfect and local. Then the following are equivalent.

(i) \( R \) is hypercyclic.

(ii) \( R \) is \( q \)-hypercyclic.

**Proof.** (i) \( \Rightarrow \) (ii). Then \( R \) is valuation. Let \( I \) be a non-zero right ideal of \( R \). Then \( E(R/I) \cong R/A \) for some right ideal \( A \) of \( R \). Let
\[ X = \text{q.i.h.} (R/I). \]

Since the submodules of \( R/A \) and hence those of \( X \) are linearly ordered, and \( R \) is right perfect, \( X \) is local. Thus \( X \) is a cyclic module, proving that \( R \) is a \( q \)-hypercyclic ring.

(ii) \( \Rightarrow \) (i) is Theorem 6.1.

**Theorem 6.5.** Let \( R \) be right perfect. Then the following are equivalent.

(i) \( R \) is hypercyclic.

(ii) \( R \) is \( q \)-hypercyclic.

**Proof.** (i) \( \Rightarrow \) (ii). Let \( R \) be hypercyclic. Then by ([12], Theorem 1.18),
\[ R = \bigoplus_{i=1}^{t} M_{n_i}(T_i), \]
where $M_{n_i}(T_i)$ is the $n_i \times n_i$ matrix ring over a local hypercyclic ring $T_i$. Since $R$ is right perfect, $T_i$ is right perfect. Thus $T_i$ is local right perfect and hypercyclic, and hence $q$-hypercyclic. Then by Theorem 6.2, $M_{n_i}(T_i)$ is $q$-hypercyclic, proving that $R$ is $q$-hypercyclic.

(ii) $\Rightarrow$ (i). Proceed as in (i) $\Rightarrow$ (ii) and use Theorem 6.4.

**Lemma 6.6.** Let $R$ be $q$-hypercyclic. Then $R$ is left perfect if and only if $R$ is right perfect.

**Proof.** If $R$ is right (or left) perfect ring then by Theorem 5.7,

$$ R = \bigoplus_{i=1}^{k} M_{n_i}(T_i), $$

where $M_{n_i}(T_i)$ are $n_i \times n_i$ matrix rings over local right (or left) perfect $qc$-rings $T_i$. Since $T_i$'s are duo, $R$ is left perfect if and only if $R$ is right perfect.

**Theorem 6.7.** The following conditions on a ring $R$ are equivalent:

(i) $R$ is right perfect and hypercyclic.

(ii) $R$ is left perfect and hypercyclic.

(iii) $R$ is uniserial.

(iv) $R$ is right perfect and $q$-hypercyclic.

(v) $R$ is left perfect and $q$-hypercyclic.

**Proof.** (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (i) is a theorem of Caldwell ([4], Theorem 1.5).

(i) $\Rightarrow$ (ii). By Theorem 6.4, $R$ is $q$-hypercyclic. Then by Lemma 6.6, $R$ is left perfect.

(i) $\Leftrightarrow$ (iv) is Theorem 6.5.

(iv) $\Leftrightarrow$ (v) is Lemma 6.6.

**References**


_Ohio University,
Athens, Ohio_