Rings Having One-Sided Ideals Satisfying a Polynomial Identity

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Introduction. The problem of determining the structure of a ring in which certain special subset satisfies a polynomial identity has recently found interest with some authors including AMITSEU, HERSTEIN, MARTINDALE and BELLUCE [9, 16, 17 and 3]. It is shown by BELLUCE and JAIN [3] that if $R$ is a prime ring which possesses a non-zero right ideal $A$ with a polynomial identity then $R$ satisfies a polynomial identity if any of the following conditions hold: (1) $l(A) = 0$, (2) $R$ is a right Goldie ring. The object of the present paper is to study rings, not necessarily prime, which possess a non-zero right ideal $A$ satisfying a polynomial identity. In contrast to the prime case, examples are given to show that (i) a ring $R$ may possess a two-sided ideal $A$ with a polynomial identity and $l(A) = 0$ but the ring itself may not satisfy any polynomial identity and (ii) a Goldie ring may fail to possess a polynomial identity even though it possesses a two-sided ideal with a polynomial identity. Section 3 is devoted to sharpen some of the results proved earlier for prime rings [3, 4]. Sufficient conditions are obtained in sections 4 and 5 that the maximal quotient rings of semiprime rings and artinian rings satisfy a polynomial identity, whenever they possess a non-zero right ideal $A$ such that $A$ satisfies a polynomial identity and $l(A) = 0$.

1. Preliminaries and Definitions. For a ring $R$ the symbols $M^\Delta$, $R^\Delta$, $L^s(R)$, $L^\Delta(R)$ and $\hat{R}$ respectively will denote as usual the singular submodule of an $R$-module $M$, the right singular ideal, the lattice of all closed right ideals, the lattice of all large right ideals and the maximal (right) quotient ring in the sense of JOHNSON [13] and we denote by $l_S(X)$ the left annihilator of a subset $X$ of $R$ in a subset $S$ of $R$. It is known that if $R$ is a ring with $R^\Delta = 0$, then $\hat{R}$ can be looked upon as $\bigcup \text{Hom}_{R}(A, R)$, where $A$ is a large right ideal of $R$, and further $\hat{R}$ is a (Von-Neumann) regular ring which as a right $R$-module is the unique maximal essential extension of $R$ as an $R$-module [11]. Thus by ECKMANN and SCHOFF [5] $\hat{R}$ is also injective as a right $R$-module. It is also proved by JOHNSON and WONG ([14], theorem 7) that $\hat{R}$ is right self-injective. Therefore by JOHNSON ([12], p. 542) each closed right ideal of $\hat{R}$ is a direct summand of $\hat{R}$. But this implies each member $A$ of $L^s(\hat{R})$ is also injective as a right $R$-module. Hence $A$ is injective hull of $A \cap R$ [5]. The lattices of closed right ideals of $\hat{R}$ and $R$ are known to be isomorphic by the mapping $A \to A \cap R$ ([12], theorem 6.8). The maximal quotient ring of a semi-prime Goldie ring is known to coincide with the classical quotient ring (cf. theorem 4.4, [13]).
We also recall the definition of a quasi-standard identity [4]. A ring $R$ is said to satisfy a quasi-standard identity (QSI) of degree $d$ if for each $d$-tuple $(r_1, \ldots, r_d)$ there exist a positive integer $n$ such that

$$\left( \sum_{g} \pm r_{g(1)} \cdots r_{g(d)} \right)^n = 0,$$

where the summation runs over all the permutations $g$ of $1, \ldots, d$ and the sign is positive or negative according as the permutation is even or odd. It was shown in [4] that a prime ring with a right singular ideal zero and uniform right ideals is a right Goldie ring if it has a quasi-standard identity. We shall obtain this result as a corollary to one of the theorems proved below.

Throughout this paper we assume that $R$ is an algebra over a field $F$. If $A$ is a non-zero right ideal satisfying some polynomial identity of degree $d$ and $l_R(A) = 0$, then since $AR$ is an algebra right ideal contained in $A$ and $l_R(AR) = 0$, we can assume that $A$ is a right ideal with a polynomial identity and $l_R(A) = 0$.

