A NOTE ON STRONG NEGATIVE PARTITION RELATIONS

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Abstract. We analyze a natural function definable from a scale at a singular cardinal, and provide a fairly simple proof that there is a strong connection between failures of square brackets partition relations at successors of singular cardinals and the club–guessing ideal $\text{id}_\mu(C, I)$.

1. Introduction

Recall that the square brackets partition relation $\kappa \rightarrow [\lambda]_\theta^\mu$ of Erdős, Hajnal, and Rado [4] asserts that for every function $F: [\kappa]^\mu \rightarrow \lambda$ (where $[\kappa]^\mu$ denotes the subsets of $\kappa$ of cardinality $\mu$), there is a set $H \subseteq \kappa$ of cardinality $\theta$ such that

$$\text{ran}(F \upharpoonright [H]^\mu) \neq \theta,$$

i.e., the function $F$ omits at least one value when we restrict it to $[H]^\mu$. Of particular interest to combinatorial set theory are the negations of square brackets partition relations, as these have strong combinatorial consequences with applications outside of pure set theory.

We will be interested in the combinatorial statement

$$\lambda \not\rightarrow [\lambda]_\lambda^2,$$

where $\lambda$ is the successor of a singular cardinal. The assertion (1.2) states that there exists a function $F: [\lambda]^2 \rightarrow \lambda$ with the property that

$$\text{ran}(F \upharpoonright [A]^2) = \lambda$$

for any unbounded subset $A$ of $\lambda$. Traditionally, more descriptive language is used when discussing (1.2) — $F$ is called a coloring, and (1.2) says that we can color the pairs of ordinals from $\lambda$ using $\lambda$ colors in such a way that all colors appear in any unbounded subset of $\lambda$, i.e., Ramsey’s Theorem fails for $\lambda$ in an incredibly spectacular way.

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The question of whether $\lambda \not\rightarrow [\lambda]^2$ (for $\lambda$ successor of singular) is an outright theorem of ZFC is still open. Much research has been devoted to this question (particularly by Shelah [5], [6], [8], [2]) and the related question of whether such a $\lambda$ can be a Jónsson cardinal. This has resulted in a complex web of conditions that tightly constrains what a potential counterexample can look like, but still no proof that a counterexample cannot exist has emerged.

In this paper, we show (assuming $\lambda = \mu^+$ for $\mu$ singular) that in many situations there is a natural coloring $c : [\lambda]^2 \rightarrow \lambda$ with the property that $c$ takes on almost every color on every unbounded $A \subseteq \lambda$. We have used two qualifying phrases in the previous sentence. The first – “in many situations” – we leave vague for now, although our theorem is general enough to cover the case where $\mu$ is singular of uncountable cofinality. The second qualifying phrase – “almost every color” – means “the set of omitted colors is small in the sense that it lies in a certain ideal associated with Shelah’s theory of guessing clubs”. The proof that the coloring has the required characteristics is a blending of techniques due to Todorčević (namely, the method of minimal walks [10]) and techniques due to Shelah (combinatorics associated with scales [5]).

This work continues research begun in [3] (where we consider colorings of finite subsets of $\lambda$ instead of coloring only the pairs from $\lambda$) and [2] (where the coloring under consideration here is implicitly used in the proof of the main result of [2]).

2. Preliminaries

The background material needed for our results divides fairly neatly into three categories: club-guessing, minimal walks, and scales. We handle each of these topics in turn.

Club Guessing

Shelah’s theory of guessing clubs has been a key ingredient in many of his theorems establishing negative square-bracket partition relations. The foundations of the theory can be found in [6], while [8], [2], and [7] provide glimpses of how useful the material can be in combinatorial set theory. We will be concerned with a particularly ubiquitous aspect of club-guessing — the ideal $\text{id}_p(\bar{C}, \bar{I})$. This ideal has a fairly complicated definition, and so its definition is the first thing we discuss.

