

SIMULTANEOUS REFLECTION AND IMPOSSIBLE IDEALS

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ABSTRACT. We prove that if $\mu^+ \rightarrow [\mu^+]_{\mu^+}^2$ holds for a singular cardinal μ , then any collection of fewer than $\text{cf}(\mu)$ stationary subsets of μ^+ must reflect simultaneously.

1. INTRODUCTION

The main theorem of this paper is easily stated:

Main Theorem. *If μ is a singular cardinal and $\mu^+ \rightarrow [\mu^+]_{\mu^+}^2$, then every collection of fewer than $\text{cf}(\mu)$ stationary subsets of μ^+ reflects simultaneously.*

Tucked within the statement of the theorem, however, are two bits of jargon which deserve a little explanation, and so we will record the needed definitions before discussing the result further.

The symbol $\mu^+ \rightarrow [\mu^+]_{\mu^+}^2$ denotes a square-brackets partition relation, and it asserts that for any function $f : [\mu^+]^2 \rightarrow \mu^+$, there is an $H \subseteq \mu^+$ of cardinality μ^+ such that the range of $f \upharpoonright [H]^2$ is a proper subset of μ^+ . Said another way, $\mu^+ \rightarrow [\mu^+]_{\mu^+}^2$ means that whenever we color the pairs of ordinals drawn from μ^+ with μ^+ colors, there is a set H of cardinality μ^+ on which the coloring omits at least one value, that is, H is homogeneous for the coloring in a very weak sense. Expressions of the form $\kappa \rightarrow [\kappa]_{\kappa}^{<\omega}$ and $\kappa \not\rightarrow [\kappa]_{\kappa}^2$ should be given the obvious meaning. We recommend Chapter XI of [6], Chapter 20 of [8], or section 8.2 of [10] for a general treatment of such relations, while the last section of [4] discusses some of what is known for successors of singular cardinals.

Moving further into the statement of the theorem, recall that a stationary set $S \subseteq \mu^+$ is said to *reflect at* $\delta < \mu^+$ if $S \cap \delta$ is a stationary subset of δ . We will follow the notation in Jech's article [9] in the Handbook of Set Theory [7] and define

$$(1.1) \quad \text{Tr}(S) := \{\delta < \mu^+ : S \text{ reflects at } \delta\}.$$

The conclusion of the theorem then asserts that whenever $\langle S_\alpha : \alpha < \kappa \rangle$ is a collection of stationary subsets of μ^+ with $\kappa < \text{cf}(\mu)$, then there is a single $\delta < \mu^+$ such that each S_α reflects at δ .

There is a connection between square-brackets partition relations and stationary reflection. For example, seminal work of Todorćević [14] demonstrated that the relation $\kappa \rightarrow [\kappa]_{\kappa}^2$ implies that all stationary subsets of κ reflect (Chapter 20 of [8] contains a nice proof of this, while Todorćević's monograph [15] is devoted to this and related matters.) It follows easily that $\kappa^+ \not\rightarrow [\kappa^+]_{\kappa^+}^2$ whenever κ is *regular*, but

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we are not so lucky at successors of singular cardinals, as it is consistent (assuming large cardinals) that every stationary subset of such a cardinal reflects. The question of whether $\mu^+ \rightarrow [\mu^+]_{\mu^+}^2$ can hold for μ singular is still very much open; this is but one of a whole family of problems asking about the extent to which “coloring theorems” necessarily hold at successors of singular cardinals.

Recent work of the author [3] (building on work of Shelah in Chapter III of [13]) lets one conclude that if $\mu^+ \rightarrow [\mu^+]_{\mu^+}^2$ for μ singular, then there is a $\theta < \mu$ such that any collection of fewer than $\text{cf}(\mu)$ stationary subsets of $\{\delta < \mu^+ : \text{cf}(\delta) \geq \theta\}$ must reflect simultaneously. The main theorem of this paper therefore gives us the logical next step in this line of research by removing the cofinality restriction on the stationary sets involved.

2. THE IDEAL EXISTS

Our proof of the main theorem makes use of a contradiction arising in the following manner: assuming the failure of a condition which implies our result, we show that there is an ideal possessing quite strong properties, and then we demonstrate that no such ideal can exist. This explains the phrase “impossible ideals” in the title of the paper, and also suggests that we should spend a little time on definitions taken from the theory of ideals. We warn the reader that unless we indicate otherwise, by “ I is an ideal on κ ”, we shall mean “ I is a proper ideal on κ containing the bounded subsets of κ ”.

Definition 2.1. Let I be an ideal on the cardinal κ .

- (1) I^+ is the collection of I -positive subsets of κ , that is, those subsets of κ that are not in I .
- (2) I^* is the dual filter to I , that is, those subsets of κ whose complements are in I .
- (3) If σ is a cardinal, then I is σ -complete if I is closed under *arbitrary* unions of length *less than* σ .
- (4) If σ is a regular cardinal, then we say I is σ -indecomposable if I is closed under *increasing* unions of length σ .
- (5) I is weakly θ -saturated for a cardinal θ if there do not exist θ disjoint members of I^+ . This is equivalent to there being no function $p : \kappa \rightarrow \theta$ with $p^{-1}(\{\alpha\}) \in I^+$ for each $\alpha < \theta$.

One more bit of standard terminology will be needed: If $\tau < \kappa$ are cardinals, then we set

$$(2.1) \quad S_\tau^\kappa := \{\delta < \kappa : \text{cf}(\delta) = \tau\}.$$

Variants of the above notation should be given the obvious interpretation. We note that if S is a stationary subset of S_τ^κ and S reflects at δ , then $\text{cf}(\delta) > \tau$. This follows easily, as ordinals of cofinality at most τ have closed unbounded subsets consisting of ordinals of cofinality less than τ .

Given all of the above terminology, the statement of the following lemma should now make sense.

