ON PERFECT PRE–IMAGES OF ω_1

TODD EISWORTH

Abstract. We show that PFA implies that every perfect pre–image of ω_1 of countable tightness contains a closed copy of ω_1.

1. Introduction

In this paper, we continue the chain of investigation of Fremlin [9], Balogh [2] and others [1] into consequences of the Proper Forcing Axiom in topology.

We will be concerned with perfect pre–images of ω_1, i.e., (Hausdorff) topological spaces X with the property that there is a closed mapping π from X onto ω_1 such that $\pi^{-1}(\{\alpha\})$ is compact for each $\alpha < \omega_1$. In particular, we want to investigate when such spaces contain closed copies of the topological space ω_1.

There are many such results known — for example, Balogh proves in [2] that PFA implies that any closed pre–image of ω_1 of character $\leq \omega_1$ contains a closed copy of ω_1. The author together with Peter Nyikos has shown [7] that the Continuum Hypothesis (CH) is consistent with the statement that all first countable closed pre–images of ω_1 contain a copy of ω_1. A result of Nyikos (proved in Fremlin's paper [9]) shows that a weak version of $\diamondsuit$ (in fact, so weak that it is compatible with Martin's Axiom) suffices to produce a first countable perfect (in fact, each fiber has size at most two) pre–image of ω_1 that does not contain a copy of ω_1.

Our proof is primarily independent of earlier work, although the result of Alan Dow [3] that PFA implies compact spaces of countable tightness contain points of first countability is used in a crucial way during our argument. The notion of forcing we use is of a type similar to that used in the papers [8], [6], and [7].

We assume that the reader has some familiarity with elementary submodels and proper forcing, as well as some knowledge of general topology. Those who are familiar with Dow's papers [4], [5], and [3] will have no problems with the arguments presented here.

Date: June 14, 2002.

1991 *Mathematics Subject Classification.* 03E75.

Key words and phrases. Proper Forcing axiom, pre–images of ω_1, countable tightness.

The author would like to thank Mark and Joyce D'Mello for their hospitality while this research was completed.

2. Traces of elementary submodels

Definition 2.1. Let X be a topological space. A subset A of X is said to be *ω–closed* if for every countable $A_0 \subseteq A$, $\overline{A}_0 \subseteq A$.

It is not hard to see that the ω–closed subsets of A behave a lot like closed sets — in fact, they are precisely the closed sets in the finer topology on X generated by the closure operation

$$\mathrm{cl}_\omega(A) = \bigcup\{\mathrm{cl}_X A_0 : A_0 \in [A]^{\aleph_0}\}.$$

(It is not hard to see that this is in fact a closure operation.) Also note that closed sets are trivially ω–closed.

Assume the following facts about the topological space X:

- X is regular
- there is a closed, continuous map π from X onto ω_1
- $|X| = \aleph_1$
- $\mathcal{F}$ is a maximal filter of ω–closed subsets of X that contains all sets of the form $\pi^{-1}(C)$ for $C \subseteq \omega_1$ closed unbounded

Our first goal will be to show that the filter $\mathcal{F}$ behaves in many ways like the filter of closed unbounded subsets of ω_1.

Definition 2.2.

(1) The phrase "*for almost all x*" means "the set of such x is in $\mathcal{F}$".

(2) If $A \subseteq X$ and $\alpha < \omega_1$, then by "*the α^{th} level of A*", we mean $A \cap \pi^{-1}(\{\alpha\})$. We denote the α^{th} level of A by $A(\alpha)$.

Lemma 2.3.

(1) If $\{x_n : n \in \omega\} \subseteq X$ is such that $\pi(x_n) < \pi(x_{n+1})$, then $\{x_n : n \in \omega\}$ has a limit point.

(2) $\mathcal{F}$ is countably closed.

Proof. The first statement is immediate because the mapping π is a closed mapping. For the second, assume $\{A_n : n \in \omega\}$ is subset of $\mathcal{F}$. We want to show that the intersection of the A_n's is in $\mathcal{F}$. Since $\mathcal{F}$ is a filter, without loss of generality $A_{n+1} \subseteq A_n$.

We first show that $\bigcap_{n\in\omega} A_n$ is non–empty. To do this, we choose $x_n \in A_n$ in such a way that $\pi(x_n) < \pi(x_{n+1})$. We can do this because we assumed that $\mathcal{F}$ contains $\pi^{-1}(C)$ for each closed unbounded C. Now we apply the first part of this lemma to get that $\{x_n : n \in \omega\}$ has a limit point x. Since the A_n's are decreasing and ω–closed, we have that x is in each A_n. To finish, a similar argument establishes that the intersection of the A_n's meets every set in $\mathcal{F}$, and hence by maximality of $\mathcal{F}$, it must be an element of $\mathcal{F}$. □

The next lemma shows that the filter $\mathcal{F}$ is closed under a generalized Cantor–Bendixson operation. We will need this fact at one point later on.

Definition 2.4. Given $A \subseteq X$, let us define a set A^+ by $x \in A^+$ if and only if there is a countable $A_0 \subseteq A$ such that $x \in \overline{A_0}$ and $\pi[A_0] \subseteq \pi(x)$.

Thus x is in A^+ if and only if there is a countable $A_0 \subseteq A$ such that x is in the closure of A_0 and $\pi(y) < \pi(x)$ for all $y \in A_0$.

Lemma 2.5. If A is in $\mathcal{F}$, then A^+ is in $\mathcal{F}$ as well.

Proof. Since A is ω–closed, it is clear that A^+ is a subset of A; we will check first that A^+ is ω–closed.

Let $B = \{x_n : n \in \omega\}$ be a countable subset of A^+, and suppose that $z \in \overline{B}$. Let $B_n \subseteq A$ be a set that witnesses x_n's membership in A^+.

Let $\delta_n = \pi(x_n)$. Since $z \in \overline{B}$, we know that without loss of generality, $\pi(z) = \sup\{\delta_n : n \in \omega\}$, and hence $\pi[B_n] \subseteq \pi(z)$ for each n. Thus the union of the B_n's will witness that z belongs to A^+.