### 2.1. Example (Amitsur)

Let $D$ be a division algebra infinite dimensional over its center $C$. Consider the ring $R$ of all triangular matrices of the form \[ \begin{pmatrix} x & y \\ 0 & k \end{pmatrix}, \] where $x$ and $y$ are in $D$ and $k$ is in $C$. $R$ has the Jacobson radical

$$N = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} : y \in D \right\}, \quad N^2 = 0 \quad \text{and} \quad R/N \cong D \oplus C.$$

Therefore $R/N$ is semi-simple artinian and it satisfies no polynomial identity since $D$ cannot satisfy any polynomial identity. Consequently, $R$ cannot satisfy any polynomial identity. But $R$ has an ideal

$$A = \left\{ \begin{pmatrix} 0 & y \\ 0 & k \end{pmatrix} : y \in D, \ k \in C \right\}$$

such that (i) $l_R(A) = 0$ and (ii) $A$ satisfies the identity $(X_1X_2 - X_2X_1)^2 = 0$.

### 2.2. Example

Let $D$ be an infinite dimensional algebra over its center $C$. Let $F$ be any field. Then $D \oplus F$ is a Goldie ring having a two-sided ideal satisfying a polynomial identity. However $D \oplus F$ satisfies no polynomial identity.

### 3. Prime Rings

We give a relationship between the degrees of polynomial identities satisfied by the right ideal $A$ in a prime ring $R$ and the ring $R$ in the theorem 1 of [3]. We will also need this result later on in sections 4 and 5.

#### 3.1. Theorem

Let $R$ be a prime ring. If $A$ is a non-zero right ideal satisfying a polynomial identity of degree $d$ and $l_R(A) = 0$, then $R$ satisfies a standard identity of degree $d$.

**Proof.** Since $A$ is a prime ring by itself, therefore, by Posner [18] the quotient ring of $A$ exists and it satisfies some multilinear identities as satisfied by $A$. But $A$ satisfies a multilinear identity of degree $d' \leq d$. Because the quotient ring of $A$ is semi-simple artinian, by Amitsur [1], the quotient ring satisfies a standard identity of degree $d'$. It is shown in the proof of theorem 1 in [3] that $R$ is embeddable in the
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quotient ring of $A$. Hence $R$ satisfies a standard identity of degree $d'$. But then $R$ also satisfies a standard identity of degree $d$, because $d' \leq d$.

We now prove that the maximal quotient ring satisfies a generalized polynomial identity (for definition see AMITSUR [2]).

3.2. Theorem. Let $R$ be a prime ring such that $R$ has a zero right singular ideal and has uniform right ideals. If there exists a non-zero right ideal $A$ in $R$ such that $A$ satisfies a quasi-standard identity then the maximal quotient ring of $R$ satisfies a generalized polynomial identity.

Proof. It is well known that under the given conditions on $R$, each right ideal contains a uniform right ideal. Let $U$ be a uniform right ideal in $A$. Then $U$ has QSI. Let $f$ be a mapping of $U$ to $\text{Hom}_R(U, U)$, given by $a \to l_a$, where $l_a$ denotes the left multiplication by $a$, $f(U)$ is then a non-zero homomorphism and $f(U)$ is a left ideal in $\text{Hom}_R(U, U)$. Thus $K = \text{Hom}_R(U, U)$ which is an integral domain has a left ideal with QSI. But this implies $K$ has a left ideal with SI and hence by theorem 3.1 $K$ has SI. Thus by AMITSUR $\hat{K}$ has SI. But by FAITH and UTUMI [cf. 6],

$$\hat{K} = \text{Hom}_R(\hat{U}, \hat{U}) = \text{Hom}_R(e\hat{R}, e\hat{R}) = e\hat{R}.$$

This implies the minimal right ideal $e\hat{R}$ has a polynomial identity and hence $\hat{R}$ has a generalized polynomial identity.

We deduce a result which is proved in [4] by using weak transitivity of $R$.

Corollary. If the ring $R$ satisfies a quasi-standard identity then the maximal quotient ring satisfies a polynomial identity and hence is a finite dimensional central simple algebra.

Proof. If $I$ is a minimal right ideal in $\hat{R}$, then $U = I \cap R$ is a uniform right ideal of $R$ such that $\hat{U} = I$. Following the proof in the theorem we can show that each minimal right ideal of $\hat{R}$ satisfies the same polynomial identity. Hence the socle has a polynomial identity. Since the maximal quotient ring is also prime, it follows by theorem 3.1 that it also satisfies a polynomial identity. This proves the corollary.