Lemma 2.2. Suppose μ is a singular cardinal and there are cardinals $\tau < \theta < \mu$, and stationary $S \subseteq S_\tau^{\mu^+}$ such that S does not reflect in an ordinal of cofinality greater than θ . Then there are an ideal I and a coloring $c : [\mu^+]^2 \rightarrow \mu^+$ such that

- I is a proper ideal on μ^+ extending the non-stationary ideal,
- I is τ -complete,
- I is σ -indecomposable for all regular $\sigma < \mu$ with $\sigma \neq \tau$, and
- for every unbounded $A \subseteq \mu^+$,

$$(2.2) \quad \text{ran}(c \upharpoonright [A]^2) \in I^*.$$

Proof. By Claim 2.4 (and Remark 2.4A) on page 126 of [13], there is a collection $\bar{C} = \langle C_\delta : \delta \in S \rangle$ such that

- C_δ is club in δ of order-type τ ,
- $\alpha \in \text{nacc}(C_\delta) \implies \text{cf}(\alpha) > \theta$, and
- for every club $E \subseteq \mu^+$ there are stationarily many $\delta \in S$ with $C_\delta \subseteq E$.

For each $\delta \in S$ let I_δ be the ideal on C_δ generated by the bounded subsets together with $\text{acc}(C_\delta)$, the *accumulation points* of C_δ and let I be the ideal $\text{id}_p(\bar{C}, \bar{I})$ from Chapter III of [13], defined by putting a subset A of μ^+ into I if there is a closed unbounded $E \subseteq \mu^+$ such that

$$(2.3) \quad \delta \in S \cap E \implies A \cap E \cap C_\delta \in I_\delta.$$

Note that each I_δ is τ -complete and σ -indecomposable for all regular $\sigma < \mu$ other than τ . It follows easily (see Observation 3.2(1) on page 139 of [13]) that I is an ideal satisfying the first three requirements demanded by Lemma 2.2.

Before we commence with the construction of the coloring c , we note the following characterization of I^* :

- ⊗ A set A is in I^* if and only if there is a club $E \subseteq \mu^+$ such that for all $\delta \in S \cap E$, there is a $\gamma^\otimes < \delta$ such that

$$(2.4) \quad E \cap \text{nacc}(C_\delta) \setminus \gamma^\otimes + 1 \subseteq A,$$

where $\text{nacc}(C_\delta) := C_\delta \setminus \text{acc}(C_\delta)$, the *non-accumulation points* of C_δ .

The coloring c is actually well-known — we essentially use Todorćević’s original square-bracket operation defined using minimal walks. We shall be a bit more precise in a moment, but first let us recall that a sequence $\bar{e} = \langle e_\alpha : \alpha < \lambda \rangle$ is called a C -sequence for the cardinal λ if e_α is closed unbounded in α for each $\alpha < \lambda$. Given $\alpha < \beta < \lambda$ the *minimal walk from β to α along \bar{e}* is defined to be the sequence

$$(2.5) \quad \beta = \beta_0 > \cdots > \beta_n = \alpha$$

obtained by setting

$$(2.6) \quad \beta_{i+1} = \min(e_{\beta_i} \setminus \alpha),$$

as long as $\beta_i > \alpha$. We will write “ $\beta_i(\alpha, \beta)$ ” instead of just “ β_i ” in the sequel to emphasize the dependence on α and β .

Our plan is to construct a certain C -sequence \bar{e} from the ideal I , and then show that if we use minimal walks along this particular \bar{e} to implement Shelah’s simplified version [12] of Todorćević’s operation, then the resulting coloring has all of the properties we need.¹

The following proposition gives us our \bar{e} ; the proof is a variant Shelah’s “ladder swallowing trick” from Chapter III of [13].

¹Other choices of coloring are possible here. For example, we could just as easily used the coloring from [2], but we wanted to demonstrate that the original colorings defined using minimal walks can also be exploited in these circumstances.

Proposition 2.3. There is a C -sequence $\langle e_\alpha : \alpha < \mu^+ \rangle$ such that

$$(2.7) \quad \delta \in S \cap e_\alpha \implies \text{nacc}(C_\delta) \subseteq \text{nacc}(e_\alpha).$$

Notice that such a C -sequence must also satisfy

$$(2.8) \quad \delta \in S \cap e_\alpha \implies C_\delta \subseteq e_\alpha,$$

as C_δ and e_α are both closed.

Proof. Let $\bar{e}^* = \langle e_\alpha^* : \alpha < \mu^+ \rangle$ be a C -sequence satisfying the following conditions:

- $\alpha = \beta + 1 \implies e_\alpha^* = \{\beta\}$,
- $\text{otp}(e_\alpha^*) = \text{cf}(\alpha)$, and
- If $S \cap \alpha$ is non-stationary in α , then $e_\alpha^* \cap S = \emptyset$

There is no problem finding such \bar{e}^* , and this choice ensures $e_\alpha^* \cap S$ is empty except for the cases where α is the successor of an ordinal in S , or S reflects at α . Note in the latter circumstance that $\tau < \text{cf}(\alpha) \leq \theta$.

We now build $\langle e_\alpha : \alpha < \mu^+ \rangle$ satisfying the needed property. We start by setting $e_\alpha = e_\alpha^*$ unless $\alpha = \delta + 1$ for some $\delta \in S$, or $S \cap \alpha$ is stationary in α . If $\alpha = \delta + 1$ for some $\delta \in S$, then we set

$$(2.9) \quad e_\alpha = e_{\delta+1} = C_\delta \cup \{\delta\}.$$

Notice that $\text{nacc}(C_\delta) \subseteq \text{nacc}(e_\alpha)$ in this trivial case.

