ON COUNTABLY COMPACT SPACES SATISFYING wD HEREDITARILY

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Abstract. We show that in the class of countably compact regular spaces, the assumption that our space satisfies Property wD hereditarily has an influence on the relationship between tightness and hereditary π–character. We close with some comments on countably compact topological groups — among other things, we show that PFA implies that all countably compact T_5 groups are metrizable.

Definition 1. A space X said to satisfy Property wD (or more succinctly, satisfy wD) if for every infinite closed discrete subspace C of X, there is a discrete collection $\{U_n : n \in \omega\}$ of open subsets of X such that $U_n \cap U_m = \emptyset$ if $m \neq n$ and each U_n meets C is exactly one point. X satisfies wD hereditarily if each subspace Y of X satisfies Property wD.

It is not hard to prove that normal spaces satisfy wD, and also that first countable countably compact spaces satisfy wD hereditarily.

Definition 2. A space X is said to be countably tight (written $t(X) = \aleph_0$) if for each $A \subseteq X$ and each point $x \in \overline{A}$, there is a countable subset B of A such that x is in the closure of B. A subset A of X is said to be sequentially closed if A contains all limit points of convergent sequences from A. A space X is said to be sequential if sequentially closed subsets of X are closed.

In the class of compact Hausdorff spaces, the relationship between having countable tightness and being sequential takes us beyond the realm of ZFC. It is not hard to see that a sequential space is is countably tight. Assuming the axiom $\diamondsuit$, there is a construction due to Fedorcuk [4] of a compact space of countable tightness that is not sequential. It is a celebrated result of Balogh [1] (who built on work of Fremlin and Nyikos, see [2]) that ZFC is consistent with the assumption that all countably tight compact spaces are sequential. It is an open problem as to whether CH is enough to build a countably tight compact space that is not sequential. The results obtained in this paper (particularly Theorem 1) have some relevance to this problem

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in the special case where the compact space satisfies wD hereditarily; this will appear in a forthcoming paper.

In the course of our investigations, we will need another cardinal invariant.

Definition 3. If X is a topological space and $x \in X$, then a collection $\mathcal{B}$ of open sets is said to be a π–base for x in X if whenever U is an open neighborhood of x, there is a $V \in \mathcal{B}$ such that $V \subseteq U$. This differs from the definition of a base in that we do not require that x is an element of V. The π–character of x in X (denoted $\pi\chi(x, X)$) is the least cardinality of a π–base for x. The π–character of X is defined by

$$\pi\chi(X) = \sup\{\pi\chi(x, X) : x \in X\} + \omega. \tag{0.1}$$

We say X is hereditarily of countable π–character ($h\pi\chi(X) = \aleph_0$) if every subspace of X has countable π–character.

It is not hard to show that $t(X) \leq h\pi\chi(X)$. A remarkable result due to Shapirovskii (see [9] or [6]) is that for compact spaces, $t(X) = h\pi\chi(X)$. This result will be used in the sequel.

We begin with an easy observation.

Lemma 4. Let X be a countably compact T_3 space. Then X satisfies wD hereditarily if and only if whenever $D \subseteq X$ is a countably infinite discrete set, there is a family of open sets $\{U_n : n \in \omega\}$ such that each U_n meets D and the sequence $\{U_n : n \in \omega\}$ converges to $\overline{D} \setminus D$.

Proof. Assume first that X satisfies wD hereditarily, and let $D \subseteq X$ be a countably infinite discrete set. Note that $K = \overline{D} \setminus D$ is a non–empty closed subset of X, and D is a closed discrete subset of $X \setminus K$. If we apply wD to D in the space $X \setminus K$, we get an infinite subset of D that expands to a discrete (in $X \setminus K$) family of open sets $\{U_n : n \in \omega\}$. Each U_n is open in X, and clearly each U_n meets D. Since X is countably compact and the sequence $\{U_n : n \in \omega\}$ is discrete in $X \setminus K$, it must be the case that every neighborhood V of K contains all but finitely many of the U_n's.

Conversely, suppose that Y is a subset of X, and $D \subseteq Y$ is closed and discrete in Y. Since D is discrete in X, there is a family of open sets $\{U_n : n \in \omega\}$ that converges to $K = \overline{D} \setminus D$ and such that each U_n meets D. Since D is discrete, without loss of generality $U_n \subseteq X \setminus K$, U_n's are disjoint, and $|U_n \cap D| = 1$.

We claim that the U_n's are discrete in $X \setminus K$. To see this, suppose $x \notin K$, and let V be a neighborhood of x such that $\overline{V} \cap K = \emptyset$. Now V can meet only finitely many of the U_n's as they converge to K, and so we can find a neighborhood of x that meets at most one of the U_n's.

Now if we let $x_n = D \cap U_n$, then we have the required subset of D that expands to a discrete family of open subsets of Y, as witnessed by the open sets $U_n \cap Y$. □

The following corollary is folklore.