Now suppose by way of contradiction that $A^+ \notin \mathcal{F}$. By the maximality $\mathcal{F}$, there is a set $B \in \mathcal{F}$ such that $B \cap A^+ = \emptyset$. Now $A \cap B \in \mathcal{F}$, so we can choose points $\{x_n : n \in \omega\}$ in $A \cap B$ such that $\pi(x_n) < \pi(x_{n+1})$. This set of points has a limit $\bar{x}$, and it is routine to verify that $\bar{x} \in A^+ \cap B$, a contradiction. □

Now we begin an investigation of how the topological structures we are looking at interact with elementary submodels. We will also be noting that various subsets of X are countable intersections of separable closed sets — although this may seem a bit strange, we will need the information in the last section of the paper.

As a start, let N be a countable elementary submodel of $H(\lambda)$ for some large regular λ, and assume $\{X, \mathcal{F}\} \in N$.

Definition 2.6. The *trace of* N, denoted $\mathrm{Tr}(N)$, is defined by

$$\mathrm{Tr}(N) = \bigcap_{A \in N \cap \mathcal{F}} \mathrm{cl}(N \cap A). \tag{2.1}$$

Proposition 2.7. $\mathrm{Tr}(N)$ is a non–empty closed subset of X. Moreover, $\mathrm{Tr}(N)$ is a countable intersection of separable closed subsets of X.

Proof. It follows immediately from the definition of $\mathrm{Tr}(N)$ that it is a countable intersection of separable closed sets (and hence closed), so we need only show that $\mathrm{Tr}(N)$ is not empty. Let $\{A_n : n \in \omega\}$ enumerate $N \cap \mathcal{F}$. By our assumptions on $\mathcal{F}$, we can choose points $x_n \in N \cap \bigcap_{i \leq n} A_i$ such that $\pi(x_n) < \pi(x_{n+1})$. Then the set $\{x_n : n \in \omega\}$ has a limit point which by construction is an element of $\mathrm{Tr}(N)$. □

Proposition 2.8. Let $\mathfrak{M} = \langle M_\alpha : \alpha < \omega_1 \rangle$ be an $\in$–increasing chain of countable elementary submodels of $H(\lambda)$, continuous at limit ordinals, such that $\{X, \mathcal{F}\} \in M_0$ and for $\alpha < \omega_1$, $\langle M_\beta : \beta < \alpha \rangle \in M_{\alpha+1}$. Then

$$\mathrm{Tr}(\mathfrak{M}) := \bigcup_{\alpha < \omega_1} \mathrm{Tr}(M) \text{ is } \omega\text{–closed.} \tag{2.2}$$

Proof. We prove by induction on $\alpha < \omega_1$ that the set

$$A_\alpha = \bigcup_{\beta \leq \alpha} \operatorname{Tr}(M_\beta) \tag{2.3}$$

is ω–closed; clearly this suffices as every countable subset of $\operatorname{Tr}(\mathfrak{M})$ is contained in A_α for some α.

The cases where $\alpha = 0$ or α is a successor ordinal are already handled by Proposition 2.7 (and the induction hypothesis), so assume that α is a limit ordinal.

Let $A_{<\alpha}$ denote $\cup_{\beta<\alpha} \operatorname{Tr}(M_\beta)$. To show that A_α is ω–closed, it suffices to prove

$$\operatorname{cl}_\omega A_{<\alpha} \setminus A_{<\alpha} \subseteq \operatorname{Tr}(M_\alpha),$$

so assume that x is a member of $\operatorname{cl}_\omega A_{<\alpha} \setminus A_{<\alpha}$.

Let U be any neighborhood of x. Since x is not in $A_{<\alpha}$, our induction hypothesis implies that U must intersect A_β for arbitrarily large $\beta < \alpha$.

Now given $B \in M_\alpha \cap \mathcal{F}$, there is some $\beta_0 < \alpha$ with $B \in M_{\beta_0}$, and hence there is a $\beta < \alpha$ such that $B \in M_\beta$ and $U \cap \operatorname{Tr}(M_\beta) \neq \emptyset$.

By the definition of $\operatorname{Tr}(M_\beta)$, we have that $U \cap M_\beta \cap B$ is non–empty, and hence $U \cap M_\alpha \cap B$ is non–empty. Since U was an arbitrary neighborhood of x and B was an arbitrary member of $M_\alpha \cap \mathcal{F}$, we have that $x \in \operatorname{Tr}(M_\alpha)$ as required. □

Proposition 2.9. If $\mathfrak{M}$ is as in the previous Proposition, then $\operatorname{Tr}(\mathfrak{M}) \in \mathcal{F}$.

Proof. We know that $\operatorname{Tr}(\mathfrak{M})$ is ω–closed, so it suffices (because of the maximality of $\mathcal{F}$) to show that it meets every set in $\mathcal{F}$. Let $B \in \mathcal{F}$ be arbitrary, and let N be a countable elementary submodel of $H(\lambda)$ that contains X, $\mathcal{F}$, $\mathfrak{M}$, and B. Note that if $\delta = N \cap \omega_1$, then $M_\delta \cap \omega_1 = \delta$ as well, and since $|X| = \aleph_1$, we have $N \cap X = M_\delta \cap X$.

For $\alpha < \delta$, $M_\alpha \in N$ and hence $M_\alpha \subseteq N$ as well. Thus $M_\delta \subseteq N$. Together with the fact that $M_\delta \cap X = N \cap X$, we have that $\operatorname{Tr}(N) \subseteq \operatorname{Tr}(M_\delta)$. Since $\operatorname{Tr}(N)$ is a non–empty subset of A (as $A \in N \cap \mathcal{F}$), we have that $A \cap \operatorname{Tr}(\mathfrak{M}) \neq \emptyset$, as required. □

Corollary 2.10. Almost every point of X is a member of $\operatorname{Tr}(N)$ for some appropriate N.

Corollary 2.11. $\mathcal{F}$ is generated by sets whose levels are all countable intersections of separable closed sets, i.e., given $A \in \mathcal{F}$, we can find $B \subseteq A$ in $\mathcal{F}$ such that each level of B is a countable intersection of separable closed sets.

Proof. Choose a tower $\mathfrak{N}$ as above with $A \in N_0$. Then $\operatorname{Tr}(\mathfrak{N})$ is as required. □

There is another closed subset of X that is natural to consider when discussing countable elementary submodels.

Definition 2.12. If N is a countable elementary submodel of $H(\lambda)$ containing X and $\mathcal{F}$, then we define the weak trace of N, denoted $\mathrm{wTr}(N)$, by the formula

$$\mathrm{wTr}(N) = \bigcap N \cap \mathcal{F}. \tag{2.4}$$

Since $\mathcal{F}$ is countably complete, we know that $\mathrm{wTr}(N)$ is (ω–closed and) a member of $\mathcal{F}$. The following fact will be used later.