It remains to consider the case when $S \cap \alpha$ is stationary in α , and this is handled by a straightforward construction of length $\text{cf}(\alpha)$ in which we produce objects $e_\alpha[\xi]$ for each $\xi < \text{cf}(\alpha)$:

Given $\alpha < \mu^+$ with $S \cap \alpha$ stationary in α , we define

$$(2.10) \quad e_\alpha[0] = e_\alpha^*,$$

$$(2.11) \quad e_\alpha[\xi + 1] = \text{closure in } \alpha \text{ of } e_\alpha[\xi] \cup \bigcup \{C_\delta : \delta \in S \cap e_\alpha[\xi]\},$$

$$(2.12) \quad e_\alpha[\xi] = \text{closure in } \alpha \text{ of } \bigcup_{\zeta < \xi} e_\alpha[\zeta], \text{ if } \xi \text{ is a limit,}$$

and finally

$$(2.13) \quad e_\alpha = \text{closure in } \alpha \text{ of } \bigcup_{\xi < \theta} e_\alpha[\xi].$$

Notice that

$$(2.14) \quad |e_\alpha[\xi + 1]| \leq |e_\alpha[\xi]| + |e_\alpha[\xi]| \cdot \tau,$$

and since $\text{cf}(\alpha)$ is a regular cardinal greater than τ , an easy argument tells us

$$(2.15) \quad |e_\alpha| = \text{cf}(\alpha).$$

Next suppose $\delta \in e_\alpha \cap S$. Since $\text{cf}(\delta) < \text{cf}(\alpha)$, it must be the case that $\delta \in e_\alpha[\xi]$ for some $\xi < \text{cf}(\alpha)$ and hence

$$(2.16) \quad C_\delta \subseteq e_\alpha[\xi + 1] \subseteq e_\alpha.$$

Moreover, since $\text{cf}(\alpha) \leq \theta$ by our assumption on S and all elements of $\text{nacc}(C_\delta)$ are of cofinality greater than θ by our choice of \bar{C} , it follows that

$$(2.17) \quad \text{nacc}(C_\delta) \subseteq \text{nacc}(e_\alpha),$$

as required. \square

Observe our assumption that S does not reflect in ordinals of cofinality greater than θ , already used to pick out the ideal I , also plays a critical role in the proof of the above proposition. Let us now fix a C -system $\bar{e} = \langle e_\alpha : \alpha < \mu^+ \rangle$ as in the preceding proposition, and turn once more to definitions needed to define the coloring c .

We will make use of several functions defined using minimal walks. These functions are all standard in this context, although the notation used tends to vary from author to author. The most basic function we consider is the function $\rho_2 : [\lambda]^2 \rightarrow \omega$ giving the length of the walk from β to α , that is,

$$(2.18) \quad \rho_2(\alpha, \beta) = \text{least } i \text{ for which } \beta_i(\alpha, \beta) = \alpha.$$

Next, for $i \leq \rho_2(\alpha, \beta)$, we set

$$\beta_i^-(\alpha, \beta) = \begin{cases} 0 & \text{if } i = 0, \\ \sup(e_{\beta_j(\alpha, \beta)} \cap \alpha) & \text{if } i = j + 1 \text{ for } j < \rho_2(\alpha, \beta). \end{cases}$$

Thus, for $0 < i < \rho_2(\alpha, \beta)$, the ordinals $\beta_i^-(\alpha, \beta)$ and $\beta_i(\alpha, \beta)$ are the two consecutive elements in $e_{\beta_{i-1}(\alpha, \beta)}$ which bracket α .

Finally, we define for $k \leq \rho_2(\alpha, \beta)$

$$(2.19) \quad \gamma_k(\alpha, \beta) := \max\{\beta_\ell^-(\alpha, \beta) : \ell \leq k\}.$$

Standard arguments show us

$$(2.20) \quad \gamma_k(\alpha, \beta) < \alpha \text{ for } k < \rho_2(\alpha, \beta)$$

and

$$(2.21) \quad \gamma_{\rho_2(\alpha, \beta)}(\alpha, \beta) < \alpha \iff \alpha \in \text{nacc}(e_{\beta_{\rho_2(\alpha, \beta)-1}(\alpha, \beta)}).$$

The main (and well-known) property of minimal walks which we need is the following:

Proposition 2.4. For $k \leq \rho_2(\alpha, \beta)$, if $\gamma_k(\alpha, \beta) < \alpha^* \leq \alpha$, then

$$(2.22) \quad \langle \beta_\ell(\alpha, \beta) : \ell \leq k \rangle = \langle \beta_\ell(\alpha^*, \beta) : \ell \leq k \rangle$$

Proof. This follows by an easy induction; or see Chapter 20 of [8]. \square

Corollary 2.5. If $\gamma_{\rho_2(\alpha, \beta)}(\alpha, \beta) < \alpha$, then for any α^* satisfying

$$(2.23) \quad \gamma_{\rho_2(\alpha, \beta)}(\alpha, \beta) < \alpha^* \leq \alpha,$$

we have

$$(2.24) \quad \langle \beta_\ell(\alpha^*, \beta) : \ell \leq \rho_2(\alpha, \beta) \rangle = \langle \beta_\ell(\alpha, \beta) : \ell \leq \rho_2(\alpha, \beta) \rangle.$$

In particular,

$$(2.25) \quad \beta_{\rho_2(\alpha, \beta)}(\alpha^*, \beta) = \alpha,$$

so the walk from β down to α^* passes through α .

Note that (2.24) implies, for example, that $\langle \beta_\ell(\alpha^*, \beta) : \ell \leq \rho_2(\alpha, \beta) \rangle$ is the same as $\langle \beta_\ell(\alpha, \beta) : \ell \leq \rho_2(\alpha, \beta) \rangle$.

We are now in a position to define the coloring c ; as we mentioned before, this is just Shelah's version of Todorćević's square-brackets function, implemented using our specially constructed C -sequence as a parameter.

