A NOTE ON JÓNSSON CARDINALS

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Abstract. We use elementary submodels to prove a few facts about Jónsson cardinals.

Definition 1. A cardinal $\lambda$ is a Jónsson cardinal if $\lambda \rightarrow [\lambda]^{\lambda}_\lambda$. This means that for any function $f : [\lambda]^{\lambda_\lambda} \rightarrow \lambda$, there is $H \in [\lambda]^{\lambda_\lambda}$ such that the range of $f \upharpoonright [H]^{\lambda_\lambda}$ is a proper subset of $\lambda$.

Jónsson cardinals have been extensively studied in the literature. Kanamori’s book [2] has an excellent survey of what is known and how Jónsson cardinals are related to large cardinals.

Proposition 2 (Folklore). A cardinal $\lambda$ is a Jónsson cardinal if and only if for every large enough regular $\chi$ and every $x \in H(\chi)$, we can find $M \prec H(\chi)$ such that

- $\{\lambda, x\} \in M$
- $|M \cap \lambda| = \lambda$
- $\lambda \not\subseteq M$

We open this paper with an application of Jónsson cardinals to topology. Recall that if $M \prec H(\chi)$ and $X \in M$ is a topological space, then $X_M$ is the topological space with underlying set $M \cap X$ and base $\{U \cap M : U \subseteq M, U$ open in $X\}$.

Theorem 1. The following statements are equivalent:

1. There is a Jónsson cardinal.
2. There are a topological space $X$ and $M \prec H(\chi)$ (for $\chi$ some large regular cardinal) with $X \in M$ such that $X_M$ is homeomorphic to $X$ but $X \neq X_M$.

Proof. The proof that (1) implies (2) is due to Junqueira and Tall [1]; it suffices to observe that if $\lambda$ is a Jónsson cardinal, then the discrete space of cardinality $\lambda$ works — we just take $M$ witnessing that $\lambda$ is Jónsson.

The proof that (2) implies (1) is more involved; we show that (2) implies that at least one of $|X|$ and $w(X)$ is a Jónsson cardinal.

Suppose that we are given $M \prec H(\chi)$ and $X \in M$ such that $X_M$ is homeomorphic to $X$ but not equal to $X$. Further suppose that $|X|$ is not a Jónsson cardinal.

Since $X_M$ is homeomorphic to $X$, we know $|M \cap X| = |X_M| = |X|$. Also, $|X| \in M$ because $X$ is. Since $|X|$ is not a Jónsson cardinal, we are forced to conclude that $|X| \subseteq M$, and hence $X \subseteq M$.

In $M$, let us fix a base $\{U_\alpha : \alpha < w(X)\}$ for the topology of $X$. The cardinal $w(X)$ is in $M$ because $X$ is. Now $\{U_\alpha : \alpha \in M \cap w(X)\}$ is a base for the topology of $X_M$. Since $X_M$ and $X$ are homeomorphic, we know $w(X_M) = w(X)$ and therefore
Putting all these facts together, we arrive at the conclusion that \( w(X) \) is Jónsson. \( \Box \)

Our next application of elementary submodels is to give a short proof of a result due independently to Tryba [6] and Woodin (unpublished).

**Theorem 2.** If \( \lambda \) is a Jónsson cardinal, then every stationary subset of \( \lambda \) reflects.

**Lemma 3.** Suppose \( M < H(\chi), \lambda \in M, |M \cap \lambda| = \lambda \), and \( \lambda \not< M \). If \( S \subseteq M \) is a stationary subset of \( \lambda \), then \( S \setminus M \) is stationary.

**Proof.** Suppose \( S \) and \( M \) are a counterexample. There is a closed unbounded set \( E \subseteq \lambda \) such that \( E \cap S \subseteq M \).

In \( M \), we can fix a partition of \( S \) into \( \lambda \) stationary subsets, i.e., there is a function \( f: S \to \lambda \) in \( M \) such that \( S_\alpha := f^{-1}(\{\alpha\}) \) is stationary for each \( \alpha < \lambda \).

Fix \( \alpha < \lambda \) such that \( \alpha \not< M \). Such \( S_\alpha \) is stationary, we know that \( E \cap S_\alpha \) is non-empty. Since \( S_\alpha \not< S \), we have \( E \cap S_\alpha \subseteq M \). Fix \( \beta \in E \cap S_\alpha \). Then since \( f \in M \) and \( \beta \in M \), \( \alpha = f(\beta) \) is in \( M \), a contradiction. \( \Box \)

**Proof of Theorem 2.** Let \( S \) be a stationary subset of \( \lambda \). We must produce \( \beta < \lambda \) such that \( S \cap \beta \) is stationary in \( \beta \).

