

# RECENT APPLICATIONS OF TOTALLY PROPER FORCING

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A totally proper poset is a partially ordered set which is proper and has the property that forcing with it does not add reals. This announcement has to do with various uses of these kinds of posets, either by themselves or in combination with other kinds of proper posets, to construct models of set theory in which various strong topological statements hold. Some of these statements have not been known to be consistent before at all, while the consistency of others was heretofore only known in models of  $\text{MA} + \neg \text{CH}$ , whereas all the models we describe here satisfy  $2^{\aleph_0} < 2^{\aleph_1}$ , and most of them satisfy CH as well.

## 1. SOME SPECIAL MODELS OF CH AND THEIR USES.

For over two decades now, one of Shelah's favorite projects has been the construction of models of the continuum hypothesis (CH) where many theorems with an  $\text{MA}(\omega_1)$ -like flavor can be shown to hold. This area of set theory, and of its applications, has recently expanded to where it seems to be at roughly the same stage where  $\text{MA}(\omega_1)$  was around 1974, when the explosion of results related a year later in Mary Ellen Rudin's

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booklet [R] was well under way. We have played a role in this expansion, beginning with a modest-seeming advance one of us helped bring about, concerning models of CH in which there are no Ostaszewski spaces [ER]. Successive refinements of the technique in [E] and [EN1] have produced far more sweeping results along these lines, such as:

**Theorem 1.** [EN1] *CH is consistent with the statement that every countably compact first countable space is either compact or contains a copy of  $\omega_1$ .*

[Throughout this announcement, “space” means “Hausdorff space”.]

In this case, we have a statement whose consistency was already known to be compatible with the usual axioms of set theory: Balogh showed in 1987 that it follows from the Proper Forcing Axiom [PFA]. On the other hand, much of the motivation for ‘the Shelah project’ comes from the fact that CH is a strong axiom in its own right: the idea is that these consistency results can be used together with CH itself to show the consistency of statements not known to be consistent in any other way. And this has indeed happened. One example is the application, by Gary Gruenhage, of CH together with the statement in Theorem 1 to show, for the first time:

**Theorem A.** [G] *It is consistent that every countably compact space with a small diagonal is compact.*

Here is another example, where a different model of CH was used. Recall that an  $\alpha_1$ -point in a space  $X$  is a point  $p$  such that, if  $\{\sigma_n : n \in \omega\}$  is a countable family of sequences converging to  $p$ , then there is a sequence  $\sigma$  converging to  $p$  such that  $\text{ran}(\sigma_n) \subseteq^* \text{ran}(\sigma)$  for all  $n$ . [As usual,  $A \subseteq^* B$  means  $A \setminus B$  is finite.] Obviously, every point of first countability (*i.e.*, every point of countable character) is an  $\alpha_1$ -point, and so is the extra point in the one-point compactification of an uncountable discrete space.

**Theorem 2.** [EN2] *Assume  $P_{11} + CH$ . Let  $Y$  be a compact Fréchet-Urysohn space. Every  $\alpha_1$ -point of  $Y$  is either of countable character or is the one relatively nonisolated point in a subspace  $E$  of  $Y$  such that  $E$  is the one-point compactification of an uncountable discrete space.*

This statement too has not been obtained in any model of  $\neg CH$  to date. But neither does it follow from CH alone: there is a counterexample under  $\diamond$  that is even an S-space [N4]. More simply, in the one-point compactification  $T + 1$  of any Souslin tree  $T$ , the extra point is an  $\alpha_1$ -point which is not of countable character, yet  $T + 1$  does not contain the one-point compactification of an uncountable discrete space.

The axiom  $P_{11}$ , defined in Section 2, is a weakening of an axiom designated  $(*)$  by Abraham and Todorčević in [AT], where they show it to be compatible with CH. Not surprisingly, their first application of  $(*)$  was to show that it implies Souslin's Hypothesis. These axioms are part of a hierarchy of axioms, mostly compatible with CH, which will be discussed in the following section.

## 2. AN AXIOM SCHEMA AND SOME MORE APPLICATIONS

The proof of Theorem 2 involves one of the less demanding axioms in a schema having to do with the following concepts.

**Definition 1.** A subset  $S$  of a poset  $P$  is *downward closed* if  $\hat{s} \subset S$  for all  $s \in S$ , where  $\hat{s} = \{p \in P : p \leq s\}$ . A collection of subsets of a set  $X$  is an *ideal* if it is downward closed with respect to  $\subset$ , and closed under finite union.

