Math 263C, WINTER 2007, Solution of Quiz 3

1. Find the radius of convergence and the interval of convergence of the series

\[ \sum_{n=1}^{\infty} \frac{x^n}{n3^n}. \]

**ANSWER:** To find radius and interval of convergence we need to use the "Limit Ratio Test":

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{n3^{n+1}} \right| = \lim_{n \to \infty} \left| \frac{x}{3} \frac{n}{n + 1} \right| = \left| \frac{x}{3} \right| \lim_{n \to \infty} \frac{n}{n + 1} = \left| \frac{x}{3} \right|.
\]

The series is convergent if and only if \( \left| \frac{x}{3} \right| < 1 \Rightarrow |x| < 3 \). We now check if we can include the endpoints \( x = -3 \) and \( x = 3 \) to the interval or not.

For \( x = -3 \) the series becomes \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \), an alternating series with \( b_n = \frac{1}{n} \).

It is clear that \( b_n > b_{n+1} \) for all \( n \). Secondly, \( \lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n} = 0 \). Hence by the Alternating Test, the series is convergent for \( x = -3 \).

For \( x = 3 \), the series becomes \( \sum_{n=1}^{\infty} \frac{1}{n} \). This series diverges by p-series theorem with \( p = 1 \).

Thus the convergence interval is \( I = [-3, 3) \) and the convergence radius is \( R = 3 \).

2. Find a power series representation for the function

\[ f(x) = \frac{1}{1 - (2x)^2}. \]

Then find the interval of convergence.

**ANSWER:** In the "basic" power series

\[ \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n, \quad R = 1, \quad I = (-1, 1) \]

we replace \( x \) by \((2x)^2\) to get

\[ \frac{1}{1 - (2x)^2} = \sum_{n=1}^{\infty} [(2x)^2]^n = \sum_{n=1}^{\infty} 4^n x^{2n}. \]
Note $[(2x)^2]^n = 4^n x^{2n}$. To find the interval of convergence we use the Limit Ratio Test:

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{4^{n+1} x^{2n+2}}{4^n x^{2n}} \right| = \lim_{n \to \infty} |4x^2| = |4x^2|.$$ 

The series is convergent if and only if $|4x^2| < 1 \Rightarrow |2x| < 1 \Rightarrow |x| < \frac{1}{2}$. The interval of convergence is $I = (-\frac{1}{2}, \frac{1}{2})$. Note, the endpoints are not included, because the series is not convergent for $x = -\frac{1}{2}$ and $x = \frac{1}{2}$. 