4. Semi-prime Rings. Lemma 4.1, which follows is well known (cf. LEVY [15]).

4.1. Lemma. If a semi-prime ring $T$ has acc on annihilator ideals then the set $M$ of annihilator (two sided) ideals contains only a finite number of maximal members whose intersection is zero.

4.2. Theorem. If $R$ is a semi-prime ring which has acc on annihilator two sided ideals and if there exists a non-zero right ideal $A$ satisfying a polynomial identity such that $l_R(A) = 0$, then the maximal quotient ring of $R$ also satisfies a polynomial identity.

Proof. Let $B$ be any two sided ideal of $R$. If for any $a$ in $R$, $Ba = 0$, then for any $a$ in $F$, $B(\alpha x) = 0$. Therefore, annihilator ideal is an algebra ideal. By 4.1 there exist a finite number of distinct maximal annihilator ideals, say, $A_1, \ldots, A_n$ with zero intersection. Thus $R_1 = R/A_1$ is a prime ring which is an algebra over $F$. If $g_1$ is the natural homomorphism of $R$ onto $R_1$, then it is easy to prove that $l_{R_1}(g_1(A)) = 0$.
Thus by theorem 3.1, $R_i$ satisfies a standard identity of degree $d$. By Posner [18], the classical quotient ring $Q(R_i)$ (which is same as $\hat{R}_i [13]$) exists and satisfies the same identity. Following the lines of proof of theorem 4.7 in [8], we can show that $Q(R)$ exists and is isomorphic to $\bigoplus \sum Q(R_i)$. Consequently, $Q(R)$ also satisfies a standard identity. Now each $Q(R_i)$, we know, is simple artinian. Hence by [7], theorem 4.4, $R$ is a semi-prime Goldie ring. But by Johnson [13], $\hat{R} = Q(R)$. This completes the proof.

A consequence of the above is the following result proved by Small [19].

**Corollary.** If the ring $R$ satisfies a polynomial identity, then the classical quotient ring $Q(R)$ also satisfies a polynomial identity.

Our next theorem is concerned with a semi-prime ring having its socle as a large right ideal. The proof depends on the following lemma which is interesting by itself.

**4.3. Lemma.** If $T$ is any semi-prime ring such that its socle $X$ is a large right ideal, then

$$\hat{T} = \bigsqcup \lim_{i} \text{Hom}_{X_i}(X_i, X_i),$$

where $X_i$ are the homogeneous components of the socle of $T$.

**Proof.** Since each right ideal of $T$ contains an idempotent, $T^\circ = 0$. Further the socle $X$ is a large right ideal of $T$, therefore $X^\circ = 0$ and hence the maximal quotient ring of $X$ and that of $T$ are same. But in a semi-prime ring a minimal right ideal is also a minimal right ideal of its socle (as a ring). Therefore the socle $X$ of $T$ is completely reducible as a right $X$-module. Consequently, each right ideal of $X$ is a direct summand of $X$. Therefore if $A$ is a large right ideal of $X$ then $A = X$. But $X = \bigsqcup \lim_{i} \text{Hom}_{X_i}(A, X)$ where $A$ is a large right ideal of $X$. This gives $X = \text{Hom}_{X_i}(X, X)$. Let $X = \bigsqcup \lim_{i} X_i$, where $X_i$ are the homogeneous components of $X$. Therefore,

$$\text{Hom}_{X}(X, X) \cong \bigsqcup \lim_{i} \text{Hom}_{X_i}(X_i, X) = \bigsqcup \lim_{i} \text{Hom}_{X_i}(X_i, X_i).$$

But $X_i$ is a direct summand of $X$, therefore, $\text{Hom}_{X_i}(X_i, X_i) = \text{Hom}_{X_i}(X_i, X_i)$. Hence $\text{Hom}_{X}(X, X) = \bigsqcup \lim_{i} \text{Hom}_{X_i}(X_i, X_i, X_i)$. This completes the proof.

**Remark.** It is worth noticing that in the above lemma, semi-primeness of $T$ can be replaced by $T^\circ = 0$ and each minimal right ideal of $T$ is a minimal right ideal of $X$ (as a ring).