Definition 2.6. Given $\alpha < \beta < \mu^+$, we let $k(\alpha, \beta)$ be the maximal $k \leq \rho_2(\alpha, \beta)$ for which

$$(2.26) \quad \langle \beta_\ell^-(\alpha, \beta) : \ell \leq k \rangle = \langle \beta_\ell^-(\gamma_k(\alpha, \beta) + 1, \alpha) : \ell \leq k \rangle,$$

and then define

$$(2.27) \quad c(\alpha, \beta) := \beta_{k(\alpha, \beta)}(\alpha, \beta).$$

Given an unbounded $A \subseteq \mu^+$, our goal is to prove that there is a club $E \subseteq \mu^+$ such that for any $\delta \in S \cap E$, there is a $\gamma^\otimes < \delta$ such that

$$(2.28) \quad E \cap \text{nacc}(C_\delta) \setminus \gamma^\otimes + 1 \subseteq \text{ran}(c \upharpoonright [A]^2).$$

In order to define E , we first fix a μ^+ -approximating sequence $\langle M_i : i < \mu^+ \rangle$ over $\{A, \bar{C}, \bar{e}\}$, which means that $\langle M_i : i < \mu^+ \rangle$ is a continuous \in -chain of elementary submodels of $H(\chi)$ (for some sufficiently large regular χ) satisfying

- μ^+ , A , \bar{C} , and \bar{e} are all in M_0 ,
- $|M_i| < \mu^+$,
- $\langle M_j : j \leq i \rangle \in M_{i+1}$ for $i < \mu^+$, and
- $M_i \cap \mu^+$ is a proper initial segment of μ^+ for each i .

Given this sequence of models, we define

$$(2.29) \quad E := \{\delta < \mu^+ : M_\delta \cap \mu^+ = \delta\}.$$

Now assume $\delta \in S$ with $E \cap \text{nacc}(C_\delta)$ unbounded in δ (clearly we need only consider such δ). Since δ must be in E , we know $\delta = \sup(M_\delta \cap \mu^+)$. Choose $\beta \in A$ greater than $M_{\delta+1} \cap \mu^+$, and define

$$(2.30) \quad \gamma^\otimes := \min(C_\delta \setminus \gamma_{\rho_2(\delta, \beta)-1}(\delta, \beta)).$$

Now suppose $\epsilon \in \text{nacc}(C_\delta) \cap E \setminus \gamma^\otimes + 1$. We will find an $\alpha \in A$ such that $c(\alpha, \beta) = \epsilon$; the proof is not too difficult, but the notation is a bit cumbersome.

We observe first that our choices imply

$$(2.31) \quad \gamma_{\rho_2(\delta, \beta)-1}(\delta, \beta) < \epsilon < \delta,$$

and so

$$(2.32) \quad \langle \beta_\ell(\delta, \beta) : \ell < \rho_2(\delta, \beta) \rangle = \langle \beta_\ell(\epsilon, \beta) : \ell < \rho_2(\epsilon, \beta) \rangle.$$

In particular,

$$(2.33) \quad \beta_{\rho_2(\delta, \beta)-1}(\delta, \beta) = \beta_{\rho_2(\delta, \beta)-1}(\epsilon, \beta);$$

to make things a bit neater, we will refer to this ordinal as β^* .

Clearly $\delta \in e_{\beta^*}$, and so by our choice of \bar{e} we know

$$(2.34) \quad C_\delta \subseteq e_{\beta^*}$$

and

$$(2.35) \quad \epsilon \in \text{nacc}(e_{\beta^*}).$$

From (2.34), we conclude

$$(2.36) \quad \beta_{\rho_2(\delta, \epsilon)}(\epsilon, \beta) = \epsilon,$$

and

$$(2.37) \quad \rho_2(\delta, \beta) = \rho_2(\epsilon, \beta),$$

while from (2.35) and (2.21) we obtain

$$(2.38) \quad \gamma_{\rho_2(\epsilon, \beta)}(\epsilon, \beta) < \epsilon.$$

We can squeeze a little more information out of our situation, as the circumstances imply

$$(2.39) \quad \gamma_{\rho_2(\epsilon, \beta)}(\epsilon, \beta) = \beta_{\rho_2(\epsilon, \beta)}^-(\epsilon, \beta).$$

This is the case because

$$(2.40) \quad \gamma^\otimes \in C_\delta \cap \epsilon \subseteq e_{\beta^*} \cap \epsilon,$$

and so

$$(2.41) \quad \gamma_{\rho_2(\epsilon, \beta)-1}(\epsilon, \beta) \leq \gamma^\otimes \leq \beta_{\rho_2(\epsilon, \beta)}^-(\epsilon, \beta).$$

Let us therefore define

$$(2.42) \quad \epsilon^- := \beta_{\rho_2(\epsilon, \beta)}^-(\epsilon, \beta) = \gamma_{\rho_2(\epsilon, \beta)}(\epsilon, \beta),$$

and reiterate that $\epsilon^- < \epsilon$ by (2.38).

We come now to an important point: we have shown that the circumstances of Corollary 2.5 hold, and therefore whenever $\epsilon^- < \alpha \leq \epsilon$, we know

$$(2.43) \quad \langle \beta_\ell(\alpha, \beta) : \ell \leq \rho_2(\epsilon, \beta) \rangle = \langle \beta_\ell(\epsilon, \beta) : \ell \leq \rho_2(\epsilon, \beta) \rangle,$$

$$(2.44) \quad \langle \beta_\ell^-(\alpha, \beta) : \ell \leq \rho_2(\epsilon, \beta) \rangle = \langle \beta_\ell^-(\epsilon, \beta) : \ell \leq \rho_2(\epsilon, \beta) \rangle,$$

and

$$(2.45) \quad \beta_{\rho_2(\epsilon, \beta)}(\alpha, \beta) = \epsilon.$$

In summary, we have found $\epsilon^- < \epsilon$ such that whenever $\epsilon^- < \alpha \leq \epsilon$, the walk from β to ϵ is an initial segment of the walk from β to α . In particular, the walk from β down to such an α must pass through the ordinal ϵ .

