 NOTE ON GENERALIZED COMMUTATIVE RINGS

By S. K. JAIN and P. K. MENON

[Received December 14, 1967]

BELLUCE-HERSTEIN-JAIN have defined [1] a ring \( R \) to be a generalized commutative ring (written as g.c. ring) if given \( a, b \in R \) there exist positive integers \( m = m(a, b), n = n(a, b) \) such that \( (ab)^n = (ba)^n \). A multiplicative semi-group \( S \) will be said to have \( H \)-property if given \( a, b \in S \) there exists a positive integer \( n = n(a, b) \) such that \( a^n b = ba^n \). In this note we provide an alternative proof to prove that the commutator ideal of a g. c. ring is nil. The lemma which we prove below has an independent interest also. It follows from the lemma that if \( G \) is a multiplicative group in which for each \( a, b \in G, (ab)^m(a, b) = (ba)^n(a, b) \) where \( m(a, b) \) and \( n(a, b) \) are positive integers then \( G \) has \( H \)-property. We assume for convenience that the ring \( R \) has unity.

**Lemma.** Let \( \mathcal{G} \) be a multiplicative group. Let for \( a, b \in \mathcal{G} \) there exist positive integers \( m, n, r \) and \( s \) depending on \( a \) and \( b \) such that \( ab^m a^{-1} = b^r \) and \( ba^s b^{-1} = a^s \). Then there exists a positive integer \( \lambda \) such that \( ab^\lambda = b^\lambda a^\lambda \).

**Proof.** If \( b \) is of finite order then the result is obvious. So let \( b \) be not of finite order. Then if \( b^n = b^n \), for positive integers \( m \) and \( n \), we must have \( m = n \). We have by hypothesis \( ab^m a^{-1} = b^n \). By induction we get \( a^r b^m a^{-r} = b^n \) for all positive integers \( r \). We write for convenience

\[ a^r b^m a^{-r} = b^n. \]  (1)
Consider the collection of all ordered pairs \((m_i, n_i)\) satisfying (1). Let \(x(r)\) and \(y(r)\) be the smallest positive integers among \(m_i\)'s and \(n_i\)'s respectively. We claim \(a^r b^{x(r)} a^{-r} = b^{y(r)}\). For, let

\[ a^r b^{x(r)} a^{-r} = b^y \quad (2) \]
\[ a^r b^{2a^{-r}} = b^{y(r)} \quad (3) \]

Raise (2) both sides by \(x\) and (3) by \(x(r)\). This would make left hand sides equal. Thus the right hand sides \(b^{2x}\) and \(b^{y(r)x(r)}\) are also equal. This by our remark in the beginning implies \(y x = y(r) x(r)\). Since \(x(r) \leq x\) and \(y(r) \leq y\), we must have \(x(r) = x\) and \(y(r) = y\). Therefore, we have

\[ a^r b^{x(r)} a^{-r} = b^{y(r)}\]

for each positive integer \(r\). (4)

Suppose we have also

\[ a^r b^{s} a^{-r} = b^s \quad (5) \]

Then (4) and (5) yield \(b^{y(r)} = b^{y(r)x(r)}\). This means \(\lambda y(r) = \mu x(r)\). So we obtain that for a given positive integer \(r\), if the relation (5) is true, then the ratio \(\frac{\lambda}{\mu}\) is constant and equals \(\frac{x(r)}{y(r)}\).

Now, if \(s\) is another positive integer,

\[ a^{r+s} b^{x(r)+s} a^{-r-s} = (a^r b^{x(r)} a^{-r}) (a^s b^{s}) \]
\[ = (a^r a^s b^{x(r)} a^{-r} a^{-s}) = (a^r b^{y(r)} a^{-s}) \]
\[ = (a^s b^{x(r)} a^{-s}) y(r) = b^{y(s)} y(r) \]

Therefore, by the remark just made before,

\[ \frac{x(r) x(s)}{y(r) y(s)} = \frac{x(r + s)}{y(r + s)} \]

If we set \(f(r) = \frac{x(r)}{y(r)}\), then \(f(r + s) = f(r)f(s)\).

This gives \(f(r) = [f(1)]^r\). So that if we can prove \(f(r) = 1\), for some \(r\), then \(f(r) = 1\) for each \(r\).

In particular we would have for \(r = 1\), the relation \(ab^{x(1)} = b^{y(1)} a\). So we now proceed to show \(f(r) = 1\) for some \(r\). So far we have not used our second hypothesis.
ON GENERALIZED COMMUTATIVE RINGS

\[ b.a^r.b^{-1} = a^r. \]  \hspace{2cm} (6)

(Note that \( r \) and \( s \) are some fixed positive integers satisfying this relation.) We conjugate (4) by \( b \) and rewrite it as \( b.a^r.b^{-1}b^{s(r)}b.a^{-r}.b^{-1} = b^{s(r)}. \) We use (6) to obtain \( a^r.b^{s(r)}a^{-r} = b^{s(r)}. \) Also \( a^r.b^{s(r)}a^{-r} = b^{s(r)}. \)

Hence we obtain an integer \( \lambda \) such that \( a^r.b^{s(r)}a^{-\lambda} = b^{s(r)}. \) Raise this both sides by \( x(\lambda) \), we get \( a^r.b^{s(r)}x(\lambda)a^{-\lambda} = b^{s(r)}x(\lambda). \) The left hand side is \( b^{s(\lambda)x(\lambda)}. \) Thus \( b^{s(\lambda)x(\lambda)} = b^{s(r)x(\lambda)}. \) But this implies \( x(\lambda) = y(\lambda). \)

Hence \( f(\lambda) = 1. \) This completes the proof.