Proposition 2.13. Let N be a countable elementary submodel of $H(\lambda)$ that contains X and $\mathcal{F}$. Let $\delta = N \cap \omega_1$. Then the δ^{th} level of $\mathrm{wTr}(N)$ is a countable intersection of separable closed sets.

Proof. Let $\{A_n : n \in \omega\}$ be a family of sets in $N \cap \mathcal{F}$ that generates $N \cap \mathcal{F}$. We may assume (by Corollary 2.11) that each level of each A_n is a countable intersection of separable closed sets. It is routine to verify that

$$\mathrm{wTr}(N)(\delta) = \bigcap_{n<\omega} A_n(\delta), \tag{2.5}$$

and therefore $\mathrm{wTr}(N)(\delta)$ is also a countable intersection of separable closed sets. □

3. Promises

In this section, we will spend a little time on the combinatorial machinery that makes our notion of forcing work. This machinery has appeared in various forms in earlier works, see for example [8], [6], and [7]

Definition 3.1. Let us say that a subset A of X is *large* if it meets every set in $\mathcal{F}$, otherwise we say that A is small.

Note that since $\mathcal{F}$ is closed under countable intersections, any countable union of small sets is small.

Definition 3.2. A *promise* is a function f whose domain is a large subset of X such that for $x \in \operatorname{dom} f$, $f(x)$ is an open neighborhood of x, i.e., f is a neighborhood assignment for a large subset of X.

Definition 3.3. If f is a promise, then we say a point y is *banned by f* if

$$\{x \in \operatorname{dom} f : y \in f(x)\} \text{ is large.} \tag{3.1}$$

We let $\operatorname{Ban} f$ be the set of all $y \in X$ that are banned by f.

Theorem 1. *If f is a promise, then* $\operatorname{Ban} f$ *is an ω–closed set that is not in $\mathcal{F}$.*

Proof. We first show that $\operatorname{Ban} f$ is ω–closed. Suppose not, and let y be a point in $\mathrm{cl}_\omega \operatorname{Ban} f$ that is not banned by f. Thus there is a countable set $A = \{y_n : n \in \omega\} \subseteq \operatorname{Ban} f$ such that $y \in \overline{A}$. Now let

$$B = \{x \in \operatorname{dom} f : y \in f(x)\}. \tag{3.2}$$

Note that B is large as y is not banned by f.

For $n \in \omega$, we let

$$B_n = \{x \in B : y_n \in f(x)\}. \tag{3.3}$$

Each B_n is small as y_n is banned by f, but since $y \in \overline{A}$, we have

$$B = \bigcup_{n \in \omega} B_n, \tag{3.4}$$

which is a contradiction.

Now suppose $\operatorname{Ban} f$ is an element of $\mathcal{F}$. For each $y \in \operatorname{Ban} f$, let A_y be a set in $\mathcal{F}$ such that

$$A_y \cap \{x \in \operatorname{dom} f : y \in f(x)\} = \emptyset. \tag{3.5}$$

Now let $\mathfrak{M} = \langle M_\alpha : \alpha < \omega_1 \rangle$ be an $\in$–chain of countable elementary submodels as in the previous section such that both the promise f and the function $y \to A_y$ are elements of M_0.

Now choose a point $x \in \operatorname{dom} f \cap \operatorname{Tr}(\mathfrak{M})$, say $x \in \operatorname{Tr}(M_\alpha) \cap \operatorname{dom} f$. By definition of $\operatorname{Tr}(M_\alpha)$, we can find a point

$$y \in f(x) \cap M_\alpha \cap \operatorname{Ban} f, \tag{3.6}$$

and this is a contradiction as $A_y \in M_\alpha \cap \mathcal{F}$ implies $x \in \operatorname{Tr}(M_\alpha) \subseteq A_y$. □

4. The notion of forcing

In this section, we will define a notion of forcing that will be used to "shoot" closed copies of ω_1 through suitable closed pre–images of ω_1. For this section, we assume the following about our topological space X.

(1) X is regular.
(2) $|X| = \aleph_1$
(3) $\pi : X \to \omega_1$ is a closed mapping
(4) $\mathcal{F}$ is a maximal filter of ω–closed subsets of X that contains $\pi^{-1}(C)$ for every closed unbounded $C \subseteq \omega_1$
(5) if $A \subseteq X$ is a countable intersection of separable closed sets, then there is a point $z \in A$ such that $\chi(z, A) = \aleph_0$, i.e., A contains a point that is first countable in the subspace topology on A

We will show that there is a (totally) proper notion of forcing P that shoots a closed copy of ω_1 through X.

Definition 4.1. A condition p is a triple (σ_p, A_p, Φ_p) such that

(1) $\operatorname{dom} \sigma_p = \alpha + 1$ for some $\alpha < \omega$
(2) σ_p is a one–to–one continuous function into X
(3) $\beta < \gamma \leq \alpha \to \pi(\sigma_p(\beta)) < \pi(\sigma_p(\gamma))$
(4) $A_p \in \mathcal{F}$
(5) Φ_p is a countable set of promises

We say that a condition q extends p, written $q \leq p$ if

(5) $\sigma_q \supseteq \sigma_p$
(6) $\Phi_q \supseteq \Phi_p$
(7) $[q] \setminus [p] \subseteq A_p$
(8) $A_q \subseteq A_p$
(9) if $f \in \Phi_p$, then

$$Y(f,q,p) := \{x \in \operatorname{dom} f : [q] \setminus [p] \subseteq f(x)\} \text{ is large,}$$

and furthermore $f \restriction Y(f,q,p) \in \Phi_q$.

Definition 4.2. Assume $p \in P$ and $D \subseteq P$ is dense. A point x is *good to p and D* if for every neighborhood U of x, there is a $q \leq p$ in D such that $[q] \setminus [p] \subseteq U$. We let $\mathrm{Good}(p, D)$ be the set of all points in X that are good to p and D.

Proposition 4.3. If $p \in P$ and $D \subseteq P$ is dense, then $\mathrm{Good}(p, D) \in \mathcal{F}$.

