RINGS IN WHICH EVERY RIGHT IDEAL IS QUASI-INJECTIVE

S. K. JAIN, S. H. MOHAMED AND SURJEET SINGH

It is well known that if every right ideal of a ring $R$ is injective, then $R$ is semi simple Artinian. The object of this paper is to initiate the study of a class of rings in which each right ideal is quasi-injective. Such rings will be called $q$-rings. It is shown by an example that a $q$-ring need not be even semi prime. A number of important properties of $q$-rings are obtained.

Throughout this paper, unless otherwise stated, we assume that every ring has unity $1 \neq 0$. If $M$ is a right $R$-module, then $\hat{M}$ will denote the injective hull of $M$. For any positive integer $n$, $R_n$ will denote the ring of all $n \times n$ matrices over the ring $R$. $R^n, J(R)$ and $B(R)$ will denote the right singular ideal, the Jacobson radical and the prime radical respectively. A ring $R$ is said to be a right duo ring if every right ideal of $R$ is two-sided. Left duo rings are defined symmetrically. By a duo ring we mean a ring which is both right and left duo ring.

It is shown that $R_n(n > 1)$ is a $q$-ring if and only if $R$ is semi-simple Artinian. Some of the main results are: (i) a prime $q$-ring is simple Artinian, (ii) a semi-prime $q$-ring is a direct sum of two rings $S$ and $T$, where $S$ is a complete direct sum of simple Artinian rings, and $T$ is a semi-prime $q$-ring with zero socle, and (iii) a semi-prime $q$-ring is a direct sum of two rings $A$ and $B$, where $A$ is a right self injective duo ring, and $B$ is semi-simple artinian.

2. Let $R$ be a right self injective ring. If $B$ is any right ideal of $R$, then $\hat{B} = eR$ for some idempotent $e$ of $R$. Let $K = \text{Hom}_R(\hat{B}, \hat{B})$. Then $K \cong eRe$. In fact every element in $K$ can be realized by the left multiplication of some element of $eRe$. By Johnson and Wong ([3], Theorem 1.1) $B$ is a quasi injective as a right $R$-module if and only if $KB = B$. Hence $B$ is quasi injective if and only if $B = KB = (eRe)B = (eRe)(eB) = B$. Hence every two-sided ideal in a right self injective ring is quasi-injective. So, the following is immediate.

2.1. Every commutative self injective ring is a $q$-ring.

Now, we give an example of a $q$-ring which is not semi-prime.

Example 2.2. Let $Z$ be the ring of integers. Set $R = Z/(4)$. It is trivial that $R$ is a $q$-ring. But $R$ is not semi-prime, since its only proper ideal is nilpotent.
In fact, \( Z(n) \) is a \( q \)-ring for every integer \( n \geq 1 \), since it is self injective (cf. Levy [5]). Also we remark that \( Z(n) \) has nonzero nilpotent ideals if \( n \) is not square free.

Next we prove

**Theorem 2.3.** The following are equivalent

1. \( R \) is a \( q \)-ring
2. \( R \) is right self injective, and every right ideal of \( R \) is of the form \( eI \), \( e \) is an idempotent in \( R \), \( I \) is a two sided ideal in \( R \).
3. \( R \) is right self-injective, and every large right ideal of \( R \) is two sided.

**Proof.** Assume (1). Therefore \( R \) is right self injective. Let \( B \) be any right ideal of \( R \). Then \( \hat{B} = eR \) for some idempotent \( e \). Since \( B \) is quasi injective \( B = \hat{B} B = eRB = eI \), where \( I = RB \), the smallest two-sided ideal of \( R \) containing \( B \). Hence (1) implies (2).

Assume (2). Let \( A \) be a large right ideal of \( R \). Then \( A = eI \), \( e^2 = e \), \( I \) is a two sided ideal. Since \( A \cap (1 - e)R = 0 \), \( (1 - e)R = 0 \). This implies that \( e = 1 \). Hence \( A = I \), proving (3).

Now assume (3). Let \( B \) be a right ideal of \( R \). If \( K \) is a complement of \( B \), then \( B \oplus K \) is large in \( R \). By assumption \( B \oplus K \) is a two-sided ideal in \( R \), hence quasi-injective. This implies \( B \) is a quasi-injective, completing the proof.