Notice that we have not made use of the club E yet, but this changes now. Let us define

$$(2.46) \quad k = \rho_2(\epsilon, \beta)$$

and

$$(2.47) \quad \bar{s} = \langle \beta_\ell^-(\epsilon, \beta) : \ell \leq k \rangle,$$

and let $\varphi(x, y)$ be the formula (with parameters including k and \bar{s}) which asserts

- $x < y$
- $y \in A$
- $\rho_2(x, y) = k$, and
- $\langle \beta_\ell^-(x, y) : \ell \leq k \rangle = \bar{s}$.

It is clear that $\varphi(\epsilon, \beta)$ holds, and all parameters needed in the definition of φ lie in the model M_ϵ . Since ϵ and δ are both in E , we know

$$(2.48) \quad \epsilon = M_\epsilon \cap \mu^+ < \delta = M_\delta \cap \mu^+ < \beta,$$

and then a standard elementary submodel argument establishes

$$(2.49) \quad (\exists^{\text{stat}} x < \mu^+) (\exists^* y < \mu^+) [\varphi(x, y)].$$

(Here the quantifier “ $\exists^{\text{stat}} x < \mu^+$ ” asserts that the set of such x is stationary in μ^+ , while “ $\exists^* y < \mu^+$ ” tells us that the set of such y is unbounded in μ^+ .)

Since (2.49) holds in the model M_ϵ , it follows that we can find ϵ_a and α such that

$$(2.50) \quad \epsilon^- < \epsilon_a < \alpha < \epsilon$$

$$(2.51) \quad \varphi(\epsilon_a, \alpha)$$

and

$$(2.52) \quad e_\epsilon \cap (\epsilon_a, \alpha) \neq \emptyset.$$

We come now to the main point:

Proposition 2.7. $c(\alpha, \beta) = \epsilon$.

Proof. We have chosen α so that $\epsilon^- < \alpha < \epsilon$, and so (2.43), (2.44), and (2.45) all hold. Thus, the proposition follows provided we establish

$$(2.53) \quad k(\alpha, \beta) = \rho_2(\epsilon, \beta).$$

We will do this in two claims:

Claim 1. If $k \leq \rho_2(\epsilon, \beta)$, then

- (1) $\gamma_k(\epsilon_a, \alpha) = \gamma_k(\alpha, \beta)$, and
- (2) $\langle \beta_\ell^- (\alpha, \beta) : \ell \leq k \rangle = \langle \beta_\ell^- (\gamma_k(\alpha, \beta) + 1, \alpha) : \ell \leq k \rangle$.

Proof. Given $k \leq \rho_2(\epsilon, \beta)$, we know

$$(2.54) \quad \gamma_k(\epsilon_a, \alpha) = \gamma_k(\epsilon, \beta)$$

as $\varphi(\epsilon_a, \alpha)$ holds, and

$$(2.55) \quad \gamma_k(\epsilon, \beta) = \gamma_k(\alpha, \beta)$$

by (2.44). Thus the first part of the claim is true. For the second part, note

$$\begin{aligned} \langle \beta_\ell^- (\alpha, \beta) : \ell \leq k \rangle &= \langle \beta_\ell^- (\epsilon, \beta) : \ell \leq k \rangle && \text{by (2.44)} \\ &= \langle \beta_\ell^- (\epsilon_a, \alpha) : \ell \leq k \rangle && \text{by } \varphi(\epsilon_a, \alpha) \\ &= \langle \beta_\ell^- (\gamma_k(\epsilon_a, \alpha) + 1, \alpha) : \ell \leq k \rangle && \text{by Proposition 2.4} \\ &= \langle \beta_\ell^- (\gamma_k(\alpha, \beta) + 1, \alpha) : \ell \leq k \rangle && \text{by part (1)}. \end{aligned}$$

□

Claim 2. If $k = \rho_2(\epsilon, \beta) + 1$, then $\langle \beta_\ell^- (\alpha, \beta) : \ell \leq k \rangle \neq \langle \beta_\ell^- (\gamma_k(\alpha, \beta) + 1, \alpha) : \ell \leq k \rangle$.

Proof. Note that for this k , we have

$$(2.56) \quad \beta_k^- (\alpha, \beta) = \sup(e_\epsilon \cap \alpha),$$

and so our choice of α guarantees

$$(2.57) \quad \epsilon_a < \beta_k^- (\alpha, \beta) = \gamma_k(\alpha, \beta).$$

Now assume by way of contradiction that the two sequences in question are equal. An argument analogous to the one in the previous claim establishes

$$\begin{aligned} \langle \beta_\ell^-(\gamma_k(\alpha, \beta) + 1, \alpha) : \ell \leq \rho_2(\epsilon, \beta) \rangle &= \langle \beta_\ell^-(\alpha, \beta) : \ell \leq \rho_2(\epsilon, \beta) \rangle \\ &= \langle \beta_\ell^-(\epsilon, \beta) : \ell \leq \rho_2(\epsilon, \beta) \rangle \\ &= \langle \beta_\ell^-(\epsilon_a, \alpha) : \ell \leq \rho_2(\epsilon, \beta) \rangle. \end{aligned}$$

But then it follows easily that

$$(2.58) \quad \langle \beta_\ell(\gamma_k(\alpha, \beta) + 1, \alpha) : \ell \leq \rho_2(\epsilon, \beta) \rangle = \langle \beta_\ell(\epsilon_a, \alpha) : \ell \leq \rho_2(\epsilon, \beta) \rangle$$

as well. In particular, the last terms of these sequences are the same, and so

$$(2.59) \quad \beta_{k-1}(\gamma_k(\alpha, \beta) + 1, \alpha) = \beta_{k-1}(\epsilon_a, \alpha) = \epsilon_a.$$