**Definition 1.** Let $S \subseteq \lambda$ be a stationary set of limit ordinals. We say $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is an $S$-club system if $C_\delta$ is closed and unbounded in $\delta$ for each $\delta \in S$. We can extend this notion to sets containing successor
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ordinals by requiring that for successor $\delta$, $C_\delta$ is a closed subset of $\delta$ containing the predecessor of $\delta$.

Notice that the above definition makes no demands on the order–type or cardinality of $C_\delta$ – all that is required is that it be closed and unbounded in $\delta$.

**Definition 2.** Suppose $\lambda$ is a regular cardinal, $S \subseteq \lambda$ is stationary, $\bar{C} = \{C_\delta : \delta \in S\}$ is an $S$–club system, and $\bar{I} = \{I_\delta : \delta \in S\}$ is a sequence such that $I_\delta$ is an ideal on $C_\delta$. The ideals $\text{id}^p(\bar{C}, \bar{I})$ and $\text{id}_p(\bar{C}, \bar{I})$ are defined as follows:

1. A set $A \subseteq \lambda$ is in $\text{id}^p(\bar{C}, \bar{I})$ if there is a closed unbounded $E \subseteq \lambda$ such that for every $\delta \in S \cap A$, we have
   
   \[ E \cap C_\delta \in I_\delta. \]

2. A set $A \subseteq \lambda$ is in $\text{id}_p(\bar{C}, \bar{I})$ if there is a closed unbounded $E \subseteq \lambda$ such that for all $\delta \in S \cap E$,
   
   \[ E \cap A \cap C_\delta \in I_\delta. \]

The club-guessing ideals defined above are not necessarily proper ideals. However, the following fact (especially statement (2)) explains their name.

**Claim 3.** Let $\lambda, \bar{C}$, and $\bar{I}$ be as in the previous definition. Then the following are equivalent:

1. $\text{id}^p(\bar{C}, \bar{I})$ is a proper ideal.
2. for every closed unbounded $E \subseteq \lambda$, the set of $\delta \in S$ for which
   
   \[ E \cap C_\delta \notin I_\delta \]
   
   is stationary.
3. $\text{id}_p(\bar{C}, \bar{I})$ is a proper ideal.

We will not need the full generality of the preceding definition, for our results are concerned with $\text{id}_p(\bar{C}, \bar{I})$ for a particular choice of $\bar{I}$ which we define below.

**Definition 4.**

1. Suppose $C$ is closed and unbounded in $\delta$. Then
   
   \[ \text{nacc}(C) := \{ \alpha \in C : \sup(C \cap \alpha) < \alpha \}, \]
   
   and $\text{acc}(C) := C \setminus \text{nacc}(C)$.\footnote{The notation “acc” and “nacc” comes from “accumulation points” and “non–accumulation points”.

(2.4)
(2) Suppose \( \lambda = \mu^+ \) for \( \mu \) singular, and let \( C_\delta \) be a closed unbounded subset of \( \delta \) for some \( \delta < \lambda \). The ideal \( J_{C_\delta}^{[\mu]} \) is generated by sets of the form

\[
\{ \gamma \in C_\delta : \text{cf}(\gamma) < \alpha \text{ or } \gamma < \beta \}
\]

for \( \alpha < \mu \) and \( \gamma < \delta \).

We note that \( J_{C_\delta}^{[\mu]} \) is an ideal of subsets of \( C_\delta \); the main point of the definition is that if \( A \) is a subset of \( C_\delta \) that is not in \( J_{C_\delta}^{[\mu]} \), then for any \( \alpha < \mu \) and \( \beta < \delta \) there is a \( \gamma \in A \) greater than \( \beta \) and of cofinality greater than \( \alpha \).

**Minimal Walks**

Suppose now that \( \overline{C} = \langle C_\alpha : \alpha < \lambda \rangle \) is a \( \lambda \)-club system for some cardinal \( \lambda \). Given \( \alpha < \beta < \lambda \), we define the **minimal walk from \( \beta \) to \( \alpha \)** along \( \overline{C} \) to be the sequence \( \beta = \beta_0 > \cdots > \beta_{n+1} = \alpha \) defined by

\[
\beta_{i+1} = \min(C_{\beta_i} \setminus \alpha).
\]

The use of “\( n+1 \)” as the index of the last step is deliberate, as the ordinal \( \beta_n \) (the penultimate step) is quite important in our proof.

