MATH 163A, Fall Quarter 2001, MIDTERM 3

Student's Name (in capital letters):

Show all your work to get full credit. No work will amount to no credit. Circle your final answers.

1. Find the absolute maximum, absolute minimum of each function on a given closed interval. (The use of a graphic calculator is not allowed.)

   (a) (3 points)  \( f(x) = x^4 - 2x^2 \) on \([-1, 2]\).

   \[ f'(x) = 4x^3 - 4x \] \[ f'(x) = 0 \Rightarrow 4x^3 - 4x = 0 \Rightarrow 4x(x^2 - 1) = 0 \] \[ \Rightarrow x = 0, \quad x = \pm 1 \quad \text{all are in } [-1, 2]. \text{ Hence} \]

   \[ f(-1) = -1 \quad \text{abs. min} \]
   \[ f(1) = -1 \quad 4 \]
   \[ f(0) = 0 \]
   \[ f(2) = 16 - 8 = 8 \quad \text{abs. max} \]

   (b) (4 points)  \( f(x) = x^3 - 3x^2 - 9x + 1 \) on \([-2, 4]\).

   \[ f'(x) = 3x^2 - 6x - 9 \Rightarrow f'(x) = 0 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow x = -1, 3 \]

   Hence:

   \[ f(-2) = -1 \]
   \[ f(-1) = 6 \quad \text{absolute max.} \]
   \[ f(3) = -26 \quad 4 \quad \text{abs. min.} \]
   \[ f(4) = -19 \]

2. Find nonnegative numbers \( x, y \) satisfying \( 2x + y = 150 \) such that the product \( P = x^2y \) is maximum, by doing the following steps:

   (a) (1 point) Solve for \( y \) from \( 2x + y = 150 \).

   \[ y = 150 - 2x \]
(b) (1 points) Substitute \( y \) (found in part (a)) in the function \( P = x^2y \) to get a function \( P(x) \) depending only on \( x \).

\[
P'(x) = x^2(150 - 2x) = x^2 150 - 2x^3
\]

(c) (4 points) Find the closed interval for \( x \), and maximize the function \( P(x) \) (this means, find the absolute maximum of \( P(x) \) on the closed interval you just found for \( x \)).

\[
x \geq 0; \quad 150 - 2x \geq 0 \quad \Rightarrow \quad 150 \leq 2x \quad \Rightarrow \quad 75 \leq x, \quad \text{interval is:} \quad [0, 75]
\]

\[
P'(x) = 300x - 6x^2 \quad \Rightarrow \quad p'(x) = 0 \quad \text{gives} \quad 6x(50 - x) = 0
\]

\[
\Rightarrow \quad x = 0, \quad x = 50
\]

\[
P(50) = 50^2(150 - 100) = 50^3 = 125,000 \quad \text{abs. max}
\]

Answer: when \( x = 50 \), \( y = 150 - 2x = 50 \), the product \( x^2y \) will be maximum.

3. Let \( f(x) = x^3 - 6x^2 \) be a given function. Do the following:

(a) (1 points) Find \( y \)-intercept.

\[
f(0) = 0
\]

(b) (3 points) Find \( x \)-intercepts.

\[
x^3 - 6x^2 = 0 \quad \Rightarrow \quad x^2(x - 6) = 0 \quad \Rightarrow \quad x = 0, \quad x = 6
\]
(c) (1 point) Find \( f'(x) \).
\[
\frac{d}{dx}(x^3 - 12x) = 3x^2 - 12x
\]

(d) (3 points) Find all critical numbers of \( f(x) \)
\[
3x^2 - 12x = 0 \implies 3x(x - 4) = 0, \quad x = 0, x = 4
\]

Make a sign chart for \( f'(x) \) to find out:

(e) (4 points) The intervals where \( f(x) \) is increasing, where it is decreasing.

(f) (3 points) Any local maximum, local minimum (give the values of them).
\[
\text{Local max} = f(0) = 0 \\
\text{local min} = f(4) = 64 - 6(16) = 64 - 96 = -32
\]

(h) (2 point) Find \( f''(x) \), and find all numbers which make \( f''(x) \) equal zero.
\[
f''(x) = 6x - 12, \quad f''(x) = 0 \implies 6x - 12 = 0 \\
\Rightarrow x = 2.
\]
Make a sign chart for $f''(x)$ to find out:

(i) (3 points) Intervals where $f(x)$ is concave up, where it is concave down.

(k) (3 points) All inflection points.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f''(x)$</th>
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<tbody>
<tr>
<td></td>
<td>$- - - - 2$</td>
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<td>$0 + + + +$</td>
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Inflection point $= (2, f(2)) = (2, -16)$

(l) (4 points) Use all information above to sketch the graph of $f(x)$. Attn: The use of a graphic calculator is not allowed.