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A NOTE ON JÓNSSON CARDINALS

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ABSTRACT. We use elementary submodels to prove a few facts about Jónssoncardinals.

Definition 1. A cardinal λ is a *Jónsson cardinal* if $\lambda \to [\lambda]^{<\aleph_0}_\lambda$. This means that for any function $f : [\lambda]^{<\aleph_0} \to \lambda$, there is $H \in [\lambda]^\lambda$ such that the range of $f \restriction [H]^{<\aleph_0}$ is a proper subset of λ

Jónsson cardinals have been extensively studied in the literature. Kanamori's book [2] has an excellent survey of what is known and how Jónsson cardinals are related to large cardinals.

Proposition 2 (Folklore)**.** A cardinal λ is a Jónsson cardinal if and only if for every large enough regular χ and every $x \in H(\chi)$, we can find $M \prec H(\chi)$ such that

- $\{\lambda, x\} \in M$
- $|M \cap \lambda| = \lambda$
- $\lambda \nsubseteq M$

We open this paper with an application of Jónsson cardinals to topology. Recall that if $M \prec H(\chi)$ and $X \in M$ is a topological space, then X_M is the topological space with underlying set $M \cap X$ and base $\{M \cap U : U \in M, U \text{ open in } X\}$.

Theorem 1. *The following statements are equivalent:*

1. *There is a Jónsson cardinal.*
2. *There are a topological space X and $M \prec H(\chi)$ (for χ some large regular cardinal) with $X \in M$ such that X_M is homeomorphic to X but $X \neq X_M$.*

Proof. The proof that (1) implies (2) is due to Junqueira and Tall [1]; it suffices to observe that if λ is a Jónsson cardinal, then the discrete space of cardinality λ works — we just take M witnessing that λ is Jónsson.

The proof that (2) implies (1) is more involved; we show that (2) implies that at least one of $|X|$ and $w(X)$ is a Jónsson cardinal.

Suppose that we are given $M \prec H(\chi)$ and $X \in M$ such that X_M is homeomorphic to X but not equal to X. Further suppose that $|X|$ is not a Jónsson cardinal.

Since X_M is homeomorphic to X, we know $|M \cap X| = |X_M| = |X|$. Also, $|X| \in M$ because X is. Since $|X|$ is not a Jónsson cardinal, we are forced to conclude that $|X| \subseteq M$, and hence $X \subseteq M$.

In M, let us fix a base $\{U_\alpha : \alpha < w(X)\}$ for the topology of X. The cardinal $w(X)$ is in M because X is. Now $\{U_\alpha : \alpha \in M \cap w(X)\}$ is a base for the topology of X_M. Since X_M and X are homeomorphic, we know $w(X_M) = w(X)$ and therefore

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$|M \cap w(X)| = w(X)$. Since $X_M \neq X$, we note that $w(X)$ cannot be a subset of M. Putting all these facts together, we arrive at the conclusion that $w(X)$ is Jónsson. □

Our next application of elementary submodels is to give a short proof of a result due independently to Tryba [6] and Woodin (unpublished).

Theorem 2. *If λ is a Jónsson cardinal, then every stationary subset of λ reflects.*

Lemma 3. Suppose $M \prec H(\chi)$, $\lambda \in M$, $|M \cap \lambda| = \lambda$, and $\lambda \nsubseteq M$. If $S \in M$ is a stationary subset of λ, then $S \setminus M$ is stationary.

Proof. Suppose S and M are a counterexample. There is a closed unbounded set $E \subseteq \lambda$ such that $E \cap S \subseteq M$.

In M, we can fix a partition of S into λ stationary subsets, i.e., there is a function $f : S \to \lambda$ in M such that $S_\alpha := f^{-1}(\{\alpha\})$ is stationary for each $\alpha < \lambda$.

Fix $\alpha < \lambda$ such that $\alpha \notin M$. Such S_α is stationary, we know that $E \cap S_\alpha$ is non-empty. Since $S_\alpha \subseteq S$, we have $E \cap S_\alpha \subseteq M$. Fix $\beta \in E \cap S_\alpha$. Then since $f \in M$ and $\beta \in M$, $\alpha = f(\beta)$ is in M, a contradiction. □

Proof of Theorem 2. Let S be a stationary subset of λ. We must produce $\beta < \lambda$ such that $S \cap \beta$ is stationary in β.