**Definition 2.** An ideal  $\mathcal{J}$  of countable subsets of a set  $X$  is *countable-covering* if  $\mathcal{J} \upharpoonright Q$  is countably generated for each countable  $Q \subset X$ . That is, for each countable subset  $Q$  of  $X$ , there is a countable subcollection  $\{J_n^Q : n \in \omega\}$  of  $\mathcal{J}$  such that every member  $J$  of  $\mathcal{J}$  that is a subset of  $Q$  satisfies  $J \subset J_n^Q$  for some  $n$ .

**Definition 3.** An ideal  $\mathcal{I}$  of countable subsets of a set  $X$  is a *P-ideal* if, whenever  $\{I_n : n \in \omega\}$  is a countable subset of  $\mathcal{I}$ , then there exists  $J \in \mathcal{I}$  such that  $I_n \subseteq^* J$  for all  $n$ .

**Definition 4.** Given an ideal  $\mathcal{I}$  of subsets of a set  $S$ , a subset  $A$  of  $S$  is *orthogonal to  $\mathcal{I}$*  if  $A \cap I$  is finite for each  $I \in \mathcal{I}$ . The  $\omega$ -*orthocomplement of  $\mathcal{I}$*  is the ideal  $\{J : |J| \leq \omega, J \text{ is orthogonal to } \mathcal{I}\}$  and will be denoted  $\mathcal{I}^\perp$ .

Two basic facts are: (1) when restricted to ideals whose members are countable,  $\omega$ -orthocomplementation is a Galois correspondence, which means that it is order-reversing (*i.e.*, if  $\mathcal{I} \subset \mathcal{J}$  then  $\mathcal{I}^\perp \supset \mathcal{J}^\perp$ ) and  $\mathcal{I} \subset \mathcal{I}^{\perp\perp}$  for all  $\mathcal{I}$ ; and (2) if  $\mathcal{J}$  is countable-covering then  $\mathcal{J}^\perp$  is a P-ideal and  $\mathcal{J} = \mathcal{J}^{\perp\perp}$ .

**Axiom Schema.** Let  $\mathcal{X}_1$  [*resp.*  $\mathcal{X}_2$ ] [*resp.*  $\mathcal{X}_3$ ] be the collection of uncountable [*resp.* stationary] [*resp.* closed unbounded (“club”)] subsets of  $\omega_1$ . Then  $P_{mn}$  [*resp.*  $CC_{mn}$ ] is the axiom that if  $\mathcal{J}$  is a P-ideal [*resp.* countable-covering ideal] on a member of  $\mathcal{X}_{\max\{m,n\}}$ , then either:

- (i) there exists  $A \in \mathcal{X}_m$  such that  $[A]^\omega \subset \mathcal{J}$ ; or
- (ii) there exists  $B \in \mathcal{X}_n$  such that  $[B]^\omega \subset \mathcal{J}^\perp$ .

We define  $wP_{mn}$  like  $P_{mn}$  except that alternative (i) ends with “[ $A$ ] $^\omega \subset \mathcal{J}^{\perp\perp}$ .”

A corollary of basic facts (1) and (2) above is that  $P_{mn} \implies wP_{mn} \implies CC_{nm}$  for all  $m, n$  [note the subscript reversal]. There are easy examples to show that  $CC_{3i}$  is false for all  $i$ ; *a fortiori*,  $wP_{i3}$  and  $P_{i3}$  are false. All the other axioms in this schema are consistent, but we do not know whether  $CC_{13}$  or any of the logically stronger consistent axioms in this schema are compatible with CH. On the other hand, they can be obtained by iterating totally proper posets with countable supports, and so they are also compatible with any of the usual “small uncountable cardinals” (except perhaps  $\mathfrak{c}$  itself) being equal to  $\omega_1$ .

Even the weakest of these axioms is actually quite strong, as the following application in [EN1] illustrates:

**Theorem 4.**  $[CC_{11}]$  *Let  $X$  be a locally countably compact  $T_3$  space such that every countable subset of  $X$  has Lindelöf closure in  $X$ . Then one of the following is true:*

- (1) *Every uncountable subset of  $X$  has a condensation point.*
- (2)  *$X$  has an uncountable closed discrete subspace.*
- (3)  *$X$  contains a perfect preimage of  $\omega_1$ .*

Quick corollaries of Theorem 4 are that  $CC_{11}$  implies  $\neg\clubsuit$  and also that it implies there are no Souslin trees. These two facts have easy direct proofs, as does the fact that  $CC_{12}$  implies every Aronszajn tree has a stationary antichain (hence cannot be collectionwise Hausdorff (cwH), thanks to the Pressing-Down Lemma). A slightly trickier fact to prove is that  $CC_{13}$  implies that every Aronszajn tree is special.