**Theorem 2.4.** Let \( n > 1 \) be an integer. Then \( R_\ast \) is a \( q \)-ring if and only if \( R \) is semi-simple Artinian.

**Proof.** Suppose that \( R \) is not semi-simple Artinian. By Lambek ([4], Proposition 2, p. 61), there exists a large right ideal \( B \) of \( R \) such that \( B \neq R \). Let \( e_{ij}, 1 \leq i, j \leq n \) be the matrix units of \( R_\ast \) and let \( E = (\sum_{i,j \leq n} a_{ij} e_{ij} : a_{ij} \in B, 1 \leq j \leq n \) and \( a_{ij} \in R, 1 \leq i, j \leq n \). It is clear that \( E \) is a right ideal in \( R_\ast \). But \( E \) is not two-sided, for \( e_{ss} \in E \) and \( e_{ss} = e_{ss} \in E \). Now, we prove that \( E \) is a large right ideal in \( R_\ast \). Let \( 0 \neq x = \sum_{i,j=1}^n b_{ij} e_{ij} \). If \( b_{ij} = 0, 1 \leq j \leq n \), then \( x \in E \). So, let \( b_{is} = 0 \) for some \( k \). Since \( B \) is large in \( R \), there exists \( a \in R \) such that \( 0 \neq b_{is} a \in B \). Then,

\[
x(a e_{kk}) = (\sum_{i,j=1}^n b_{ij} e_{ij})(a e_{kk}) = \sum_{i=1}^n b_{is} a e_{is} \in E.
\]

Hence, \( 0 \neq x(a e_{kk}) \in E \). Therefore \( E \) is a large right ideal in \( R_\ast \) which is not two-sided, and by Theorem 2.3, \( R_\ast \) is not a \( q \)-ring. This proves "only if" part. Other part is obvious.

We are now ready to show the existence of right self injective rings which are not \( q \)-rings.
EXAMPLE 2.5. Let $R$ be a right self injective ring which is not semi-simple (we can take $R = \mathbb{Z}/(4)$). Let $n > 1$ be an integer. By Utumi ([6], Th. 8.3) $R_n$ is right self injective. But $R_n$ is not a $q$-ring, by the above theorem.

Next we prove

THEOREM 2.6. A simple ring is a $q$-ring if and only if it is Artinian.

Proof. Let $R$ be a simple $q$-ring. Let $B$ be a large right ideal in $R$. Then $B$ is two-sided, and hence $B = R$. This proves that $R$ does not contain any proper large right ideal. Hence $R$ is Artinian. The converse is trivial.

Now, we give an example of a right self injective simple ring which is not a $q$-ring.

EXAMPLE 2.7. Let $S$ be a noncommutative integral domain which is not a right Ore domain (cf. Goldie [1]). Let $R = \hat{S}$. Then $R$ is a right self injective simple regular ring which is not Artinian. By the above theorem $R$ is not a $q$-ring.

LEMMA 2.8. Let $R$ be a $q$-ring. Then $B(R)$ is essential in $J(R)$ as a right $R$-module.

Proof. Since $R$ is self injective, $J(R) = R'$, by Utumi ([6], Lemma 4.1). Let $0 \neq x \in J(R)$. There exist a large right ideal $E$ of $R$ such that $xE = 0$. Then $xE \subseteq P$ for every prime ideal $P$ of $R$. Since $R$ is a $q$-ring, $E$ is two-sided. This implies that either $x \in P$ or $E \subseteq P$.

Let $\{P_i\}_{i \in I}$ be the set of all prime ideals of $R$ such that $x \in P_i$ for every $i \in I$, and $\{P_j\}_{j \in J}$ be the set of all prime ideals of $R$ such that $x \in P_j$ for every $j \in J$. Let $X = \bigcap_{i \in I} P_i$, and $Y = \bigcap_{j \in J} P_j$. $X \neq 0$, since $0 \neq x \in X$. On the other hand, $E \subseteq P_j$ for every $j \in J$. Thus $E \subseteq Y$, which implies that $Y$ is large in $R$. Therefore $B(R) = X \cap Y = (0)$. Moreover, there exists $a \in R$ such that $0 \neq xa \in Y$. This implies that $0 \neq xa \in X \cap Y = B(R)$, completing the proof.