Thus,

$$(2.60) \quad \beta_k(\gamma_k(\alpha, \beta) + 1, \alpha) \leq \beta_{k-1}(\gamma_k(\alpha, \beta) + 1, \alpha) = \epsilon_a,$$

and the conjunction of (2.57) and (2.60) yields a contradiction. \square

Claim 1 and Claim 2 together imply $c(\alpha, \beta) = \epsilon$, and so the proof of Proposition 2.7 is complete. \square

The argument culminating with the proof of Proposition 2.7 demonstrates that \otimes holds for $\text{ran}(c \upharpoonright [A]^2)$ whenever $A \subseteq \mu^+$ is unbounded in μ^+ , and thus

$$(2.61) \quad \text{ran}(c \upharpoonright [A]^2) \in I^*$$

for any such A , as required. \square

Although the following corollary looks a lot like Lemma 2.2, there are important differences, as the square-brackets partition relation which we assume lets us demand more from our ideal I , and this additional information will play a critical role in the next section.

Corollary 2.8. Assume μ is a singular cardinal for which $\mu^+ \rightarrow [\mu^+]_{\mu^+}^2$ holds, and μ^+ has a stationary subset S such that

$$(2.62) \quad \sup\{\text{cf}(\delta) : S \cap \delta \text{ stationary in } \delta\} < \mu,$$

then there is an ideal I such that

- I is a proper ideal on μ^+ extending the non-stationary ideal,
- I is $\text{cf}(\mu)$ -indecomposable,
- I is σ -indecomposable for all sufficiently large regular $\sigma < \mu$, and
- I is weakly μ -saturated.

Proof. Our first move is to show that there is a stationary set satisfying (2.62) contained in $S_\tau^{\mu^+}$ for some $\tau \neq \text{cf}(\mu)$. If S satisfies (2.62) and $S \setminus S_{\text{cf}(\mu)}^{\mu^+}$ is stationary, then this is an immediate consequence of Fodor's Lemma, so assume that we are given an $S \subseteq S_{\text{cf}(\mu)}^{\mu^+}$ which satisfies (2.62). By a well-known result of Todorcevic, if $\mu^+ \rightarrow [\mu^+]_{\mu^+}^2$ then every stationary subset of μ^+ must reflect, and it follows that $\text{Tr}(S)$ is a stationary set consisting of ordinals of cofinality greater than $\text{cf}(\mu)$. Since $\text{Tr}(\text{Tr}(S)) \subseteq \text{Tr}(S)$, we know that $\text{Tr}(S)$ does not reflect in an ordinal of cofinality greater than θ , and an application of Fodor's Lemma tells us that there is a τ (necessarily different from $\text{cf}(\mu)$) such that $\text{Tr}(S) \cap S_\tau^{\mu^+}$ is stationary.

Thus, we may assume that we have a stationary set S satisfying (2.62) contained in $S_\tau^{\mu^+}$ for some $\tau \neq \text{cf}(\mu)$. Now let I and c be as in the conclusion of Lemma 2.2. Since $\tau \neq \text{cf}(\mu)$, we know that I is $\text{cf}(\mu)$ -indecomposable. Moreover, I is σ -indecomposable for all regular $\sigma < \mu$ other than τ . Thus, we need only check that I is weakly μ -saturated.

This follows easily: if there is a partition $p : \mu^+ \rightarrow \mu$ such that $p^{-1}(\{\alpha\}) \in I^+$ for each $\alpha < \mu$, then the composition $p \circ c$ shows us that $\mu^+ \dashv\vdash [\mu^+]_\mu^2$. An elementary argument (see the introductory section of [1]) would then give us $\mu^+ \dashv\vdash [\mu^+]_{\mu^+}^2$, which contradicts our assumptions. \square

3. THE IDEAL CANNOT EXIST

We come now to the heart of the matter: we show that there is no ideal with the properties listed in Corollary 2.8, and then show that this is enough to obtain our main theorem. We begin this section with more terminology, and then pass on to two lemmas about ideals on successors of singular cardinals.

Definition 3.1. Let I be an ideal on the cardinal κ .

- (1) $\text{Indec}(I) = \{\tau < \kappa : \tau = \text{cf}(\tau) \text{ and } I \text{ is } \tau\text{-indecomposable}\}$.
- (2) $\text{Wsat}(I)$ is the least cardinal θ for which I is weakly θ -saturated. Note that $\text{Wsat}(I) \leq \kappa^+$.

Lemma 3.2. Suppose μ is singular and J is a weakly μ -saturated ideal on μ^+ . Then there is a set $A \in J^+$ and $\theta < \mu$ such that the ideal $J \upharpoonright A := \{B \subseteq \mu^+ : B \cap A \in J\}$ is weakly θ -saturated.

Proof. Let us assume for each $A \in J^+$ and $\theta < \mu$ that $J \upharpoonright A$ is not weakly θ -saturated, and we will prove J is not weakly μ -saturated. Our assumptions imply that J is not weakly $\text{cf}(\mu)$ -saturated so there is a partition $\langle A_\alpha : \alpha < \text{cf}(\mu) \rangle$ of μ^+ into J -positive sets. Let $\langle \mu_\alpha : \alpha < \text{cf}(\mu) \rangle$ be an increasing sequence of cardinals cofinal in μ .

For each $\alpha < \text{cf}(\mu)$, the ideal $J \upharpoonright A_\alpha$ fails to be weakly μ_α -saturated, and this means that A_α can be partitioned into μ_α disjoint J -positive sets. Since the sets A_α are disjoint, we easily obtain a collection of μ disjoint J -positive sets and so J is not weakly μ -saturated. \square

Lemma 3.3. Let μ be a singular cardinal and let J be a $\text{cf}(\mu)$ -indecomposable ideal on μ^+ . If J is weakly θ -saturated for some $\theta < \mu$, then $\text{Indec}(J)$ must be bounded in μ .