Axiom  $CC_{22}$  is a consequence of  $PFA^+$  in addition to being compatible with CH. It plays a key role in the research announced in [N3] which combines semi-proper forcing with some large cardinal axioms to produce models of  $PFA^+$  with some far-reaching structure theorems for locally compact  $T_5$  spaces. In contrast, the following applications of some of these axioms are all compatible with CH, and have been obtained from  $MA(\omega_1)$  earlier:

**Theorem 5.**  $[CC_{12}]$  *Let  $X$  be locally compact, locally connected, and countably tight. If  $X$  is either strongly cwH or locally ccc and cwH, and every Lindelöf subset of  $X$  has Lindelöf closure, then either:*

- (1)  *$X$  is paracompact, or*
- (2)  *$X$  has a closed subspace which is a perfect preimage of  $\omega_1$ .*

**Theorem 6.**  $[CC_{12}]$  *Let  $X$  be a locally compact, perfectly normal, cwH space. If every Lindelöf subspace of  $X$  has Lindelöf closure, then  $X$  is paracompact.*

**Corollary.**  $[CC_{12}]$  *Let  $M$  be a perfectly normal manifold. Then either  $M$  is metrizable, or it contains a countable subset with nonmetrizable closure.*

In view of this corollary, it is hardly surprising that the main example of [RZ] is separable. Another example in [RZ], whose construction was only outlined, was not separable; however, it too had a separable nonmetrizable subspace, and this corollary shows this was unavoidable for a “CH alone” example.

The following results in [EN2] were likewise obtained earlier from  $MA(\omega_1)$ , but are also compatible with any of the usual “small uncountable cardinals” (except perhaps  $\mathfrak{c}$  itself) being equal to  $\omega_1$ . They involve the concept of a Type I space, which in a locally compact setting can be characterized by being of Lindelöf degree  $\leq \aleph_1$  and having the property that every Lindelöf subset has Lindelöf closure. Part of the utility of the concept is that every cwH tree of height  $\leq \omega_1$  is the topological direct sum of Type I trees. Also, if there are no Q-sets, then every normal tree of cardinality and height  $\leq \omega_1$  is of Type I.

**Theorem 7.** [CC<sub>13</sub>] *Every locally compact, locally countable Type I space either contains a perfect preimage of  $\omega_1$  or is the countable union of discrete subspaces.*

**Theorem 8.** [CC<sub>13</sub>] *Every perfectly normal, locally compact, Type I space is subparacompact.*

**Theorem 9.** [CC<sub>13</sub>] *Every Type I tree either has an uncountable branch or is a countable union of antichains.*

Finally, here is a case where the utility of our Axiom Schema and  $MA(\omega_1)$  was noticed simultaneously:

**Theorem 10.** [N2] *If there are no Kurepa trees, and either  $MA(\omega_1)$  or  $CC_{13}$  holds, then every cwH tree of height  $< \omega_2$  is monotone normal.*

Monotone normality is a very strong property where trees are concerned, being equivalent to the tree being the topological direct sum of copies of ordinals. In [N2] it is shown that the nonexistence of Kurepa trees can be relaxed to the condition that every Kurepa tree has an Aronszajn subtree; it would be

nice to know whether this weaker axiom can be shown compatible with either  $\text{MA}(\omega_1)$  or  $CC_{13}$  without the use of large cardinal axioms.

### 3. A MODEL OF $2^{\aleph_0} < 2^{\aleph_1}$ WITH SOME PROMISING PROPERTIES

Another use of these axioms and techniques is to take advantage of the fact that totally proper posets at worst produce “very innocuous” reals when they are iterated transfinitely using countable supports. For instance, they will not affect any of the well-known small uncountable cardinals like  $\mathfrak{b}$  or  $\mathfrak{d}$  or  $\mathfrak{s}$  or  $\mathfrak{u}$ ; nor will they add Q-sets. If they are combined carefully with other posets which do add reals of one kind but not another, one can frequently achieve results with an MA-like flavor in models that do not share many other characteristics of MA-like models. Here is a result of this kind, answering a problem posed in [N1]:

**Theorem 11.** [ENS] *It is consistent that there is no  $T_5$ , locally compact, separable space of cardinality  $\aleph_1$ .*

This has a nice corollary relevant to the theory of Boolean algebras. Recall that a *thin-tall* space is a scattered space of cardinality  $\aleph_1$  in which the Cantor-Bendixson derivatives are all countable and the  $\omega_1$ st level is empty. Any thin-tall space is separable, and so we have:

**Corollary** *It is consistent that there is no  $T_5$  thin-tall locally compact scattered space.*

As observed in [N1], the topological statement in Theorem 11 is false in any model with a Q-set, hence in any model of  $\text{MA}(\omega_1)$ ; on the other hand, CH also implies it is false, with the Kunen ‘line’ as a counterexample. In fact, in [N1] it was shown that the statement in Theorem 11 is equivalent to the simultaneous nonexistence of both Q-sets (that is, of uncountable subsets of  $\mathbb{R}$  in which every subset is an  $F_\sigma$  in the

relative topology) and of locally compact, locally countable,  $T_5$  S-spaces. Now,  $\mathfrak{b} = \aleph_1$  implies the existence of such S-spaces (in fact, of perfectly normal ones: as Todorćević showed [T, Chapter 2], the Kunen ‘line’ essentially exists in models of  $\mathfrak{b} = \aleph_1$ ) and these are also thin-tall spaces. So the Corollary to Theorem 11 has a neat implication for the various normality-like properties of thin-tall locally compact spaces. There are elementary constructions of  $T_3$  thin-tall locally compact spaces just from ZFC. There are also constructions of normal ( $T_4$ ) examples from ZFC, but they are not so elementary: if  $\mathfrak{b} = \aleph_1$  we have the above construction, but if  $\mathfrak{b} > \aleph_1$  then the existing construction is so different that we do not know whether it can yield a normal space in all models of  $\mathfrak{b} = \aleph_1$ . And now we know that the existence of both  $T_5$  and perfectly normal (sometimes labeled  $T_6$ ) examples is ZFC-independent.

The ground model for the forcing for Theorem 11 satisfies CH and hence  $\mathfrak{b} = \aleph_1$ , so it is not surprising that one of the key ingredients of the iteration was the interweaving of posets that add dominating reals (specifically, Laver reals). To insure against Q-sets, the ground model satisfied  $2^{\aleph_1} > \aleph_2$  and the iteration added only  $\aleph_2$  reals. To destroy all relevant S-spaces, posets that would ordinarily force  $P_{11}$  were interweaved with the Laver reals and with posets somewhat like those in [E]. These posets handled locally compact first countable spaces of cardinality  $\aleph_1$  in intermediate models (which satisfied CH), with the first kind of poset insuring that the non-Lindelöf ones would acquire either uncountable discrete subspaces or countably compact noncompact subspaces, while the second kind then forced perfect preimages of  $\omega_1$  into them, which thus gave uncountable discrete subspaces by another route. In this way, the final model gave:

**Theorem 12.**  $2^{\aleph_0} < 2^{\aleph_1}$  is compatible with the statement that every locally compact, first countable space of countable spread is hereditarily Lindelöf; in particular, it is compatible with the nonexistence of first countable, locally compact S-spaces.

These results have elegant translations in terms of Boolean algebras via Stone duality. The Stone space  $\mathcal{S}(A)$  of a Boolean algebra  $A$  is hereditarily Lindelöf iff every ideal of  $A$  is countably generated, and first countable iff every maximal ideal is countably generated. Also,  $\mathcal{S}(A)$  is of countable spread iff every minimal set of generators for an ideal is countable. (An ideal is said to be minimally generated if it has a generating set  $D$  such that no member of  $D$  is in the ideal generated by the remaining members.) Hence we have, for example:

**Corollary**  $2^{\aleph_0} < 2^{\aleph_1}$  is consistent with the following statement: if a Boolean algebra  $A$  has the property that every minimal set of generators for an ideal is countable, and every maximal ideal of  $A$  is countably generated, then every ideal of  $A$  is countably generated.

The statement in this corollary also follows from  $\text{MA}(\omega_1)$ , by a 1978 theorem of Szentmiklóssy. On the other hand, it has also long been known to be incompatible with CH: there is a simple construction of a first countable, zero-dimensional compactification of the Kunen ‘line’ which is also an S-space, and the complement of the Kunen ‘line’ is locally countable and uncountable, hence the ideal of its compact open subsets is not countably generated.

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