Hence, we have the following

THEOREM 2.9. A $q$-ring is regular if and only if it is semi-prime.
Proof. The result follows by the above lemma, and Utumi ([6], Corollary 4.2).

Theorem 2.10. Let \( V \) be a vector space over a division ring \( D \), and let \( R = \text{Hom}_d(V, V) \). Then \( R \) is a \( q \)-ring if and only if \( V \) is of finite dimension over \( D \).

Proof. The "if" part is obvious. Conversely, suppose that \( V \) is of infinite dimension over \( D \). Let \( X = \{ x_1, x_2, \ldots \} \) be a denumerable set of linearly independent elements of \( V \). \( X \) can be extended to a basis \( X \cup Y \) of \( V \). Let \( F \) be the ideal in \( R \) consisting of all elements of finite rank. Let \( \sigma \in R \) be defined by \( \sigma(x_i) = x_{i+1} \), \( \sigma(x_{m+1}) = 0 \) for every \( i \), and \( \sigma(y) = 0 \) for every \( y \in Y \). Let \( E = \sigma R + F \). Then \( F \subseteq E \). Since \( F \) is a two-sided ideal in \( R \), \( F \) is large. Therefore \( E \) is a large right ideal in \( R \). We proceed to prove that \( E \) is not two-sided. Let \( \lambda, \mu \in R \) be defined by \( \lambda(x_i) = x_i \) for every \( i \), and \( \lambda(y) = 0 \) for every \( y \in Y \), \( \mu(x_i) = x_i \), \( \mu(x_{m+1}) = 0 \), \( \mu(y) = 0 \) for every \( y \in Y \). Let \( \lambda = \lambda \sigma \lambda \). Then \( \lambda(x_i) = x_i \) for every \( i \). Hence \( X \subseteq \lambda(V) \). We assert that \( \lambda \in E \); for otherwise, let \( \lambda = \sigma r + f \), \( r \in R \), \( f \in F \). Then \( \lambda(V) = \sigma r(V) + f(V) \). But since \( f \) is of finite rank, there exists an integer \( n \) such that \( x_{m+1} \in f(V) \). Also, by definition of \( r \), \( x_{m+1} \in \sigma(V) \). Hence \( x_{m+1} \in \sigma(V) + f(V) \), which is a contradiction. Thus \( \lambda \in E \), as desired. However \( \lambda \in R \sigma R \subseteq RE \). Hence \( E \) is not a two-sided ideal. Therefore, by Theorem 2.3, \( R \) is not a \( q \)-ring. This completes the proof.

We remark that the above theorem is also a consequence of Theorem 2.4.

The right (left) socle of a ring \( R \) is defined to be the sum of all minimal right (left) ideals of \( R \). It is well known that in a semi-prime ring \( R \), the right and left socles of \( R \) coincide, and we denote any of them by \( \text{soc} \, R \).

Lemma 2.11. A semi-prime \( q \)-ring \( R \) with zero socle is strongly regular.

Proof. Let \( M \) be a maximal right ideal in \( R \). Either \( M \) is a direct summand of \( R \) or \( M \) is large in \( R \). If \( M \) is a direct summand of \( R \), then its complement is a minimal right ideal. This implies that \( \text{soc} \, R = 0 \), a contradiction. Therefore, every maximal right ideal is large, hence two-sided. By Lemma 2.8, \( \mathcal{J}(R) = 0 \). Thus \( R \) is isomorphic to a subdirect sum of division rings, which implies that \( R \) has no nonzero nilpotent elements. Since \( R \) is regular, by Theorem 2.9, \( R \) is strongly regular.
LEMMA 2.12. A prime q-ring has nonzero socle.

Proof. Let R be a prime q-ring. If possible, let soc R = 0. By the above lemma, R is strongly regular. Hence R is a division ring, and soc R = R contradicting our assumption. Therefore soc R ≠ (0).

THEOREM 2.13. A prime ring R is a q-ring if and only if R is simple Artinian.