Proof. Note that $\mathrm{Good}(p, D)$ is a closed set by its definition. Suppose the proposition fails, so $B := X \setminus \mathrm{Good}(p, D)$ is large. For each $x \in B$, choose a neighborhood U_x such that there is no $q \leq p$ in D satisfying $[q] \setminus [p] \subseteq U_x$. The function f with domain B sending x to U_x is a promise (it's domain is large), and also

$$p' := (\sigma_p, A_p, \Phi_p \cup \{f\}) \tag{4.1}$$

is a condition in P that extends p. Since D is dense, we can find a condition $q \leq p'$ in D. By definition of extension,

$$Y(f,q,p') := \{x \in \operatorname{dom} f : [q] \setminus [p'] \subseteq f(x)\} \text{ is large.} \tag{4.2}$$

For $x \in Y(f, q, p')$, we have

$$[q] \setminus [p] = [q] \setminus [p'] \subseteq f(x) = U_x, \tag{4.3}$$

contradicting our choice of U_x. □

We will need a slight strengthening of the preceding definition and proposition.

Definition 4.4. Let $p \in P$, $D \subseteq P$ dense, and $A \in \mathcal{F}$ be given. A point x is good to p, D, and A if for every neighborhood U of x, there is $q \leq p$ in D such that $[q] \setminus [p] \subseteq U \cap A$. We let $\mathrm{Good}(p, D, A)$ be the set of all points in X that are good to p, D, and A.

Corollary 4.5. $\mathrm{Good}(p, D, A) \in \mathcal{F}$.

Proof. Note that $\mathrm{Good}(p, D, A)$ is a closed subset of A. Now define

$$q = (\sigma_p, A_p \cap A, \Phi_p).$$

Clearly q is an extension of p in P. One finds upon checking the relevant definitions that

$$\mathrm{Good}(q, D) \subseteq \mathrm{Good}(p, D, A), \tag{4.4}$$

and since the former set is in $\mathcal{F}$, the corollary is proved. □

Definition 4.6. Recall that a collection $\mathcal{B}$ is a π–network for the point z in the topological space X if for each open neighborhood U of z, there is a non–empty $B \in \mathcal{B}$ such that $B \subseteq U$. We do not require that B is open, or that $z \in B$.

Definition 4.7. A point $x \in X$ is nice to p, D, and A if there is a countable family of conditions $\{q_n : n \in \omega\}$ such that

- $q_n \leq p$
- $q_n \in D$
- $\{[q_n] \setminus [p] : n \in \omega\}$ is a π–network for x in A

We let $\mathrm{Nice}(p, D, A)$ be the set of all points in X that are nice to p, D, and A.

Proposition 4.8. $\mathrm{Nice}(p, D, A)$ is an ω–closed subset of $\mathrm{Good}(p, D, A)$.

Proof. The fact that $\mathrm{Nice}(p, D, A)$ is a subset of $\mathrm{Good}(p, D, A)$ follows by the definitions involved. Now suppose $\{a_n : n \in \omega\} \subseteq \mathrm{Nice}(p, D, A)$ and z is in the closure of $\{a_n : n \in \omega\}$. For each n, let $\mathcal{A}_n$ be a family witnessing that a_n is nice to p, D, and A. Then it is easily seen that the union of the $\mathcal{A}_n$'s witnesses that z is nice to p, D, and A. □

The next theorem is crucial; it has a good claim to being the heart of the entire paper.

Theorem 2. *Suppose $p \in P$, $D \subseteq P$ is dense, and $A \in \mathcal{F}$. Then almost every point in X is nice to p, D, and A.*

Proof. Our proof of this result will fill the remainder of this section. Suppose by way of contradiction that the theorem fails. Since $\mathrm{Nice}(p, D, A)$ is ω–closed, there is a set $E \in \mathcal{F}$ such that no point in D is nice to p, D, and A.

Let N be a countable elementary submodel of $H(\lambda)$ that contains X, P, $\mathcal{F}$, p, D, A, and E, and let $\delta = N \cap \omega_1$.

The δ^{th} level of $\mathrm{wTr}(N)$ is a countable intersection of separable closed sets, hence there is a point $z \in \mathrm{wTr}(N)(\delta)$ such that $\chi(z, \mathrm{wTr}(N)(\delta)) = \aleph_0$. We will show that z is nice to p, D, and A — this will be a contradiction as $z \in \mathrm{wTr}(N) \subseteq E$.

Fix a family $\{U_n : n \in \omega\}$ of open neighborhoods of z such that

- $\pi[U_0] \subseteq \delta + 1$
- $\overline{U}_{n+1} \subseteq U_n$
- $\{U_n \cap \mathrm{wTr}(N) : n \in \omega\}$ is a neighborhood base for z in $\mathrm{wTr}(N)(\delta)$.

Note that since $\pi^{-1}[\delta + 1]$ is clopen in X, without loss of generality $\pi[U_n] \subseteq \delta + 1$ for each n, and $\{U_n \cap \mathrm{wTr}(N) : n \in \omega\}$ is a neighborhood base for z in $\mathrm{wTr}(N)$.

Claim 4.9. Let V be any open neighborhood of z. Then there is an $m < \omega$ and $B \in N \cap \mathcal{F}$ such that $U_m \cap B \subseteq V$.

The proof of this lemma is a bit convoluted, so we will break it down into smaller pieces to aid in comprehension.

Proof. Since X is regular, we can find an open neighborhood W of z such that $\overline{W} \subseteq V$. By our choice of $\{U_n : n \in \omega\}$ there is an m for which $\overline{U}_m \cap \mathrm{wTr}(N) \subseteq W$.

Subclaim 1. It suffices to show that there is a $B \in N \cap \mathcal{F}$ such that

$$\pi[U_m \cap B \setminus W] \subseteq \{\delta\}. \tag{4.5}$$

Proof. Suppose we have a set $B \in N \cap \mathcal{F}$ satisfying (4.5). The set B^+ (see Lemma 2.5) is also an element of $N \cap \mathcal{F}$, and since $B^+ \subseteq B$, we have

$$\pi[U_m \cap B^+ \setminus W] \subseteq \{\delta\}. \tag{4.6}$$

We claim that $U_m \cap B^+ \setminus W$ is a subset of $\overline{W}$. To see this, suppose it fails. Then there is a point $y \in U_m \cap B^+ \setminus W$ that is not in $\overline{W}$. Note that $\pi(y) = \delta$. Now $U_m \setminus \overline{W}$ is an open neighborhood of y, and since $y \in B^+$, there is some point $w \in U_m \cap B \setminus \overline{W}$ with $\pi(w) < \pi(y) = \delta$. This point w witnesses that (4.5) is false, so we have a contradiction.

