Solutions of a homogeneous linear system

Let \( AX = 0 \) be a linear system of \( m \) homogeneous equations in \( n \) unknown. Let

\[
X = \begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix},
\]

and let \( C_1, C_2, \ldots, C_n \) be columns of \( A \). Then this homogeneous linear system (= homogeneous LS) can be written as

\[
[C_1 \ C_2 \ \ldots \ C_n] \begin{bmatrix}
x_1 \\
\vdots \\
x_n
\end{bmatrix} = \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix}_{m \times 1} \iff x_1 C_1 + \cdots + x_n C_n = 0.
\]

Let \( C(A) \) be the column space of \( A \), that means \( C(A) \) is generated by all the columns of \( A \), in symbol: \( C(A) = \langle C_1, C_2, \ldots, C_n \rangle \). Without loss of generality, we can assume that \( \{C_1, C_2, \ldots, C_r\} \) is a maximal set of linearly independent columns. Hence \( \{C_1, C_2, \ldots, C_r\} \) is a basis of \( C(A) \). It follows that each \( C_i \), \((r+1 \leq i \leq n)\) is a linear combination of \( C_1, C_2, \ldots, C_r \), that is:

\[
(*) \quad C_i = \alpha_1^{(i)} C_1 + \cdots + \alpha_r^{(i)} C_r \iff \alpha_1^{(i)} C_1 + \cdots + \alpha_r^{(i)} C_r - C_i = 0
\]

where \( r+1 \leq i \leq n \) and \( \alpha_j^{(i)} \in \mathbb{R} \). Note that this representation is unique. Put

\[
U_{r+1} = \begin{bmatrix}
\alpha_1^{(r+1)} \\
\vdots \\
\alpha_r^{(r+1)} \\
-1 \\
0 \\
\vdots \\
0
\end{bmatrix}, \quad U_{r+2} = \begin{bmatrix}
\alpha_1^{(r+2)} \\
\vdots \\
\alpha_r^{(r+2)} \\
0 \\
-1 \\
0 \\
\vdots \\
0
\end{bmatrix}, \quad \cdots, \quad U_n = \begin{bmatrix}
\alpha_1^{(n)} \\
\vdots \\
\alpha_r^{(n)} \\
0 \\
\vdots \\
0 \\
-1
\end{bmatrix}.
\]

Then from \((*)\) we see that these \( U_i \) are solutions of the homogeneous LS. Moreover the following theorem holds.

**Theorem.** Let \( N(A) \) be the collection of all solutions of the above homogeneous linear system \( AX = 0 \). Then \( N(A) \) is a subspace of \( \mathbb{R}^n \), and \( \{U_{r+1}, \ldots, U_n\} \) is a basis of \( N(A) \). Consequently, every solution of this homogeneous LS has the form \( X = \lambda_{r+1} U_{r+1} + \cdots + \lambda_n U_n \) where \( \lambda_j \) are any real numbers.