MATH 263A, Fall Quarter 2002, MIDTERM 3

Student’s Name (in capital letters):

Show all your work to get full credit. No work will amount to no credit. Circle your final answers.

1. (6 points) Let \( f(x) = \ln\left(\frac{2x}{x+2}\right) \) for \( x > 0 \). Find a sign chart for \( f'(x) \) to find the intervals where \( f(x) \) is increasing where it is decreasing.

\[
\frac{f'(x)}{x+2} = \frac{x+2}{2x} \cdot \frac{2(x+2)-2x}{(x+2)^2} = \frac{(x+2)(4)}{2x(x+2)^2} = \frac{4}{2x(x+2)}
\]

\[\begin{array}{c|c}
 x & > 0 \\
 f'(x) & + \\
 f(x) & \nearrow \\
\end{array}\]

\( f'(x) > 0 \) for all \( x > 0 \).

2. Let \( f(x) = \sqrt{2x} \).

(a) (3 points) Find a formula of the distance \( d \) from the point \( (4,0) \) to a point on the graph of \( f(x) \).

\[
d = \sqrt{(4-x)^2 + (0-\sqrt{2x})^2}
\]

(b) (5 points) Find value(s) of \( x \) for that \( d^2 \) is shortest (= minimum of \( d^2 \)).

\[
f(x) = d^2 = (4-x)^2 + 2x = x^2 - 6x + 16
\]

\[
f'(x) = 2x - 6 \Rightarrow \text{Critical } \#_1 \text{ is } x = 3
\]

\[\begin{array}{c|c|c}
 x & 3 & + \\
 f'(x) & - & 0 + \\
 f(x) & \searrow & \nearrow \\
\end{array}\]

Hence at \( x=3 \), \( d^2 \) is min (= global min).

The point on the graph should be \( (3, \sqrt{6}) \).

3. Let \( f(x) = x^3 - 12x^2 + 5 \) be a given function.

(a) (4 points) Find all critical numbers of the function \( f(x) \).

\[
f'(x) = 3x^2 - 24x, \quad f'(x) = 0 \Rightarrow 3x(x-8) = 0
\]

\[\Rightarrow x = 0, \quad x = 8\]
(b) (4 points) Make a sign chart for \( f'(x) \) to find out the intervals where \( f(x) \) is increasing, where \( f(x) \) is decreasing.

\[
\begin{array}{c|cccc}
  x & 0 & 2 & 8 \\
  f'(x) & + & 0 & - & 0 & + \\
\end{array}
\]

\[ L_{\text{max}} \] \quad \[ L_{\text{min}} \]

(c) (4 points) Look at the sign chart in (b) to see at which values of \( x \), \( f(x) \) has local extrema. Then evaluate any local minimum, local maximum of \( f(x) \).

\[
\text{local max } = f(0) = 5 \\
\text{local min } = f(8) = -251
\]

4. Let \( f(x) = x^4 - 4x^3 + 5 \) be a function.

(a) (2 point) Find \( f''(x) \)

\[
\begin{align*}
f'(x) &= 4x^3 - 12x^2 \\
f''(x) &= 12x^2 - 24x
\end{align*}
\]

(b) (5 points) Make a sign chart for \( f''(x) \) to find concavity of \( f(x) \) and all possible inflection points of \( f(x) \).

\[
f''(x) = 0 \Rightarrow 12x(x-2) = 0 \Rightarrow x = 0, 2
\]

\[
\begin{array}{c|cccc}
  x & 0 & 2 \\
  f''(x) & + & 0 & - & 0 & + \\
  f(x) & \bigcup & \bigcap & \bigcup & \bigcup \\
\end{array}
\]

Inflection pts: \((0, 5)\) \((2, -11)\)
5. Evaluate the global extrema of each function at the given interval.

(a) (3 points) \( f(x) = 2x^3 + 9x^2 + 12x - 4 \) on \([-2, 2]\).

\[
\frac{d}{dx}f(x) = 6x^2 + 18x + 12 \Rightarrow f'(x) = 0 \Rightarrow x^2 + 3x + 2 = 0 \Rightarrow x = -1, x = -2
\]
\[
f(-1) = -8 \quad \text{global minimum on} \quad [-2, 2]
\]
\[
f(2) = 72 \quad \text{global maximum}
\]

(b) (4 points) \( f(x) = \frac{x^2 + 1}{x^3 + 2} \) on \([-1, 3]\).

\[
\frac{d}{dx}f(x) = \frac{2x(x^2+2)-(x^2+1)(2x)}{(x^2+2)^2} = \frac{2x^3 + 4x - 2x^3 - 2x}{(x^2+2)^2}
\]
\[
\Rightarrow f'(x) = \frac{2x}{(x^2+2)^2}, \quad f'(x) = 0 \Rightarrow x = 0
\]
\[
f(-1) = \frac{2}{3}
\]
\[
f(0) = \frac{1}{2} = \text{global min on} \quad [-1, 3]
\]
\[
f(3) = \frac{10}{11} = \text{global max}
\]