We make use of standard facts about minimal walks. In particular, suppose \( \delta \) is a limit ordinal, \( \delta < \beta < \lambda \), and \( \beta = \beta_0 > \cdots > \beta_{n+1} = \delta \) is the minimal walk from \( \beta \) to \( \delta \). For \( i < n \), we know that \( \alpha \notin C_{\beta_i} \), and so

\[
\gamma^* := \max\{ \max(C_{\beta_i} \cap \delta) : i < n \} < \delta.
\]

Suppose now that \( \gamma^* < \alpha < \delta \), and let \( \beta = \beta_0^* > \cdots > \beta_{n^*+1} = \alpha \) be the minimal walk from \( \beta \) to \( \alpha \). From the definition of \( \gamma^* \) it follows that

\[
\beta_i = \beta_i^* \text{ for } i \leq n,
\]

i.e., the walks from \( \beta \) to \( \delta \) and from \( \beta \) to \( \alpha \) agree up to and including the step before the former reaches its destination. A proper discussion of minimal walks and their applications is beyond the scope of this paper. We refer the reader to [10], [1], or [9] for more information.

**Lemma 5.** Let \( \lambda = \mu^+ \) for \( \mu \) a singular cardinal, and suppose \( \overline{C} = \langle C_\delta : \delta \in S \rangle \) is an \( S \)-club system for some stationary \( S \subseteq \lambda \). If \( \sup\{|C_\delta| : \delta \in S\} < \mu \), then there is a \( \lambda \)-club system \( \overline{e} = \langle e_\alpha : \alpha < \lambda \rangle \) such that

\begin{enumerate}
\item \( |e_\alpha| < \mu \) for all \( \alpha \), and
\item \( \delta \in e_\alpha \cap S \implies C_\delta \subseteq e_\alpha. \)
\end{enumerate}
Proof. Let $\sigma < \mu$ be a regular cardinal distinct from $\text{cf}(\mu)$, and let $\langle e_\alpha^*: \alpha < \lambda \rangle$ be a $\lambda$-club system such that $\text{otp}(e_\alpha^*) = \text{cf}(\alpha)$.

For $\alpha < \lambda$, we define a sequence $\langle e_\alpha[i]: i < \sigma \rangle$ as follows:

- $e_\alpha[0] = e_\alpha^*$
- $e_\alpha[i+1]$ is the closure in $\alpha$ of $e_\alpha[i] \cup \bigcup \{C_\delta: \delta \in e_\alpha[i] \cap S\}$
- for limit $i < \sigma$, $e_\alpha[i]$ is the closure in $\alpha$ of $\bigcup_{j<i} e_\alpha[j]$.

Now we define $e_\alpha$ to be the closure in $\alpha$ of $\bigcup_{i<\sigma} e_\alpha[i]$; it is straightforward to see that $\langle e_\alpha: \alpha < \lambda \rangle$ has all of the required properties (note that $|e_\alpha| < \mu$ because we have a uniform bound on the cardinalities of the $C_\delta$’s).

We sometimes refer to conclusion (2) of the preceding lemma by saying that $\bar{e}$ swallows $\bar{C}$. The following corollary explains why the cardinalities of the $e_\alpha$’s are of concern, and also hints at the importance of the ideal $J_{C_\delta}^{[\mu]}$ as well.

Corollary 6. Suppose $\bar{e}$ and $\bar{C}$ are as in the preceding lemma. If $\delta \in S \cap C_\alpha$, then

$$\text{acc}(e_\alpha) \cap C_\delta \in J_{C_\delta}^{[\mu]}.$$  

Proof. This follows from the definition of $J_{C_\delta}^{[\mu]}$ because any member of $\text{acc}(e_\alpha)$ must have cofinality $\leq |e_\alpha| < \mu$. 