Proof. By Theorem 2.9, and the above lemma, R is a prime regular ring with nonzero socle. Hence, by Johnson ([2], Th. 3.1), \( R = \text{Hom}_D(V, V) \), where V is some vector space over a division ring D. But then R = Hom_D(V, V), since R is right self injective. By Theorem 2.10, V has finite dimension over D. Let \( V: D = n \). Then \( R \cong D_n \), completing the proof.

LEMMA 2.14. Let \( \{R_i\}_{i \in I} \) be a finite set of rings. Then the direct sum \( \bigoplus_{i \in I} R_i \) is a q-ring if and only if each \( R_i \) is a q-ring.

The proof is obvious.

That Lemma 2.14 is not true for an infinite number of rings is shown by the following example which is due to Storrer.

EXAMPLE 2.15. Let R be a 2 × 2-matrix ring over a field F. Let \( \{R_i\}_{i \in I} \) be an infinite family of copies of R and let \( S = \pi R, \alpha \in I \). Let \( E \) be the right ideal of S consisting of those elements \([x_i]_S \) of S such that all but finite \( x_i \)'s are matrices with first row zero. Since \( R_i \subset E \) for all \( \alpha \in I \), E is a large right ideal of S. To show that E is not two-sided, consider \([x_i]_S \in E \) where \( x_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) for all \( \alpha \in I \).

Let \([y_i]_S \in S \) be such that \( y_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) for all \( \alpha \in I \). Then \([y_i]_S[x_i]_S = [z_i]_S \), where \( z_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \). But then \([z_i]_S \in E \), and E is not two-sided. Hence, by Theorem 2.3, S is not a q-ring.

Example 2.15 also suggests the following.

THEOREM 2.16. Let \( \{R_i\}_{i \in I} \) be a family of simple Artinian rings and let \( R \) be their complete direct sum. Then \( R \) is a q-ring if and only if all \( R_i \)'s excepting a finite number of them are division rings.

The above theorem shows, in particular, that a regular q-ring may not be Artinian.

LEMMA 2.17. Let R be a semi-prime q-ring such that soc R is
large in $R$. Then $R$ is a complete direct sum of simple Artinian rings.

Proof. Since $\text{soc } R$ is large, every nonzero right ideal of $R$ contains a minimal right ideal. Also $R$ is regular, by Theorem 2.9. Hence by Johnson ([2], Th. 3.1), $R$ is a complete direct sum of rings $R_i$, where each $R_i$ is the ring of all linear transformations of some vector space $V_i$ over a division ring $D_i$. But then by Lemma 2.14 and Theorem 2.10, each $R_i$ is a simple Artinian ring. This completes the proof.

In the following two theorems we assume that every ring has a unity element which may be equal to zero.

**Theorem 2.18.** Let $R$ be a semi-prime q-ring. Then $R = S \oplus T$, where $S$ is a complete direct sum of simple Artinian rings and $T$ is a semi-prime q-ring with zero socle.

Proof. Let $F = \text{soc } R$. Since $R' = 0$, $\hat{F} = \{x \in R; xE \subseteq F$ for some large right ideal $E$ of $R\}$. Then it is immediate that $\hat{F}$ is a two-sided ideal in $R$. Since $R$ is self injective, $\hat{F} = eR$ for some idempotent $e$. Then $e$ is central, since $R$ is regular. Let $S = eR$ and $T = (1 - e)R$. Hence $R = S \oplus T$. By Lemma 2.14, both $S$ and $T$ are q-rings. Further, it can be easily verified that (i) $S$ is a semi-prime ring, $\text{soc } S = F$, and $F$ is large in $S$, and (ii) $T$ is a semi-prime ring with zero socle. By the above lemma $S$ is a complete direct sum of simple Artinian rings, completing the proof.

As a consequence of Lemma 2.11, Theorem 2.16 and Theorem 2.18 we have the following.

**Theorem 2.19.** A semi-prime ring $R$ is a q-ring if and only if $R = A \oplus B$, where $A$ is a right self injective duo ring and $B$ is semi-simple Artinian.

**References**

4. J. Lambek, Lectures on rings and modules.
RINGS IN WHICH EVERY RIGHT IDEAL IS QUASI-INJECTIVE


Received March 5, 1969.

University of Delhi India