To finish the proof of the claim, note that we have proved

$$U_m \cap B^+ \subseteq \overline{W} \subseteq V, \tag{4.7}$$

so U_m together with B^+ will serve to show the conclusion of Lemma 4.9. □

Subclaim 2. In order to prove Lemma 4.9, it suffices to show that there is a set $B \in N \cap \mathcal{F}$ and an $\alpha < \delta = N \cap \omega_1$ such that

$$\pi[U_m \cap B \setminus W] \cap [\alpha, \delta) = \emptyset. \tag{4.8}$$

Proof. Suppose we have a set $B \in N \cap \mathcal{F}$ satisfying (4.8). Note that α is an element of N, hence so is the set $B' = B \cap \pi^{-1}([\alpha, \omega_1))$. Note that B' is also an element of $\mathcal{F}$. Since $\pi[U_m] \subseteq \delta + 1$, we have

$$\pi[U_m \cap B' \setminus W] \subseteq \{\delta\}, \tag{4.9}$$

and so we now appeal to Subclaim 1. □

To finish the proof of our lemma, we show that there are a set $B \in N \cap \mathcal{F}$ and $\alpha < \delta$ as in the statement of Subclaim 2

Suppose by way of contradiction that this fails. Then for each $B \in N \cap \mathcal{F}$ and $\alpha \in N \cap \omega_1$, we can find a point y in $U_m \cap B \setminus W$ such that $\alpha \leq \pi(y) < \delta$.

Let $\{B_n : n \in \omega\}$ list $N \cap \mathcal{F}$, and choose a sequence of points $\{y_n : n \in \omega\}$ such that

- $y_n \in U_m \setminus W$,
- $y_n \in \bigcap_{j \leq n} B_j$, and
- $\pi(y_n) < \pi(y_{n+1}) < \delta$.

There are no obstacles to this because of our assumptions.

The set $\{y_n : n \in \omega\}$ has a limit point $\bar{y}$ by Lemma 2.3. Now $\bar{y} \in \mathrm{wTr}(N)$ because each member of $N \cap \mathcal{F}$ is ω–closed and contains all but finitely many of the y_n's. Also, $\bar{y} \in \overline{U}_m$ because each y_n is in U_m. Thus

$$\bar{y} \in \overline{U}_m \cap \mathrm{wTr}(N) \subseteq W. \tag{4.10}$$

Since W is an open set disjoint to $\{y_n : n \in \omega\}$, it follows immediately that

$$W \cap \overline{\{y_n : n \in \omega\}} = \emptyset, \tag{4.11}$$

and this is a contradiction. Thus there are $B \in N \cap \mathcal{F}$ and $\alpha < \delta$ as in the statement of Subclaim 2, and Claim 4.9 is proved. □

Corollary 4.10 (Corollary to Claim 4.9)**.** The sets of the form $U_m \cap B$, where $m < \omega$ and $B \in N \cap \mathcal{F}$ is a subset of A, are a π–network for z in A.

Lemma 4.11. Given $m < \omega$ and $B \in N \cap \mathcal{F}$ with $B \subseteq A$, we can find $q \leq p$ in D such that $[q] \setminus [p] \subseteq U_m \cap B$.

Proof. Since p, D, and A are elements of N, the set $\mathrm{Good}(p, D, A) \in N \cap \mathcal{F}$. Since $z \in \mathrm{wTr}(N)$, we know that z is good to p, D, and A. Since U_m is an open neighborhood of z, the definition of goodness gives us a q (not necessarily an element of N) as required. □

Since $N \cap \mathcal{F}$ is countable, by combining Corollary 4.10 and Lemma 4.11, we see that z is an element of $\mathrm{Nice}(p, D, A)$. As mentioned before, this gives us a contradiction and proves Theorem 2. □

5. Total Properness

Our goal in this section is to prove that the notion of forcing introduced in the last section is totally proper. We take a moment to recall the relevant definitions.

Definition 5.1. A notion of forcing P is totally proper if whenever we are given $N \prec H(\lambda)$ countable (with λ "large enough") such that $P \in N$, and $p \in N \cap P$, we can find $q \leq p$ such that for every dense open subset D of P that is in N, there is some $p' \in N \cap D$ with $q \leq p$. Such a q is said to be totally (N, P)–generic.

It is shown in [8] that a notion of forcing is totally proper if and only if it is proper and forcing with it adds no new countable subsets to the ground model.

Theorem 3. *If X, π, $\mathcal{F}$, and P are as in the previous section, then P is totally proper.*

We will postpone the proof of this theorem for a little bit as there are several preliminary results required.

Let us fix objects N, δ, z, $\{U_n : n \in \omega\}$ and $\{A_n : n \in \omega\}$ such that

(1) N is a countable elementary submodel of $H(\lambda)$ for some large λ

(2) $\{X, \mathcal{F}, P\} \in N$
(3) $\delta = N \cap \omega_1$
(4) $z \in \mathrm{Tr}(N)$ satisfies $\chi(z, \mathrm{Tr}(N)) = \aleph_0$
(5) $\{U_n : n \in \omega\}$ is a family of open neighborhoods of z in X such that
   - $\pi[U_0] \subseteq \delta + 1$
   - $\overline{U}_{n+1} \subseteq U_n$
   - $\{U_n \cap \mathrm{Tr}(N) : n \in \omega\}$ is a neighborhood base for z in $\mathrm{Tr}(N)$

(6) $\{B_n : n \in \omega\} \subseteq N \cap \mathcal{F}$ satisfies
   - $B_{n+1} \subseteq B_n$
   - if $B \in N \cap \mathcal{F}$, then there is an n such that $B_n \subseteq B$ (so the B_n's generate $N \cap \mathcal{F}$)

Definition 5.2. A target is a pair (U, A) where U is a neighborhood of z and $A \in N \cap \mathcal{F}$. A target (V, B) refines (U, A) if $V \subseteq U$ and $B \subseteq A$.

The main facts we need about targets will be provided by Propositions 5.3 and 5.5 below. Once these facts have been established, we will show that our forcing is totally proper.