Since \( \lambda \) is a Jónsson cardinal, we can find \( M < H(\chi) \) such that

- \( \{S, \lambda\} \in M \)
- \( |M \cap \lambda| = \lambda \)
- \( \lambda \not< M \)

By our lemma, we can find \( \delta \in S \setminus M \) such that \( \delta = \sup(M \cap \delta) \) (as the set \( \{\delta < \lambda : \delta = \sup(M \cap \delta)\} \) is club in \( \lambda \)). Let \( \beta_\delta = \min(M \cap \lambda \setminus \delta) \); clearly \( \delta < \beta_\delta \).

**Claim 4.** \( S \cap \beta_\delta \) is a stationary subset of \( \beta_\delta \).

**Proof.** The proof is by contradiction. If this fails, then there is a closed unbounded \( C \subseteq \beta_\delta \) disjoint from \( S \). Since \( S \) and \( \beta_\delta \) are both in \( M \), we may assume that \( C \in M \).

Given \( \alpha < \delta \), we can find \( \beta \in M \) such that \( \alpha < \beta < \delta \) because \( \delta = \sup(M \cap \delta) \). Since \( M \models "C \text{ is unbounded in } \delta" \), we can find \( \gamma \in M \cap C \) such that \( \beta < \gamma \). By choice of \( \beta_\delta \), we see that \( \gamma < \delta \). Since \( \alpha \) was an arbitrary ordinal \( < \delta \), we have shown that \( \delta \) is a limit point of \( C \). As \( C \) is closed, we have \( \delta \in C \), a contradiction as \( C \cap S \) was supposed to be empty. \( \Box \)

The proof of Lemma 3 can be easily generalized to other ideals.

**Lemma 5.** Suppose \( M < H(\chi) \) with \( \lambda \in M \). Let \( I \subseteq M \) be an ideal on \( \lambda \) such that there is a function \( f: \lambda \to \lambda \) with \( f^{-1}(\{\alpha\}) \not< I \) for each \( \alpha < \lambda \). If \( \lambda \setminus M \subseteq I \), then \( \lambda \subseteq M \).

**Proof.** Without loss of generality, the function \( f \) is in \( M \). Given \( \alpha < \lambda \), the set \( f^{-1}(\{\alpha\}) \) is not in \( I \). Since \( \lambda \setminus M \subseteq I \), this means that there is \( \beta \in \lambda \setminus M \) with \( f(\beta) = \alpha \). Since \( f \) and \( \beta \) are in \( M \), \( \alpha \) must be in \( M \) as well. As \( \alpha < \lambda \) was arbitrary, we conclude \( \lambda \subseteq M \). \( \Box \)
We now exploit this lemma by connecting the question of whether the successor of a singular cardinal can be Jónsson to a question on whether a certain ideal possesses a weak form of saturation. This approach is implicit in much of Shelah’s work in [4]. We note that the question of whether the successor of a singular cardinal can be Jónsson is still very much an open question (see [5] for example).

For the remainder of the paper, assume that \( \lambda = \mu^+ \) for some singular cardinal \( \mu \), and we let \( S \) be a stationary subset of \( \lambda \setminus \mu \) such that \( \sup \{ cf(\delta) : \delta \in S \} < \mu \). We let \( C = \langle C_\delta : \delta \in S \rangle \) be such that \( C_\delta \) is club in \( \delta \) with order–type \( cf(\delta) \). For \( \delta \in S \), we define an ideal \( I_\delta \) of subsets of \( C_\delta \) by

\[
A \text{ is not in } I_\delta \iff (\forall \alpha \in \delta)(\forall \beta < \mu)(\exists \gamma \in nacc(C_\delta))[\gamma > \alpha \text{ and } cf(\gamma) > \beta].
\]

Here \( nacc(C_\delta) \) is the set of non–accumulation points of \( C_\delta \), i.e., those \( \alpha \in C_\delta \) such that \( \sup(\alpha \cap C_\delta) < \alpha \). It is not hard to see that \( I_\delta \) is an ideal of subsets of \( C_\delta \), and we let \( I = (I_\delta : \delta \in S) \).

**Definition 6.** The ideal \( id_\mu(C, I) \) is defined by putting \( A \in id_\mu(C, I) \) if and only if there is a closed unbounded \( E \subseteq \lambda \) such that for every \( \delta \in S \cap E \), \( A \cap E \cap C_\delta \in I_\delta \).