**4.4. Theorem.** Let $R$ be a semi-prime ring such that its socle is a large right ideal.

If there exists a non-zero right ideal $A$ satisfying a polynomial identity of degree $d$ and $I_R(A) = 0$, then the maximal quotient ring of $R$ satisfies a standard identity of degree $d$.

**Proof.** Let $A$ be the given right ideal with polynomial identity of degree $d$ such that $I_R(A) = 0$. Let $A_i = A \cap S_i$, where $S_i$ are the homogeneous components of $S$. Let $x_i$ be in $S_i$ such that $x_i A_i = 0$. This implies $x_i (\sum A_j) = 0$. Therefore $x_i A = 0$, because trivially $\sum A_j \subseteq A$. i.e. $A$ is an essential extension of $\sum A_j$ as $R$-modules.
and further $R^\Delta = 0$. But then $x_4 = 0$. Hence $I_{S_4}(A_4) = 0$. Since $S_4$ is a simple ring, by theorem 3.1, $S_4$ satisfies a standard identity of degree $d$. Hence $S_4$ is a full matrix ring $D_n^{(0)}$ over a division ring $D^{(0)}$. This implies $\text{Hom}_{S_4}(S_4, S_4)$ is isomorphic to $S_4$ and thus satisfies a standard identity of degree $d$. If we apply lemma 2 to $R$, we obtain $\hat{R} = \prod \text{Hom}_{S_4}(S_4, S_4)$. Consequently $\hat{R}$ satisfies a standard identity of degree $d$.

The following simple example shows that the hypothesis in the above theorem is sufficient but not necessary.

4.5. Example. Let $\mathbb{Z}$ be the ring of integers and $F$ be a field. Then $R = F \oplus \mathbb{Z}$ is a semi-prime ring with a polynomial identity but the socle is not large.

5. Artinian Rings. We assume now that $R$ is a right artinian ring with zero right singular ideal. Let $S$ denote the (right) socle of $R$, which is a large right ideal in $R$.

5.1. Lemma. Any minimal right ideal $I$ of $R$ is a minimal right ideal of $S$ and conversely.

Proof. Let $I$ be any minimal right ideal of $R$. Let $0 \neq x \in I$. Then $xS \neq 0$, since $R^\Delta = 0$. Therefore $xS$ is a non-zero right ideal of $R$ contained in $I$. Thus $xS = I$ and this gives that $I$ is a minimal right ideal of $S$. Conversely let $J$ be a minimal right ideal of $S$. Then again $JS = J$ and therefore $JS = J$. But $JS$ is a right ideal of $R$. Therefore $J$ is also a minimal right ideal of $R$.

5.2. Lemma. Any minimal right ideal of $R$ contained in a homogeneous component $S_4$ of $S$ is a minimal right ideal of $S_4$ and conversely.

Proof. It follows from lemma 5.1 and the fact that $S_4$ is a direct summand of $S$.

5.3. Lemma. $\hat{S} = \hat{R} = \text{Hom}_{S}(S, S) = \sum \hat{S}_4$, where $S_4$ are homogeneous components of $S$.

The proof follows from the 5.2 and the remark following 4.3.

We are now in a position to prove one of the main results of this paper.

5.4. Theorem. Let $R$ be a right artinian ring such that it has a zero right singular ideal. If there exist a non-zero right ideal $A$ satisfying a polynomial identity and $I_{R}(A) = 0$ then the maximal quotient ring of $R$ satisfies a polynomial identity and is therefore a finite direct sum of finite dimensional central simple algebras.