Proposition 5.3. If (U, A) is a target and $f \in N$ is a promise, then we can refine (U, A) to a target (V, B) such that

$$\{x \in \mathrm{dom}\, f : (N \cap V \cap B) \cup \{z\} \subseteq f(x)\} \text{ is large.} \tag{5.1}$$

Proof. By our choice of $\{U_n : n \in \omega\}$, we know that for $x \in \mathrm{dom}\, f$,

$$z \in f(x) \iff (\exists n < \omega)\overline{U}_n \cap \mathrm{Tr}(N) \subseteq f(x). \tag{5.2}$$

Since $z \notin \mathrm{Ban}\, f$, there is an i such that

$$\{x \in \mathrm{dom}\, f : \overline{U}_i \cap \mathrm{Tr}(N) \subseteq f(x)\} \text{ is large.} \tag{5.3}$$

Lemma 5.4. If W is an open set such that $\overline{U}_i \cap \mathrm{Tr}(N) \subseteq W$, then there is a $B \in N \cap \mathcal{F}$ such that $N \cap U_i \cap B \subseteq W$.

Proof. The proof of this is very similar to that of Claim 4.9, but much easier. We first show that there is a $B \in N \cap \mathcal{F}$ such that

$$(N \cap U_i \cap B) \setminus W \subseteq \pi^{-1}[\alpha] \text{ for some } \alpha \in N \cap \omega_1. \tag{5.4}$$

By way of contradiction, suppose this fails, so for every $B \in N \cap \mathcal{F}$ and $\alpha \in N \cap \omega_1$, we can find $y \in (N \cap U_i \cap B) \setminus W$ with $\pi(y) > \alpha$.

Because of our choice of the family $\{B_n : n \in \omega\}$, we can choose points y_n for $n \in \omega$ such that

- $y_n \in (N \cap U_i \cap B_n) \setminus W$
- $m < n \longrightarrow \pi(x_m) < \pi(x_n)$.

Now $\{y_n : n \in \omega\}$ has a limit point $\bar{y}$ by Lemma 2.3. Note that $\bar{y} \in \mathrm{Tr}(N)$ by the definition of $\mathrm{Tr}(N)$. Also, $\bar{y} \in \overline{U}_i$, hence

$$\bar{y} \in \overline{U}_i \cap \mathrm{Tr}(N) \subseteq W. \tag{5.5}$$

This is a contradiction, as $W \cap \overline{\{y_n : n \in \omega\}} = \emptyset$.

To finish, note that for $\alpha \in N \cap \omega_1$, $B \setminus \pi^{-1}[\alpha] \in N \cap \mathcal{F}$, so we can achieve

$$N \cap U_i \cap B \subseteq W, \tag{5.6}$$

as required. □

Now back to the proof of Proposition 5.3. We know

$$\{x \in \operatorname{dom} f : \overline{U}_i \cap \operatorname{Tr}(N) \subseteq f(x)\} \text{ is large.} \tag{5.7}$$

By the previous lemma,

$$\overline{U}_i \cap \operatorname{Tr}(N) \subseteq f(x) \Longrightarrow \exists B' \in N \cap \mathcal{F} \text{ such that } N \cap U_i \cap B' \subseteq f(x). \tag{5.8}$$

Since $N \cap \mathcal{F}$ is countable, there is a single $B \in N \cap \mathcal{F}$ (without loss of generality $B \subseteq A$) for which

$$\{x \in \operatorname{dom} f : \overline{U}_i \cap \operatorname{Tr}(N) \subseteq f(x) \text{ and } N \cap U_i \cap B \subseteq f(x)\} \text{ is large.} \tag{5.9}$$

Letting $V = U_i$, we find that (V, B) is a target that refines (U, A) for which

$$\{x \in \operatorname{dom} f : (N \cap V \cap B) \cup \{z\} \subseteq f(x)\} \text{ is large.} \tag{5.10}$$

□

Proposition 5.5. Let (V, A) be a target, and assume that $p \in N \cap P$ and D is a dense subset of P that is an element of N. Then we can find a condition $q \leq p$ in $N \cap D$ such that $[q] \setminus [p] \subseteq N \cap V \cap A$.

Proof. Note that $\operatorname{Nice}(p, D, A)$ is an element of N, and by Theorem 2 it is also a member of $\mathcal{F}$. Since $z \in \operatorname{Tr}(N)$ and V is an open neighborhood of z, there is a point $y \in N \cap U \cap \operatorname{Nice}(p, D, A)$. For this particular y, we have a countable family of conditions $\{q_n : n \in \omega\}$ witnessing that y is nice to p, D, and A. There must be such a family in N by elementarity (as $y \in N$!), and since the family is countable, we know that each q_n is an element of N. Another application of elementarity tells us that there is an n for which $[q_n] \setminus [p] \subseteq V \cap A$. Since $[q_n] \setminus [p]$ is a countable set in N, it is a subset of N. Thus $[q_n] \setminus [p] \subseteq N \cap V \cap A$, and $q = q_n$ works. □

Proof of Theorem 3. Let N be a countable elementary submodel of $H(\lambda)$ that contains X, $\mathcal{F}$, π, and P. Let p be an arbitrary member of $N \cap P$. We must produce a totally (N, P)–generic $q \leq p$.

By our assumptions on X, there is a point $z \in \operatorname{Tr}(N)$ such that $\chi(z, \operatorname{Tr}(N)) = \aleph_0$. Let $\{U_n : n \in \omega\}$ be a family of open neighborhoods of z such that

- $\overline{U}_{n+1} \subseteq U_n$
- $\{U_n \cap \operatorname{Tr}(N) : n \in \omega\}$ is a neighborhood base for z in $\operatorname{Tr}(N)$

Let $\{B_n : n \in \omega\}$ be a decreasing family of members of $N \cap \mathcal{F}$ that generates $N \cap \mathcal{F}$.

We need one more lemma to help us in our proof of total properness. In some sense, this says that it is easy to force a sequence diagonalizing $N \cap \mathcal{F}$ to converge to our fixed point $z \in \operatorname{Tr}(N)$.

Lemma 5.6. Suppose $\{x_i : i \in \omega\}$ is a subset of $N \cap X$ with the property that for each n,

$$\{x_i : i \in \omega\} \setminus (U_n \cap B_n) \text{ is finite.} \tag{5.11}$$

Then $\{x_i : i \in \omega\}$ converges to z.

Proof. First, note that any infinite subset of $\{x_n : n \in \omega\}$ has a limit point. This follows because $\pi(x_n) < \delta = N \cap \omega_1$ for each n (as $x_n \in N$), and for each $\alpha < \delta$, $\pi(x_n) < \alpha$ for at most finitely many n.