Proof. Assume by way of contradiction that our ideal J satisfies the following:

$$(3.1) \quad \text{cf}(\mu) \in \text{Indec}(J)$$

$$(3.2) \quad \sup(\text{Indec}(J)) = \mu, \text{ and}$$

$$(3.3) \quad \text{Wsat}(J) < \mu.$$

By Theorem 2 of [3], there is a function $f^* : \mu^+ \rightarrow \mu^+$ such that for any $\tau \in \text{Indec}(J)$ greater than $\text{Wsat}(J)$, and stationary $S \subseteq S_\tau^{\mu^+}$,

$$(3.4) \quad \{\delta < \mu^+ : S \cap f^*(\delta) \text{ is stationary in } f^*(\delta)\} \in J^*.$$

Taking this with our assumptions (3.2) and (3.3), we see that there are arbitrarily large regular $\tau < \mu$ such that

$$(3.5) \quad \{\delta < \mu^+ : S_\tau^{\mu^+} \cap f^*(\delta) \text{ is stationary in } \delta\} \in J^*.$$

Since $S_\tau^{\mu^+}$ can only reflect in ordinals of cofinality greater than τ , it must be the case that $\text{cf}(f^*(\delta)) > \tau$ for almost every $\delta < \mu^+$, and so

$$(3.6) \quad \{\delta < \mu^+ : \text{cf}(f^*(\delta)) \leq \theta\} \in J \text{ for each cardinal } \theta < \mu.$$

Now let $\langle \mu_\alpha : \alpha < \text{cf}(\mu) \rangle$ be an increasing sequence of cardinals cofinal in μ . For each $\alpha < \text{cf}(\mu)$, define

$$(3.7) \quad A_\alpha := \{\delta < \mu^+ : \text{cf}(f^*(\delta)) \leq \mu_\alpha\}.$$

It is clear that the sequence $\langle A_\alpha : \alpha < \text{cf}(\mu) \rangle$ is increasing with union all of μ^+ . Since $\text{cf}(\mu) \in \text{Indec}(J)$, it follows that there is an α such that

$$(3.8) \quad \{\delta < \mu^+ : \text{cf}(f^*(\delta)) \leq \mu_\alpha\} \in J^+.$$

But this contradicts (3.6), and the proof is finished. \square

Theorem 1. *Suppose μ is singular and there is a stationary $S \subseteq \mu^+$ with*

$$(3.9) \quad \sup\{\text{cf}(\delta) : S \cap \delta \text{ is stationary in } \delta\} < \mu.$$

Then $\mu^+ \not\rightarrow [\mu^+]_{\mu^+}^2$.

Proof. Suppose by way of contradiction that the theorem fails for some singular cardinal μ . Corollary 2.8 tells us that μ^+ carries an ideal I satisfying all of the following:

- I is a proper ideal on μ^+ extending the non-stationary ideal,
- I is $\text{cf}(\mu)$ -indecomposable,
- I is σ -indecomposable for all sufficiently large regular $\sigma < \mu$, and
- I is weakly μ -saturated.

By Lemma 3.2, there is an I -positive set A and a cardinal $\theta < \mu$ such that $J := I \upharpoonright A$ is weakly θ -saturated. Note that easily $\text{Indec}(I) \subseteq \text{Indec}(J)$ so J is also $\text{cf}(\mu)$ -indecomposable and σ -indecomposable for all sufficiently large regular $\sigma < \mu$. But then J is a counterexample to Lemma 3.3, and we have our contradiction. \square

We come at last our main result, which we restate for the convenience of the reader:

Main Theorem. *If μ is a singular cardinal and $\mu^+ \rightarrow [\mu^+]_{\mu^+}^2$, then every collection of fewer than $\text{cf}(\mu)$ stationary subsets of μ^+ reflects simultaneously.*

Proof. We have assumed $\mu^+ \rightarrow [\mu^+]_{\mu^+}^2$, and so Theorem 3 of [3] gives us a regular $\theta < \mu$ such that every collection of fewer than $\text{cf}(\mu)$ stationary subsets of $\{\delta < \mu^+ : \text{cf}(\delta) \geq \theta\}$ must reflect simultaneously. Now suppose $\langle S_\alpha : \alpha < \alpha^* \rangle$ is a collection of stationary subsets of μ^+ with $\alpha^* < \text{cf}(\mu)$. Assuming Theorem 1, an elementary argument implies

$$(3.10) \quad T_\alpha := \text{Tr}(S_\alpha) \cap S_{\geq \theta}^{\mu^+}$$

is stationary for each $\alpha < \alpha^*$. But then there is an ordinal δ such that all of the T_α reflect at δ . But then each S_α reflects at δ as well, and the proof is complete. \square

4. ODDS AND ENDS

In this final section we collect a few remarks on variants and generalizations of the arguments presented here, and formulate a couple of open questions. We make some additional assumptions on the background of the reader in this section, but our terminology is standard.

We begin with a result that is due to Shelah (“Proof of 3.3 in Case γ ” on pages 150 and 151 of [13]); our formulation is more general than his, but obtaining this generality requires only minor modifications to his argument.

Proposition 4.1. If μ is singular and μ^+ carries a proper ideal I (containing all bounded subsets of μ^+) that is $\text{cf}(\mu)$ -indecomposable and weakly μ^+ -saturated, then $\text{pp}(\mu) = \mu^+$.

Proof. Since $\text{pp}(\mu) > \mu^+$, we know (see Theorem 6.3 on page 99 of [13], for example) that there are $\langle A_\alpha : \alpha < \mu^+ \rangle$ and $\langle h_\alpha : \alpha < \mu^+ \rangle$ such that

- $A_\alpha \subseteq \mu$
- $\text{otp}(A_\alpha) = \text{cf}(\mu)$
- each h_α is a function with $\text{dom}(h_\alpha) = \alpha$ and $h_\alpha(\beta) \in [A_\beta]^{<\text{cf}(\mu)}$
- for each $\alpha < \mu^+$, $\langle A_\beta \setminus h_\alpha(\beta) : \beta < \alpha \rangle$ is pairwise disjoint.