Corollary 5. Suppose X is a countably compact space satisfying wD hereditarily. If $S = \{x_n : n \in \omega\}$ is a non–trivial sequence converging to a point x, then S has an infinite subsequence that expands to a family $\{U_n : n \in \omega\}$ that converges to x.

Proof. By the lemma, there is a family of open sets $\{U_n : n \in \omega\}$ every member of which meets S such that $\{U_n : n \in \omega\}$ converges to $\bar{S} \setminus S = \{x\}$. □

It is an open problem as to whether "X is sequential" and "countably compact subsets of X are closed" are equivalent in compact spaces. The equivalence is known to hold if $2^{\mathfrak{t}} > \mathfrak{c}$, so in particular it holds if either $2^{\aleph_0} < 2^{\aleph_1}$ or MA holds. The following corollary tells us that in the class of compact Hausdorff spaces that satisfy wD hereditarily, the equivalence holds in ZFC. (Note that obviously a sequential space has the property that countably compact subspaces are compact.)

Corollary 6. Assume X is a countably compact space that satisfies wD hereditarily. If X is not sequential, then there is a closed $A \subseteq X$ and a point x such that $x \in \overline{A \setminus \{x\}}$ and $A \setminus \{x\}$ is countably compact.

Proof. Let $B \subseteq X$ be sequentially closed but not closed, and let $A = \overline{B}$. Choose any point $x \in A \setminus B$, and we claim that $A \setminus \{x\}$ is countably compact.

If this is not the case, then we can choose an infinite closed discrete $D \subseteq A \setminus \{x\}$. Since A is countably compact, this means that D has to converge to x, and hence by Corollary 5 D has a subsequence that expands to a sequence of open sets $\{U_n : n \in \omega\}$ that converges to x. Each of these open sets must meet B, and thus we get a sequence of points in B that converges to x, contradicting the assumption that B is sequentially closed. □

The following theorem is the main result of this note. Again, we use Lemma 4 in a crucial fashion.

Theorem 1. *Let X be a countably compact T_3 space that satisfies wD hereditarily. Let $A \subseteq X$ be countable, and suppose $x \in \overline{A} \setminus A$. If x has countable π–character in the space $\overline{A} \setminus A$, then x has countable π–character in X, and furthermore there is a countable π–base for x in X, every member of which meets A.*

Proof. Since X is T_3 and $\pi\chi(x, \overline{A} \setminus A) = \aleph_0$, there is a countable family $\{V_n : n \in \omega\}$ of open sets that each meet $\overline{A} \setminus A$ and such that whenever V is a neighborhood of x, there is an n with $\overline{V}_n \cap (\overline{A} \setminus A) \subseteq V$.

For each n, let $A_n = \overline{V}_n \cap A$. Each A_n is countable, and since $V_n \cap (\overline{A}_n \setminus A_n)$ is non–empty, we know each A_n is not closed. Thus we can find an infinite $D_n \subseteq A_n$ that is closed and discrete as a subset of A_n. This means that D_n is a discrete subset of X, and $\overline{D}_n \cap A = D_n$.

Let $K_n = \overline{D}_n \setminus D_n$. Since $\overline{D}_n \cap A = D_n$, we know $K_n \subseteq \overline{V}_n \cap (\overline{A} \setminus A)$. Now apply the preceding lemma to get a countable family $\mathcal{S}_n$ of open sets, each of which meet D_n, that converges to K_n.

Now we define $\mathcal{B}_{x,A} = \bigcup_{n\in\omega} \mathcal{S}_n$. Clearly $\mathcal{B}_{x,A}$ is a countable family of open sets, each of which meets A; so we need only verify that $\mathcal{B}_{x,A}$ is a π–base for x.

Let V be any neighborhood of x. There is some n with $\overline{V}_n \subseteq V$, hence $K_n \subseteq V$. Since $\mathcal{S}_n$ converges to K_n, this means that V contains all but finitely many members of $\mathcal{S}_n$. Since V was an arbitrary neighborhood of x, we have that $\mathcal{B}_{x,A}$ is a π–base for x in X. □

Corollary 7. Let X be a countably compact T_3 space satisfying wD hereditarily such that $t(X) = h\pi\chi(X) = \aleph_0$. If $A \subseteq X$ and $x \in \overline{A \setminus \{x\}}$, then x has a countable π–base, every member of which meets A.

Proof. Since $t(X) = \aleph_0$, there is a countable $A_0 \subseteq A \setminus \{x\}$ with $x \in \overline{A}_0$. By assumption, x has countable π–character in $\overline{A}_0 \setminus A_0$, and now we can apply the theorem. □

Corollary 8. Let X be a countably tight compact space satisfying wD hereditarily. If $A \subseteq X$ and $x \in \overline{A \setminus \{x\}}$, then x has a countable π–base in X, every member of which meets A.