Thus it suffices to show that z is the unique limit point of $\{x_n : n \in \omega\}$. To see this, let z' be any limit point of $\{x_n : n \in \omega\}$. For each $B \in N \cap \mathcal{F}$, we have that

$$\{x_n : n \in \omega\} \setminus (N \cap B) \text{ is finite.} \tag{5.12}$$

This means that z' is an element of $\mathrm{Tr}(N)$. By similar reasoning, we see that $z' \in \overline{U}_n$ for each n, i.e.,

$$z' \in \mathrm{Tr}(N) \cap \bigcap_{n \in \omega} \overline{U}_n. \tag{5.13}$$

Our choice of $\{U_n : n \in \omega\}$ now tells us that $z' = z$. □

Let $\{D_n : n \in \omega\}$ list the dense subsets of P that are elements of N. Let V be an open neighborhood of z with the property that $\pi[\overline{V}] \subseteq \delta + 1$. We shall construct, in ω stages, objects p_n, V_n, and A_n that satisfy

(1) $p_0 = p$, $V_0 = V$, $A_0 = X$
(2) (V_n, A_n) is a target (so $z \in V_n$)
(3) $p_{n+1} \in N \cap D_n$
(4) $p_{n+1} \leq p_n$
(5) (V_{n+1}, A_{n+1}) refines (V_n, A_n)
(6) $[p_{n+1}] \setminus [p_n] \subseteq N \cap V_n \cap A_n$
(7) $V_n \subseteq U_n$ and $A_n \subseteq B_n$
(8) if f is a promise appearing in Φ_{p_i} for some i, then there is an $n_f \geq i$ for which

$$\{x \in Y(f, p_{n_f+1}, p_i) : N \cap U_{n_f+1} \cap A_{n_f+1} \subseteq f(x)\} \text{ is large.} \tag{5.14}$$

We say that f is *taken care of at stage* $n_f + 1$.

We have the objects p_0, V_0, and A_0 already, so suppose we have completed stage n of the construction. We will be handed objects p_n, V_n, and A_n, as well as some promise $f \in \Phi_{p_i}$ (for some $i \leq n$) that must be taken care of at this stage.

Let $f' = f \restriction Y(f, p_n, p_i)$. Since f, p_n, and p_i are in N, we known $f' \in N$ as well. By definition of extension, we know that $Y(f, p_n, p_i)$ is large, and f' is a promise in Φ_{p_n}.

Apply Proposition 5.3 to the target (V_n, A_n) and promise f'. This gives us a target (V_{n+1}, A_{n+1}) refining (V_n, A_n) and such that

$$\{x \in \operatorname{dom} f' : (N \cap V_{n+1} \cap A_{n+1}) \cup \{z\} \subseteq f(x)\} \text{ is large.}$$

By definition, this means

$$\{x \in Y(f, p_n, p_i) : (N \cap V_{n+1} \cap A_{n+1}) \cup \{z\} \subseteq f(x)\} \text{ is large.}$$

We can shrink V_{n+1} and A_{n+1} if necessary to obtain $V_{n+1} \subseteq U_{n+1}$ and $A_{n+1} \subseteq B_{n+1}$ — this does us no harm. Now apply Proposition 5.5 to p_n, D_n, and (V_{n+1}, A_{n+1}). This gives us a condition $p_{n+1} \leq p_n$ such that

- $p_{n+1} \in N \cap D_n$
- $[p_{n+1}] \setminus [p_n] \subseteq N \cap V_{n+1} \cap A_{n+1}$.

The following claim will complete the proof that our notion of forcing is totally proper, as the condition q will be a totally (N, P)–generic extension of p.

Claim 5.7. The sequence $\{p_n : n \in \omega\}$ has a lower bound q.

Proof. First, let us define

$$\sigma_q = \bigcup_{n \in \omega} \sigma_{p_n} \cup \{\langle \delta, z \rangle\}. \tag{5.15}$$

Because of the way our construction was defined, if $\{\delta_n : n \in \omega\}$ is increasing and cofinal in δ, Lemma 5.6 implies that the sequence $\{\sigma_q(\delta_n) : n \in \omega\}$ converges to z. Thus σ_q is a continuous map from $\delta + 1$ into X.

let $A_q = \bigcap_{n \in \omega} A_{p_n}$, and define

$$[q] = \{z\} \cup \bigcup_{n \in \omega} [p_n]. \tag{5.16}$$

Suppose $f \in \Phi_{p_i}$ for some i. Define

$$Y(f, q, p_i) = \{x \in \operatorname{dom} f f : [q] \setminus [p_i] \subseteq f(x)\}. \tag{5.17}$$

Let n_f be the stage in our construction where f was taken care of, so (V_{n_f+1}, A_{n_f+1}) was chosen so that

$$\{x \in Y(f, p_{n_f}, p_i) : (N \cap V_{n_f+1} \cap A_{n+1}) \cup \{z\} \subseteq f(x)\} \text{ is large.}$$

Our construction guarantees that this set is a subset of $Y(f, q, p_i)$, hence $Y(f, q, p_i)$ is large. Thus, if we define

$$\Phi_q = \bigcup_{n \in \omega} \Phi_{p_n} \cup \bigcup_{n \in \omega} \{f \restriction Y(f, q, p_n) : f \in \Phi_{p_n}\}, \tag{5.18}$$

then $q = (\sigma_q, A_q, \Phi_q)$ is an element of P that extends each p_n, as required. □

Thus P is totally proper, and Theorem 3 is established. □

Proposition 5.8. Forcing with P adjoins a subset of X that is homeomorphic to ω_1.

Proof. Because of the way conditions are defined, it suffices to prove that for each $\alpha < \omega_1$, the set of conditions q such that $\sigma_p(\alpha)$ is defined is dense in P. We prove this by induction on α, so suppose we have the result for each $\beta < \alpha$.

Let $p \in P$ be arbitrary. Our induction hypothesis together with the fact that P is totally proper tells us that without loss of generality $\alpha \subseteq \operatorname{dom} \sigma_p$ — we can always extend p to a condition satisfying this.