Since λ is a Jónsson cardinal, we can find $M \prec H(\chi)$ such that

- $\{S, \lambda\} \in M$
- $|M \cap \lambda| = \lambda$
- $\lambda \nsubseteq M$

By our lemma, we can find $\delta \in S \setminus M$ such that $\delta = \sup(M \cap \delta)$ (as the set $\{\delta < \lambda : \delta = \sup(M \cap \delta)\}$ is club in λ). Let $\beta_\delta = \min(M \cap \lambda \setminus \delta)$; clearly $\delta < \beta_\delta$.

Claim 4. $S \cap \beta_\delta$ is a stationary subset of β_δ.

Proof. The proof is by contradiction. If this fails, then there is a closed unbounded $C \subseteq \beta_\delta$ disjoint from S. Since S and β_δ are both in M, we may assume that $C \in M$.

Given $\alpha < \delta$, we can find $\beta \in M$ such that $\alpha < \beta < \delta$ because $\delta = \sup(M \cap \delta)$. Since $M \models$ "C is unbounded in δ", we can find $\gamma \in M \cap C$ such that $\beta < \gamma$. By choice of β_δ, we see that $\gamma < \delta$. Since α was an arbitrary ordinal $< \delta$, we have shown that δ is a limit point of C. As C is closed, we have $\delta \in C$, a contradiction as $C \cap S$ was supposed to be empty. □

□

The proof of Lemma 3 can be easily generalized to other ideals.

Lemma 5. Suppose $M \prec H(\chi)$ with $\lambda \in M$. Let $I \in M$ be an ideal on λ such that there is a function $f : \lambda \to \lambda$ with $f^{-1}(\{\alpha\}) \notin I$ for each $\alpha < \lambda$. If $\lambda \setminus M \in I$, then $\lambda \subseteq M$.

Proof. Without loss of generality, the function f is in M. Given $\alpha < \lambda$, the set $f^{-1}(\{\alpha\})$ is not in I. Since $\lambda \setminus M \in I$, this means that there is $\beta \in \lambda \setminus M$ with $f(\beta) = \alpha$. Since f and β are in M, α must be in M as well. As $\alpha < \lambda$ was arbitrary, we conclude $\lambda \subseteq M$. □

We now exploit this lemma by connecting the question of whether the successor of a singular cardinal can be Jónsson to a question on whether a certain ideal possesses a weak form of saturation. This approach in implicit in much of Shelah's work in [4]. We note that the question of whether the successor of a singular cardinal can be Jónsson is still very much an open question (see [5] for example).

For the remainder of the paper, assume that $\lambda = \mu^+$ for some singular cardinal μ, and we let S be a stationary subset of $\lambda \setminus \mu$ such that $\sup\{\mathrm{cf}(\delta) : \delta \in S\} < \mu$. We let $\bar{C} = \langle C_\delta : \delta \in S \rangle$ be such that C_δ is club in δ with order–type $\mathrm{cf}(\delta)$. For $\delta \in S$, we define an ideal I_δ of subsets of C_δ by

$$A \text{ is } \textit{not} \text{ in } I_\delta \iff (\forall \alpha < \delta)(\forall \beta < \mu)(\exists \gamma \in \mathrm{nacc}(C_\delta))\big[\gamma > \alpha \text{ and } \mathrm{cf}(\gamma) > \beta\big].$$

Here $\mathrm{nacc}(C_\delta)$ is the set of non–accumulation points of C_δ, i.e., those $\alpha \in C_\delta$ such that $\sup(\alpha \cap C_\delta) < \alpha$. It is not hard to see that I_δ is an ideal of subsets of C_δ, and we let $\bar{I} = \langle I_\delta : \delta \in S \rangle$.

Definition 6. The ideal $\mathrm{id}_p(\bar{C}, \bar{I})$ is defined by putting $A \in \mathrm{id}_p(\bar{C}, \bar{I})$ if and only if there is a closed unbounded $E \subseteq \lambda$ such that for every $\delta \in S \cap E$, $A \cap E \cap C_\delta \in I_\delta$.