Proof. Let $A_4 = A \cap S_4$ where $A$ is the given right ideal with polynomial identity. It follows on the same lines as in the proof of theorem 4.4 that $I_{S_4}(A_4) = 0$. Now $S_4$ is a finite direct sum of mutually isomorphic right ideals of $R$ (and therefore of $S_4$ because of lemma 5.2). Let $S_4 = \sum A_4$, where $A_4$ are minimal right ideals of $S_4$ which are isomorphic to each other. Each $A_4$ is a direct summand and therefore it is a closed right ideal of $S_4$. In the lattice $L(S_4)$ of closed right ideals of $S_4$, we have $S_4 = \bigvee A_4$ an irredundant decomposition into atoms of $L(S_4)$. Let $B_4$ be the members of $L(S_4)$ corresponding to $A_4$ in the isomorphism between the lattices
$\text{L}^*(\mathcal{S}_t)$ and $\text{L}^*(\mathcal{S}_t)$. Then we get $\mathcal{S}_t = \bigvee B_{ij}$, an irredundant decomposition of $\mathcal{S}_t$ into atoms of $\text{L}^*(\mathcal{S}_t)$. This gives $\mathcal{S}_t = \bigoplus_{j} B_{ij}$. As explained in section 2, each $B_{ij}$ is injective hull of $A_{ij}$ as $R$-module. Thus $B_{ij}$ are mutually isomorphic as $R$-modules. But $R^\Delta = 0$ implies that they are mutually isomorphic as $\hat{R}$-module. Hence $\mathcal{S}_t$ is a direct sum of finite number of mutually isomorphic minimal right ideals of $\mathcal{S}_t$. Since $\mathcal{S}_t$ is also regular it is a full matrix ring over a division ring

$$D^{(0)} = \text{Hom}_{\mathcal{S}_t}(B_{ij}, B_{ij})$$

where $B_{ij}$ is a minimal right ideal of $\mathcal{S}_t$.

But if $f \in \text{Hom}_{\mathcal{S}_t}(B_{ij}, B_{ij})$, then restricting $f$ to the irreducible $S_t$-module $A_{ij}$, we have $f(A_{ij}) = 0$ or $f(A_{ij})$ is isomorphic to $A_{ij}$. Since $B_{ij}$ is also uniform as an $S_t$-module, $A_{ij} \cap f(A_{ij}) = 0$, when $f(A_{ij}) \neq 0$. Thus $A_{ij} = f(A_{ij})$. Further if $f$ is in $\text{Hom}_{\mathcal{S}_t}(A_{ij}, A_{ij})$ then $f$ can be uniquely extended to $\text{Hom}_{\mathcal{S}_t}(B_{ij}, B_{ij})$, because $B_{ij}$ is injective hull of $A_{ij}$ as $S_t$-module and trivially singular submodule of $A_{ij}$ is zero. Hence $\text{Hom}_{\mathcal{S}_t}(A_{ij}, A_{ij}) = \text{Hom}_{\mathcal{S}_t}(B_{ij}, B_{ij})$. However,

$$\text{Hom}_{\mathcal{S}_t}(B_{ij}, B_{ij}) = \text{Hom}_{\mathcal{S}_t}(B_{ij}, B_{ij})$$

since $S_t^\Delta = 0$. Therefore we obtain $\text{Hom}_{\mathcal{S}_t}(A_{ij}, A_{ij}) = \text{Hom}_{\mathcal{S}_t}(B_{ij}, B_{ij}) = D^{(0)}$.

Let $N_t$ be the radical of $\mathcal{S}_t$. $N_t$ is nilpotent. Thus $A_t \subset N_t$, as $\mathcal{S}_t(A_t) = 0$. Hence there exists a minimal right ideal $I_t$ of $\mathcal{S}_t$ contained in $A_t$ such that $I_t \cap N_t = 0$. Then $I_t = e_t S_t$ for some idempotent $e_t$ in $S_t$ and we have

$$D^{(0)} = \text{Hom}_{\mathcal{S}_t}(A_{ij}, A_{ij}) = \text{Hom}_{\mathcal{S}_t}(I_t, I_t) = e_t S_t e_t$$

Now $e_t S_t e_t \subseteq A_t \subseteq A$. Therefore $e_t S_t e_t$ satisfies some PI as satisfied by $A$. Consequently $D^{(0)}$ also satisfies a PI. Then KAPLANSKY's theorem states that $D^{(0)}$ is finite dimensional over its center. Then $\mathcal{S}_t = D^{(0)}_m$ is also finite dimensional over its center and $\mathcal{S}_t$ satisfies some standard identity of degree say $m$. If we set

$$m = \max(m_1, m_2, \ldots, m_k),$$

then each $\mathcal{S}_t$ satisfies the standard identity of degree $m$. Now by lemma 5, $\hat{R} = \bigoplus \mathcal{S}_t$. Hence $\hat{R}$ also satisfies the standard identity of degree $m$.

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