Proof. By Shapirovskii's theorem, compact spaces of countable tightness are hereditarily of countable π–character. Now the previous corollary gives us the required π–base. □

The preceding corollary does not hold for countably tight compact spaces in general. A counterexample that is provided by the Thomas plank. We are indebted to Peter Nyikos for this example:

Let W be an uncountable discrete space and let $W + \infty$ denote its one–point compactification. Let $X = (W + \infty) \times (\omega + 1)$; X is Fréchet hence countably tight. Let x_n be the point $\langle \infty, n\rangle$; the points $\{x_n : n \in \omega\}$ converge to the point $x = \langle \infty, \omega\rangle$. Now no infinite subset of the convergent sequence expands to a π–base for x in X — if U_n is a neighborhood of x_n, then U_n contains all but finitely many points of the nth row. Given an infinite $I \subseteq \omega$ and open sets $\{U_n : n \in I\}$ such that U_n is a neighborhood of x_n, we can find a point $z \in W$ such that $\langle z, n\rangle \in U_n$ for all $n \in I$. Now the open set $Y = X \setminus [\{z\} \times (\omega + 1)]$ is a neighborhood of x that does not contain any of the U_n's.

Theorem 2. *Let X be an ω–bounded T_3 space that satisfies wD hereditarily. Then $t(X) = \aleph_0$ if and only if $h\pi\chi(X) = \aleph_0$.*

Proof. Since $t(X) \leq h\pi\chi(X)$, we need only prove one direction. Assume $t(X) = \aleph_0$ and let $x \in Y \subseteq X$; we must prove that x has countable π–character in Y.

If x is isolated in Y there is nothing to prove, so we can assume that x is a limit point of Y. Since X is countably tight, there is a countable $A \subseteq Y \setminus \{x\}$

such that $x \in \mathrm{cl}_X A$. Since X is ω–bounded, we know that $K = \mathrm{cl}_X A$ is compact. Since K is also countably tight, we can apply Shapirovskii's theorem to conclude that $h\pi\chi(K) = \aleph_0$ and hence $\pi\chi(x, K \setminus A) = \aleph_0$. Now Theorem 1 tells us that there is a countable π–base $\{U_n : n \in \omega\}$ for x in X, every member of which hits A. The family $\{U_n \cap Y : n \in \omega\}$ is then a π–base for x in Y. □

Corollary 9. Assume PFA holds and X is a countably compact T_5 space. Then $t(X) = \aleph_0$ if and only if $h\pi\chi(X) = \aleph_0$.

Proof. By a result of Nyikos [8], if PFA holds then every separable, countably compact T_5 space is compact. This implies that X is ω–bounded. The fact that X satisfies wD hereditarily follows from the hereditary normality of X. Now we can apply the previous theorem. □

We remark that Hajnal and Juhasz [5] have constructed from CH a countably compact, countably tight non–compact T_5 space in which no point is of countable π–character, and so the conclusion of the previous corollary is independent of ZFC. The space constructed by Hajnal and Juhasz also happens to be a topological group, so we take a moment to point out some consequences of the results of this paper in the area of topological groups. The only facts about topological groups that we use are that a topological group is metrizable if and only if it is first countable, and that for topological groups character and π–character are the same. Proofs of both of these can be found in Comfort's survey paper [3].

Theorem 3. *An infinite countably compact topological group is metrizable if and only if it satisfies wD hereditarily and it contains an infinite compact subset of countable tightness.*

Proof. An infinite first countable, countably compact space satisfies wD hereditarily and contains convergent sequences, so necessity is established. For the other direction, let G be a countably compact topological group that satisfies wD hereditarily and that contains an infinite compact subset K such that $t(K) = \aleph_0$. Since G is homogeneous, it suffices to produce a point $x \in G$ such that $\pi\chi(x, G) = \aleph_0$.

Since K is infinite and countably tight, we can find a point $x \in K$ and a countable set $A \subseteq K \setminus \{x\}$ such that $x \in \overline{A}$. By Shapirovskii's theorem, we know that $h\pi\chi(K) = \aleph_0$ and hence $\pi\chi(x, \overline{A} \setminus A) = \aleph_0$. Now we can forget about K and apply Theorem 1 to conclude that x has countable π–character in G. □

Corollary 10. Assume PFA holds. Then all countably compact T_5 groups are metrizable.

Proof. Let G be such a topological group. If G is finite, then it has the discrete topology and hence is trivially metrizable, so we may assume G is infinite. Under PFA, the closure of a countable subset of a countably compact T_5 space is compact and Fréchet, hence countably tight—this is

Theorem 1.4 of Nyikos' paper [8]. Thus G contains an infinite compact subset of countable tightness. Since hereditarily normal spaces satisfy wD hereditarily, we can apply Theorem 3 to conclude that G is metrizable. $\square$

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