Said another way, if \( A \notin id_\mu(C, I) \), then for every club \( E \subseteq \lambda \) there is \( \delta \in S \cap E \) such that \( A \cap E \cap nacc(C_\delta) \) is large in the sense that it is not in the ideal \( I_\delta \). Shelah’s work in [4] shows that in many cases the ideal \( id_\mu(C, I) \) is non–trivial — given \( S \), we can find \( C \) such that \( \lambda \notin id_\mu(C, I) \).

The proposition we state next is new, although it lurks in the background throughout much of Chapter IV of [3]. It ties together many of the proofs there.

**Proposition 7.** Let \( \lambda = \mu^+ \) where \( \mu \) is singular, and let \( S \) be a stationary subset of \( S_\kappa^\lambda \) for some \( \kappa < \lambda \). Let \( M < H(\chi) \) with \( \{ \lambda, S, C \} \in M \), and assume \( |M \cap \lambda| = \lambda \). Then \( \lambda \setminus M \in id_\mu(C, I) \).

**Proof.** Suppose this is not the case. Let \( E = \{ \delta < \lambda : \delta = \sup(M \cap \delta) \} \); since \( |M \cap \lambda| = \lambda \) we know that \( E \) is closed unbounded in \( \lambda \). Since we assume \( \lambda \setminus M \notin id_\mu(C, I) \), there is a \( \delta \in S \cap E \) with \( (\lambda \setminus M) \cap E \cap C_\delta \notin I_\delta \). This means that we can find points in \( nacc(C_\delta) \cap E \) with cofinality arbitrarily large beneath \( \mu \) that are not in \( M \).

Note that we have no guarantee that \( \delta \) and \( C_\delta \) are in \( M \); to get around this, let us define \( \beta_\delta := \min(M \cap \lambda \setminus \delta) \). (So \( \beta_\delta = \delta \) if \( \delta \in M \).) In \( M \), we can fix \( C \) such that \( C \) is club in \( \beta_\delta \) and \( otp(C) = cf(\beta_\delta) \). Note that since \( S \subseteq \lambda \setminus \mu \), we know that \( \beta_\delta \) is singular and \( cf(\beta_\delta) < \mu \). We define

\[
C^* = \bigcup_{\beta \in C \cap S} C_\beta.
\]

Since \( C \) and \( S \) are in \( M \), the set \( C^* \) is in \( M \) as well. Also, note that \( C_\delta \) is a subset of \( C^* \). By our assumption on \( S \), there is some \( \gamma < \mu \) such that \( |C_\delta| < \gamma \) for each \( \delta \in S \). This together with the fact that \( |C| < \mu \) is enough to guarantee that \( |C^*| < \mu \).

Since \( (\lambda \setminus M) \cap E \cap C_\delta \notin I_\delta \), there is \( \alpha \in E \cap nacc(C_\delta) \) such that \( cf(\alpha) > |C^*| \) and \( \alpha \notin M \). Since \( cf(\alpha) > |C^*| \), we know that \( \alpha \in nacc(C^*) \) as well. This means that there is a \( \beta \in M \cap \lambda \) such that

\[
\sup(C^* \cap \alpha) < \beta < \alpha.
\]

This implies that \( \alpha \) can be defined as the least member of \( C^* \) that is above \( \beta \); since \( C^* \) and \( \beta \) are in \( M \), we conclude \( \alpha \in M \). This is a contradiction of our choice of \( \alpha \). \qed
We can now draw some conclusions about the possibility of the successor of a singular cardinal being Jónsson. For example, if $\lambda = \mu^+$ and $M \prec H(\chi)$ satisfies

- $|M \cap \lambda| = \lambda$, and
- $\lambda \notin M$,

then $M$ will contain a stationary set $S \subseteq \lambda$ and an $S$–club system $\bar{C}$ such that the ideal $\text{id}_p(\bar{C}, \bar{I})$ is non–trivial. Since $\text{id}_p(\bar{C}, \bar{I}) \in M$ and $\lambda \setminus M \in \text{id}_p(\bar{C}, \bar{I})$, Lemma 5 tells us that whenever we partition $\lambda$ into $\lambda$ sets, at least one of the pieces of the partition must be in $\text{id}_p(\bar{C}, \bar{I})$. The power of this lies in our ability to prove that in certain situations, it is possible to partition $\lambda$ into $\lambda$ disjoint sets, none of which are in $\text{id}_p(\bar{C}, \bar{I})$ and thus show that $\lambda$ is not a Jónsson cardinal — this is the essence of many results in Shelah’s book [3].

**References**

5. Saharon Shelah, *On what I do not understand (and have something to say)*, Fundamenta Mathematicae **166** (2000), 1–82.

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