A REMARK ON PRIMITIVE RINGS
AND J-PIVOTAL MONOMIALS

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1. For quite some time it has been a problem of great interest to
study rings satisfying a polynomial identity. The latter concept was
generalized by Drazin [1] to the notion of a pivotal monomial (PM).
The main structure theorem for rings satisfying a PM is that a right
primitive ring with a right PM is a full matrix ring over a division ring.
Later Amitsur [2] generalized the concept of a PM ring to that of a
J-pivotal monomial ring (JPM). This note concerns the structure of
primitive rings having a right ideal $I, I \neq 0$, where $I$ is a JPM ring.
The main theorem is: Let $R$ be a primitive ring. Then $R$ has a non-
zero socle iff for some right ideal $I \neq 0, I$ is a right JPM ring.

2. Let $R$ be a ring. We say that $R$ is (right) $J$-pivotal monomial
(JPM) ring of degree $d$ if and only if each (right) primitive homomorphic
image of $R$ is isomorphic to a full ring of $n \times n$ matrices, $n \leq d$, over
some division ring. Notice that the above definition of a JPM ring is
only an equivalent form of the original definition given by Amitsur [2].
For the purpose of this note the above definition is more desirable.

We shall require the following results concerning JPM and PM rings.
(1) If $U$ is a right ideal in a JPM ring of degree $d$ then $U$ is a JPM
ring of degree $h \leq d, [2]$; (2) a primitive PM ring of degree $d$ is a full matrix
ring over a division ring.

Theorem. Let $R$ be a primitive ring. Then $R$ has a non-zero socle
if and only if there is a non-zero right ideal $I$ of $R$ which is a JPM ring.

Proof. Assume $I \neq 0$ a right ideal, $I$ a JPM ring. Let $I'$ denote the
radical of $I$. Then $I/I'$ is primitive, hence $I/I' \cong D_n$, $D_n$ the ring of
$n \times n$ matrices over a division ring $D$. Since $D_n$ has idempotents so does
$I/I'$ and thus so has $I$. Let $e \in I$ be a non-zero idempotent. Then $eR$
is a right ideal of $I$, hence $eR$ is a JPM ring. Hence we may assume that $I$
has the form $eR$ for some idempotent $e \neq 0$. We shall show $I$ contains

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a minimal right ideal, proceeding by cases on n. Assume n = 1. If f \not= 0, f^2 = f then in I/I^o we have f - e = 0. Thus f - e \in I^o and (f^2 - e)^2 = 0, so we obtain (f - e)^2 = 0. Since ef = f = f^2 we then have e - fe = 0. Hence f(eR) = f\,eR = eR = I. Since fI \subseteq fR \subseteq I we see that fR = I. Now if J \subseteq I is a non-zero right ideal of R then like I, J contains an idempotent f \not= 0. By the above we have I = fR \subseteq J \subseteq I, so J = I. Hence I is minimal. Suppose now that n > 1. Since n is the maximal size of sets of orthogonal idempotents in D_n, then in I there are orthogonal idempotents e_1, e_2, \ldots, e_n. It's clear that e_1, e_2, \ldots, e_n is also a maximal set of orthogonal idempotents for I. Suppose e_i R is not minimal. Let J be a non-zero right ideal of I properly contained in e_i R. J is a right ideal contained in I, hence J is a JPM ring. Thus J/J^o \cong D_k, D^o a division ring. If k = 1, then, as in the case of n = 1, J is minimal. Assume \ k > 1; then J contains at least two orthogonal idempotents f_i, f_j. For i > 1, e_i J \subseteq e_i e_i R = 0, hence e does not belong to J. Thus f_i \not= e_i for i > 1, j = 1, 2. Let a_i = f_i e_i, i = 1, 2. Then a_i^2 = f_i e_i f_i e_i = f_i^2 e_i = f_i e_i = a_i. Also a_i \not= 0 since a_i = 0 implies f_i J = 0 which in turn implies f_i^2 = 0. Moreover, a_i a_j = f_i e_i f_j e_j = f_i e_j e_i = 0 if i \not= j, and a_i a_i = f_i e_i e_i f_i e_i = e_i e_i f_i e_i = 0 for i > 1. Hence a_1, a_2, e_3, \ldots, e_n is a set of m + 1 orthogonal idempotents for I which is impossible. This contradiction shows that R contains a minimal right ideal. Conversely, suppose I = eR is a minimal right ideal, e^2 = e \not= 0. Then eRe is a division ring. Hence for all x, y \in R there is an r \in eRe so that exe = eye = eye = exe = r. Letting e(\lambda) be the monomial \lambda a_{\lambda}, \lambda_i indeterminates and letting a(\lambda) = \lambda a_{\lambda}, we see that for \ x_i, x_j \in I, \pi(x) I \subseteq \pi(x) I. Hence I is a strongly right PM-ring and thus a JPM-ring.

As a consequence of the above theorem we have

**Corollary 1.** Let R be a primitive ring having at most finitely many orthogonal idempotents. Let I \not= 0 be a right ideal of R, I a JPM ring. Then R possesses a pivotal monomial.

**Proof.** By the above theorem R has a non-zero socle. Since R has only finitely many orthogonal idempotents it follows that R is a full ring of n x n matrices over a division ring. Hence R possesses the PM, \pi(\lambda) = \lambda^m in a single variable.

As a second consequence we have

**Corollary 2.** Let R be a primitive ring having at most finitely many orthogonal idempotents and let I \not= 0 be a right ideal satisfying a polynomial identity. Then R satisfies a polynomial identity.
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Proof. As in Corollary 1 we obtain \( R \) is isomorphic to \( D_n, \) \( D \) a division ring. Letting \( I = eR, \ e^2 = e \) we can choose \( D = eRe; \) since \( D \leq I, \) \( D \) satisfies a polynomial identity and thus is finite-dimensional over its center. Thus \( D_n \) is finite-dimensional and so satisfies a polynomial identity.

The condition of \( R \) having at most finitely many orthogonal idempotents cannot be removed from the hypotheses of Corollary 1 or 2. For an example see [3].

REFERENCES


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