Thus, if $\delta \in e_\alpha \cap S$, then not only is $C_\delta \subseteq e_\alpha$, but $J_{C_\delta}^{[\mu]}$–almost all points of $C_\delta$ are not accumulation points of $e_\alpha$.

Scales

Definition 7. Let $\mu$ be a singular cardinal. A scale of length $\beta$ for $\mu$ is a triple $(\vec{\mu}, \vec{f}, I)$ where

1. $\vec{\mu} = \langle \mu_i : i < \text{cf}(\mu) \rangle$ is an increasing sequence of regular cardinals such that $\sup_{i < \text{cf}(\mu)} \mu_i = \mu$.
2. $I$ is an ideal on $\text{cf}(\mu)$.
3. $\vec{f} = \langle f_\alpha : \alpha < \beta \rangle$ is a sequence of functions such that
   - $f_\alpha \in \prod_{i < \text{cf}(\mu)} \mu_i$.
   - If $\gamma < \delta < \beta$ then $f_\gamma <_I f_\beta$.
   - If $f \in \prod_{i < \text{cf}(\mu)} \mu_i$ then there is an $\alpha < \beta$ such that $f <_I f_\alpha$.

\footnote{Our convention for successor ordinals is that $e_{\alpha+1}^* = \{\alpha\}$.

\footnote{In general, the notation $f <_I g$ (where $I$ is an ideal) means that $\{i \in \text{dom}(I) : g(i) \leq f(i)\} \in I$.}
In this paper, we are concerned exclusively with the special case of the previous definition where scales are of length $\mu^+$ and $I$ is the ideal of bounded sets.

**Definition 8.** Let $\mu$ be a singular cardinal. A scale for $\mu$ is a pair $(\vec{\mu}, \vec{f})$ such that $(\vec{\mu}, \vec{f}, J^{bd})$ is a scale of length $\mu^+$ for $\mu$, where $J^{bd}$ denotes the ideal of bounded subsets of cf($\mu$). We let $<^*$ abbreviate $<_{J^{bd}}$.

It is an important theorem of Shelah [5] that scales exist for any singular $\mu$. If $\mu$ is singular and $(\vec{\mu}, \vec{f})$ is a scale for $\mu$, then there is a natural way to color the pairs of ordinals $\alpha < \beta < \mu^+$ using cf($\mu$) colors, namely

$$\Delta(\alpha, \beta) = \begin{cases} \max(\{i < \text{cf}(\mu) : f_\beta(i) \leq f_\alpha(i)\}) & \text{if there is a maximal such } i, \\ 0 & \text{otherwise.} \end{cases}$$

The use of max instead of sup is deliberate, as it makes the proof of our main theorem go a little smoother. Although we do not use it, we mention that the coloring $\Delta$ defined above serves as an example that $\mu^+ \not\rightarrow [\mu^+]^2_{\text{cf}(\mu)}$ holds for any singular $\mu$.

3. THE COLORING THEOREM

We are now in a position to state and prove the main result of the paper.

**Theorem 1.** Suppose

1. $\lambda = \mu^+$ for $\mu$ a singular cardinal
2. $S$ is a stationary subset of $\lambda$ consisting of ordinals of cofinality $\text{cf}(\mu)$
3. $\bar{C} = \langle C_\delta : \delta \in S \rangle$ is an $S$–club system such that
   - $\sup\{|C_\delta| : \delta \in S\} < \mu$, and
   - $\text{id}_p(\bar{C}, \bar{I})$ is a proper ideal, where $I_\delta = J^{bd}_{C_\delta}$.