**Theorem.** Let \( R \) be a g.c. ring. Let \( a, b \in R \). If \( a \) and each \( b^k \), where \( k \) is a positive integer, are quasi-regular then there exists a positive integer \( n = n(a, b) \), such that \( ab^n = b^n.a \).

**Proof.** If \( b \) is nilpotent then the result is obvious. So let \( b \) be not nilpotent. Then if \( b^m = b^n \), for positive integers \( m \) and \( n \), we must have \( m = n. \) For, otherwise, let \( m > n. \) Then \( b^n(1 - b^{m-n}) = 0. \) Since \( b^{m-n} \) is q.r., we get \( b^n = 0 \), a contradiction. Let \( x = b(1 - a)^{-1}, y = (1 - a)b. \) By hypothesis we have (after a little simplification) integers \( m, n \), such that \( (1 - a)b^m(1 - a)^{-1} = b^n. \)

This is same as (1) in the lemma, with \( 1 - a \) in place of \( a \). Since \( 1 - a \) has an inverse the argument in the lemma yields

\[ (1 - a)^y b^{s(r)}(1 - a)^{-r} = b^{s(r)} \] \hspace{2cm} (A)

and we want to prove \( x(r) = y(r) \) for some \( r. \) Again by hypothesis we have integers \( r \) and \( s \), such that

\[ (1 - b)(1 - a)^y(1 - b)^{-1} = (1 - a)^y. \] \hspace{2cm} (B)

Multiply the equation (A) on the right by \( 1 - b \) and on the left by \( 1 - b \). Then we get

\[ (1 - b)(1 - a)^y(1 - b)^{-1}b^{s(r)}(1 - b)^{-1}b^{s(r)}(1 - b)^{-1} = b^{s(r)}. \]

Applying the equation (B), we obtain \( (1 - a)^y b^{s(r)}(1 - a)^{-r} = b^{s(r)}. \)

But this yields as in the lemma that there exists a positive integer \( \lambda \), such that \( (1 - a)b^{s(\lambda)} = b^{s(\lambda)}(1 - a). \)

Hence \( (1 - a)b^{s(1)} = b^{s(1)}(1 - a) \), which gives \( ab^{s(1)} = b^{s(1)}a. \)

This completes the proof.
Corollary 1. If $R$ is g.c. division ring then $R$ has the $H$-property.

Proof. If $a$ or some power of $b$ is identity then trivially there exists a positive integer $n$ such that $ab^n = b^na$. In case neither $a$ nor any power of $b$ is identity then both $a$ and each power of $b$ is quasi-regular. Thus the theorem would give the result.

Corollary 2. If $R$ is g.c. division ring then $R$ is a field.

Follows from Corollary 1 and Herstein [2].

Corollary 3. A semi-simple g.c. ring is commutative.

The proof is usual deduction from the division ring.

Corollary 4. If $R$ is a non-semi-simple g.c. ring then the Jacobson radical $J(R)$ has the $H$-property.

Proof follows from the theorem.

Corollary 5. If $R$ is a g.c. ring having no non-zero nil ideals, then $R$ is commutative.

Proof. $J(R)$ as a ring in its own right also has no non-zero nil ideals. Thus by Corollary 4 and Herstein [2], $J(R)$ is commutative. Since $R/J(R)$ is also a g.c. ring and is semi-simple, it is commutative by Corollary 3. Let $a, b \in J(R)$ and $x, y \in R$. Then $(ax)(by) = (by)(ax)$. This yields $(b(ax))y = (a(by))x$, so that $ab(xy - yx) = 0$. This means $J^2(R).C(R) = 0$, where $C(R)$ is a commutator ideal. Since $R/J(R)$ is commutative, $C(R) \subset J(R)$. Thus we get $C^2(R) = 0$. Hence $C(R) = 0$. So $R$ is commutative.

Corollary 6. If $R$ is a g.c. ring then the commutator ideal of $R$ is nil.

The proof is now obvious.

Remark. We point out that the proof of the main result in [1], namely, the commutator ideal of a g.c. ring is nil, can also be shortened. The Theorem 3 therein proves that in a g.c. ring if $1 - ab, 1 - ba$, and $1 - a$ have inverses then there exists a positive integer $n$ such that $a^n - b = ba^n$. This shows then a g.c. division
ring has $H$-property and hence the theorem 1 in [1] does not need a separate argument.

REFERENCES


University of Delhi
Delhi-7
and
B-73, Rama Krishna Puram
New Delhi-22