Define

$$B = \bigcup \{\operatorname{Ban} f : f \in \Phi_p\}. \tag{5.19}$$

Since each $\operatorname{Ban} f$ is an ω–closed subset of X that is not in $\mathcal{F}$, the fact that Φ_p is countable (and $\mathcal{F}$ is countably complete) implies there is a set $A \in \mathcal{F}$ disjoint to each $\operatorname{Ban} f$ for $f \in \Phi_p$. Choose $\gamma \in A \cap A_p$. It is straightforward to see that if we define

$$\sigma_q = \sigma_p \cup \{\langle \alpha, \gamma \rangle\}, \tag{5.20}$$

$$A_q = A_p \cap A, \tag{5.21}$$

and

$$\Phi_q = \Phi_p \cup \bigcup \big\{ f \restriction \{x \in \operatorname{dom} f : \gamma \in g(x)\} : f \in \Phi_p \big\}, \tag{5.22}$$

then $q = (\sigma_q, A_q, \Phi_q)$ is a condition in P that extends p and that satisifies $\alpha \in \operatorname{dom} \sigma_q$. $\square$

Note that since our construction forces the copy of ω_1 to hit a closed unbounded set of levels of X, the copy of ω_1 will be closed in X as well.

6. Putting it all together

In this section, we prove the theorem stated in the abstract — PFA implies that perfect pre–images of ω_1 of countable tightness contain closed copies of ω_1.

Assume that PFA holds, and that X is a perfect pre–image of ω_1 of countable tightness. By a result of Dow [3], we know that in this model all compact spaces of countable tightness contain a point of first countability. If $\pi : X \to \omega_1$ is perfect and $A \subseteq X$ is closed, then $A(\delta)$ (the δ^{th} level of of A) is compact for each δ, and hence must contain a point of first countability. In particular, every closed subset of X contains a point of (relative) first countability — any point that is first countable in the first non–empty level of A is first countable in all of A.

Now let P_1 be the notion of forcing that collapses the cardinal $|X|$ to $\aleph_1$ using countable conditions. It is well–known that P_1 is countably closed and hence in the generic extension no new countable subsets of the ground model have appeared.

Let $G \subseteq P_1$ be generic. In the model $V[G_1]$, our space X is still regular, and the mapping $\pi : X \to \omega_1$ is still continuous. Since every countable subset of X is already in the ground model, we know that π remains a closed mapping. By our choice of P_1, we know that $|X| = \aleph_1$.

Claim 6.1. In $V[G]$, if $A \subseteq X$ is a countable intersection of separable closed sets, then there is a point $z \in A$ such that $\chi(z, A) = \aleph_0$.

Proof. Let $B \subseteq X$ be closed and separable, say $B = \overline{B}_0$ where B_0 is countable. Then B_0 is in the ground model, and hence so is B. Thus V and $V[G]$ see the same separable closed subsets of X. Similarly, given a sequence $\{A_n : n \in \omega\}$ of separable closed subsets of X, the sequence $\{A_n : n \in \omega$ must be in the ground model as well — forcing with P_1 adds no know countable sequences of elements from the ground model. Thus $\bigcap_{n\in\omega} A_n$ is in the ground model.

So, given that A is a countable intersection of separable closed subsets of X, we know that A is in the ground model, and hence by Dow's result, A contains a point z such that $\chi(z, A) = \aleph_0$ in the ground model. But then $\chi(z, A) = \aleph_0$ in $V[G]$ as well. □

Thus in $V[G_1]$, our space X satisfies

- X is regular
- $|X| = \aleph_1$
- if $A \subseteq X$ is a countable intersection of separable closed subsets of X, then A contains a point z satisfying $\chi(z, A) = \aleph_0$

Thus if (in $V[G_1]$) we let $\mathcal{F}$ be a maximal filter of ω–closed subsets of X and that contains $\pi^{-1}(C)$ for every closed unbounded $C \subseteq \omega_1$, the pair $(X, \mathcal{F})$ satisfies all the requirements needed for the totally proper notion of forcing detailed in the previous sections to exist.

Let $\dot{P}_2$ be a P_1–name for the notion of forcing that shoots a copy of ω_1 through X. The notion of forcing $P_1 * \dot{P}_2$ is totally proper, and it adjoins a one–to–one continuous function $\dot{f}$ from ω_1 into X.

For $\alpha < \omega_1$, let D_α be the set of conditions in $P_1 * P_2$ that decide a particular value for the function $\dot{f} \restriction \alpha + 1$. Since $P_1 * \dot{P}_2$ is totally proper, the set D_α is dense in $P_1 * P_2$ — this is where total properness is really needed, as it implies that each countable piece of the function $\dot{f}$ is already in the ground model.

Let $G \subseteq P_1 * \dot{P}_2$ be a filter that meets each D_α, and let $f : \omega_1 \to X$ be the function we get by interpreting the name $\dot{f}$ using G. Given a strictly increasing sequence of countable ordinals $\langle \delta_n : n \in \omega \rangle$ converging to δ, the fact that G meets D_δ tells us that the sequence $\langle f(\delta_n) : n \in \omega \rangle$ converges to $f(\delta)$ — any condition in $G \cap D_\delta$ will explicitly force this to be true. Given this, it is not hard to see that the range of f is homeomorphic to ω_1, and since X is countably tight, this copy of ω_1 is closed in X.

References

[1] Z. Balogh, A. Dow, D. H. Fremlin, and P. J. Nyikos, *Countable tightness and proper forcing*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), no. 1, 295–298. MR **89e:**03088

[2] Zoltán Balogh, *On compact Hausdorff spaces of countable tightness*, Proc. Amer. Math. Soc. **105** (1989), no. 3, 755–764. MR **89h:**03088
[3] Alan Dow, *An introduction to applications of elementary submodels to topology*, Topology Proc. **13** (1988), no. 1, 17–72. MR **91a:**54003
[4] ______, *Set theory in topology*, Recent progress in general topology (Prague, 1991), North-Holland, Amsterdam, 1992, pp. 167–197. MR 1 229 125
[5] ______, *More set-theory for topologists*, Topology Appl. **64** (1995), no. 3, 243–300. MR **97a:**54005
[6] Todd Eisworth, *CH and first countable, countably compact spaces*, Topology Appl. **109** (2001), no. 1, 55–73. MR 1 804 563
[7] Todd Eisworth and Peter J. Nyikos, *On first countable closed pre–images of* ω_1, in preparation.
[8] Todd Eisworth and Judith Roitman, *CH with no Ostaszewski spaces*, Trans. Amer. Math. Soc. **351** (1999), no. 7, 2675–2693. MR **2000b:**03182
[9] D. H. Fremlin, *Perfect pre-images of* ω_1 *and the PFA*, Topology Appl. **29** (1988), no. 2, 151–166. MR **89i:**03096

Department of Mathematics, University of Northern Iowa, Cedar Falls, IA, 50613
E-mail address: `eisworth@uni.edu`