Said another way, if $A \notin \mathrm{id}_p(\bar{C}, \bar{I})$, then for every club $E \subseteq \lambda$ there is $\delta \in S \cap E$ such that $A \cap E \cap \mathrm{nacc}(C_\delta)$ is large in the sense that it is not in the ideal I_δ. Shelah's work in [4] shows that in many cases the ideal $\mathrm{id}_p(\bar{C}, \bar{I})$ is non–trivial — given S, we can find $\bar{C}$ such that $\lambda \notin \mathrm{id}_p(\bar{C}, \bar{I})$.

The proposition we state next is new, although it lurks in the background throughout much of Chapter IV of [3]. It ties together many of the proofs there.

Proposition 7. Let $\lambda = \mu^+$ where μ is singular, and let S be a stationary subset of S^λ_κ for some $\kappa < \lambda$. Let $M \prec H(\chi)$ with $\{\lambda, S, \bar{C}\} \in M$, and assume $|M \cap \lambda| = \lambda$. Then $\lambda \setminus M \in \mathrm{id}_p(\bar{C}, \bar{I})$.

Proof. Suppose this is not the case. Let $E = \{\delta < \lambda : \delta = \sup(M \cap \delta)\}$; since $|M \cap \lambda| = \lambda$ we know that E is closed unbounded in λ. Since we assume $\lambda \setminus M \notin \mathrm{id}_p(\bar{C}, \bar{I})$, there is a $\delta \in S \cap E$ with $(\lambda \setminus M) \cap E \cap C_\delta \notin I_\delta$. This means that we can find points in $\mathrm{nacc}(C_\delta) \cap E$ with cofinality arbitrarily large beneath μ that are not in M.

Note that we have no guarantee that δ and C_δ are in M; to get around this, let us define $\beta_\delta := \min(M \cap \lambda \setminus \delta)$. (So $\beta_\delta = \delta$ if $\delta \in M$.) In M, we can fix C such that C is club in β_δ and $\mathrm{otp}(C) = \mathrm{cf}(\beta_\delta)$. Note that since $S \subseteq \lambda \setminus \mu$, we know that β_δ is singular and $\mathrm{cf}(\beta_\delta) < \mu$. We define

$$C^* = \bigcup_{\beta \in C \cap S} C_\beta.$$

Since C and S are in M, the set C^* is in M as well. Also, note that C_δ is a subset of C^*. By our assumption on S, there is some $\gamma < \mu$ such that $|C_\delta| < \gamma$ for each $\delta \in S$. This together with the fact that $|C| < \mu$ is enough to guarantee that $|C^*| < \mu$.

Since $(\lambda \setminus M) \cap E \cap C_\delta \notin I_\delta$, there is $\alpha \in E \cap \mathrm{nacc}(C_\delta)$ such that $\mathrm{cf}(\alpha) > |C^*|$ and $\alpha \notin M$. Since $\mathrm{cf}(\alpha) > |C^*|$, we know that $\alpha \in \mathrm{nacc}(C^*)$ as well. This means that there is a $\beta \in M \cap \lambda$ such that

$$\sup(C^* \cap \alpha) < \beta < \alpha.$$

This implies that α can be defined as the least member of C^* that is above β; since C^* and β are in M, we conclude $\alpha \in M$. This is a contradiction of our choice of α. □

We can now draw some conclusions about the possibility of the successor of a singular cardinal being Jónsson. For example, if $\lambda = \mu^+$ and $M \prec H(\chi)$ satisfies

- $|M \cap \lambda| = \lambda$, and
- $\lambda \nsubseteq M$,

then M will contain a stationary set $S \subseteq \lambda$ and an S–club system $\bar{C}$ such that the ideal $\mathrm{id}_p(\bar{C}, \bar{I})$ is non–trivial. Since $\mathrm{id}_p(\bar{C}, \bar{I}) \in M$ and $\lambda \setminus M \in \mathrm{id}_p(\bar{C}, \bar{I})$, Lemma 5 tells us that whenever we partition λ into λ sets, at least one of the pieces of the partition must be in $\mathrm{id}_p(\bar{C}, \bar{I})$. The power of this lies in our ability to prove that in certain situations, it *is* possible to partition λ into λ disjoint sets, none of which are in $\mathrm{id}_p(\bar{C}, \bar{I})$ and thus show that λ is not a Jónsson cardinal — this is the essence of many results in Shelah's book [3].

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