For each $\alpha < \mu^+$, let $g_\beta : \text{cf}(\mu) \rightarrow A_\alpha$ be the increasing enumeration of A_α . Given $\beta < \mu^+$, we define a sequence of sets $\langle X_\beta^\gamma : \gamma < \text{cf}(\mu) \rangle$ by setting

$$(4.1) \quad X_\beta^\gamma := \{\alpha < \mu^+ : \beta < \alpha \wedge h_\alpha(\beta) \subseteq g_\beta[\gamma]\},$$

that is, X_β^γ is the set of those $\alpha > \beta$ for which the set $h_\alpha(\beta)$ is contained in the first γ elements of A_β . It is clear that for each β , the sequence $\langle X_\beta^\gamma : \gamma < \text{cf}(\mu) \rangle$ is increasing with union $(\beta, \mu^+) \in I^+$. Since I is $\text{cf}(\mu)$ -indecomposable, there must be an ordinal $\gamma_\beta < \text{cf}(\mu)$ for which

$$(4.2) \quad X_\beta^{\gamma_\beta} \in I^+.$$

If we choose $x_\beta \in A_\beta \setminus g_\beta[\gamma_\beta]$, it follows that

$$(4.3) \quad Y_\beta := \{\alpha < \mu^+ : \beta < \alpha \wedge x_\beta \in A_\beta \setminus h_\alpha(\beta)\} \in I^+.$$

Since $x_\beta < \mu$ for each $\beta < \mu^+$, there is an x^* such that

$$(4.4) \quad |\{\beta < \mu^+ : x_\beta = x^*\}| = \mu^+.$$

Fix such an x^* , and define $Z := \{\beta < \mu^+ : x_\beta = x^*\}$.

Given $\alpha < \beta$ in Z , it follows immediately that $Y_\alpha \cap Y_\beta = \emptyset$, and therefore the collection $\{Y_\alpha : \alpha \in Z\}$ witnesses that I is not weakly μ^+ -saturated. \square

If μ is singular and $\text{pp}(\mu) = \mu^+$, then $\mu^+ \rightarrow [\mu^+]_{\mu^+}^2$ by a result of Todorćević (see Lemma 9.36 of [15], for example), and so we obtain the following corollary:

Corollary 4.2. If μ is singular and $\mu^+ \rightarrow [\mu^+]_{\mu^+}^2$, then there is no proper ideal on μ^+ (containing the bounded subsets of μ^+) which is simultaneously $\text{cf}(\mu)$ -indecomposable and weakly μ^+ -saturated.

Note that the preceding corollary can replace Lemma 3.3 in the proof of our main theorem, but the proof of Lemma 3.3 does not require any cardinal arithmetic assumptions.

Corollary 4.3. If μ is singular and μ^+ carries a $(\text{cf}(\mu))^+$ -complete uniform ultrafilter, then $\text{pp}(\mu) = \mu^+$. In particular, if μ is singular of countable cofinality and $\text{pp}(\mu) > \mu^+$, then there is no uniform countably complete ultrafilter on μ^+ and μ must be less than the first (strongly) compact cardinal.

Proof. Immediate, as ideals dual to countably complete ultrafilters are weakly 2-saturated and \aleph_0 -indecomposable. \square

Notice that the above corollary implies Solovay's theorem that the singular cardinals hypothesis must hold above a compact cardinal.

Lemma 3.3 also tells us a lot about weak-saturation properties of club-guessing ideals on successors of singular cardinals in ZFC. For example, suppose μ is singular and $\tau < \mu$ is a regular cardinal. Given a stationary $S \subseteq S_{\tau}^{\mu^+}$, standard club-guessing results give us a sequence $\bar{C} = \langle C_\delta : \delta \in S \rangle$ such that

- C_δ is club in δ of order-type τ , and
- for every club $E \subseteq \mu^+$, there are stationarily many $\delta \in S$ with $C_\delta \subseteq E$.

Let I_δ be the ideal on C_δ generated by the bounded subsets together with $\text{acc}(C_\delta)$. The ideal $\text{id}_p(\bar{C}, \bar{I})$ is obtained just as in the proof of Lemma 2.2, that is, a set $A \subseteq \mu^+$ is in $\text{id}_p(\bar{C}, \bar{I})$ if there is a club $E \subseteq \mu^+$ such that $A \cap E \cap C_\delta \in I_\delta$ for all $\delta \in S \cap E$.

It is not hard to show that $\text{id}_p(\bar{C}, \bar{I})$ is a τ -complete proper ideal on μ^+ that extends the non-stationary ideal and is also σ -indecomposable for all regular $\sigma \neq \tau$. Now Lemma 2.2 tells us ideals of this form with $\tau \neq \text{cf}(\mu)$ are *never* weakly μ -saturated. Can we achieve weak μ -saturation for such ideals in the case $\tau = \text{cf}(\mu)$? This is still open, but if μ is singular and $\mu^+ \rightarrow [\mu^+]_{\mu^+}^2$, then for every stationary $S \subseteq S_{\text{cf}(\mu)}^{\mu^+}$ there must exist \bar{C} and \bar{I} as above for which the ideal $\text{id}_p(\bar{C}, \bar{I})$ is weakly θ -saturated for some $\theta < \mu$ (see [2] and [5]).

In an earlier version of this paper, we concluded with a question asking about the extent to which our square-brackets assumption in the main theorem can be weakened. In particular, we asked if the main theorem continues to hold if we replace $\mu^+ \rightarrow [\mu^+]_{\mu^+}^2$ by the weaker assumption that Shelah's principle $\text{Pr}_1(\mu^+, \mu^+, \mu^+, \text{cf}(\mu))$ fails. We are happy to report that recent work of Assaf Rinot [11] shows that this is indeed the case.

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