Then there is a function $c : [\lambda]^2 \rightarrow \lambda$ with the property that for every unbounded $A \subseteq \lambda$,

$$\lambda \setminus \text{ran}(c \upharpoonright [A]^2) \in \text{id}_p(\bar{C}, \bar{I}).$$

**Proof.** The definition of $c$ is quite straightforward. Let $(\vec{f}, \vec{\mu})$ be a scale for $\mu$, and let $\vec{e} = \langle e_\alpha : \alpha < \lambda \rangle$ be a $\lambda$–club system that swallows $\bar{C}$ just as in Lemma 5. Given $\alpha < \beta < \lambda$, let $\beta = \beta_0 > \cdots > \beta_{n+1} = \alpha$ be

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the minimal walk from $\beta$ to $\alpha$ along $\vec{e}$, and define $c(\alpha, \beta)$ to be $\beta_i$ for the least $i \leq n + 1$ with

\[ (3.2) \quad \Delta(\alpha, \beta_i) \neq \Delta(\alpha, \beta). \]

(Note that there must be such an $i$, as $\Delta(\alpha, \alpha)$ has been set to $\infty$.) This recipe defines a function $c : [\lambda]^2 \to \lambda$ which we claim has the required properties.

Toward showing this, let $A$ be an arbitrary unbounded subset of $\lambda$. In order to verify (3.1), we must produce a closed unbounded $E \subseteq \lambda$ such that for each $\delta \in S \cap E$, there is a $\gamma^* < \delta$ and $\theta^* < \text{cf}(\beta^*)$ for which

\[ (3.3) \quad \{ \beta^* \in E \cap \text{nacc}(C_\delta) : \gamma^* < \beta^* \text{ and } \theta^* < \text{cf}(\beta^*) \} \subseteq \text{ran}(c \upharpoonright [A]^2). \]

Choose now a regular cardinal $\chi$ much larger than any of the cardinals currently under consideration, and let $\mathfrak{A}$ denote the structure $\langle H(\chi), \in, <, \chi \rangle$. Define

\[ (3.4) \quad x := \{ \kappa, \lambda, S, \vec{S}, \vec{C}, \vec{f}, \vec{\mu}, A \}, \]

(One should think of $x$ as a parameter coding “everything relevant”.) and for each $\nu < \lambda$, define $M_\nu$ to be the Skolem hull of $\kappa + 1 \cup C_\nu \cup \{ x, \nu \}$ in the structure $\mathfrak{A}$, i.e.,

\[ (3.5) \quad M_\nu = \text{Sk}_\mathfrak{A}(\kappa + 1 \cup C_\nu \cup \{ x, \nu \}). \]

It is clear that $|M_\nu| = \kappa + \text{cf}(\nu)$, so for each $\nu < \lambda$ the characteristic function $\chi_\nu$ of $M_\nu$, defined by

\[ (3.6) \quad \chi_\nu(i) = \begin{cases} \sup(M_\nu \cap \mu_i) & \text{if } |M_\nu| < \mu_i, \\ 0 & \text{otherwise} \end{cases} \]

is a member of $\prod_{i < \kappa} \mu_i$. Now define

\[ (3.7) \quad h(\nu) = \text{least } \zeta \text{ such that } \chi_\nu <^* f_\zeta. \]

The function $h$ is defined for all $\nu$ because $\vec{f}$ is a scale in $\prod_{i < \kappa} \mu_i$.

**Lemma 9.** For all sufficiently large $i < \kappa$, if we are given $\xi < \mu_i$ and $\zeta < \mu_{i+1}$, then there are unboundedly many $\alpha \in A$ for which

\[ (3.8) \quad \xi < f_\alpha(i) \text{ and } \zeta < f_\alpha(i + 1). \]

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5As is usual in these arguments, $H(\chi)$ denotes the collection of all sets hereditarily of cardinality less than $\chi$ and $<_\chi$ denotes some fixed well-ordering of $H(\chi)$ in order-type $\chi$. 
Proof. If not, then there is an unbounded set \( I \subseteq \kappa \) (without loss of generality satisfying \( i \in I \to i + 1 \notin I \)) such that a “bad pair” \((\xi, \zeta)\) exists for every \( i \in I \). Define a function \( f \in \prod_{i<\kappa} \mu_i \) by

\[
(3.9) \quad f(i) = \begin{cases} \xi_i & i \in I, \\ \zeta_i & i - 1 \in I, \\ 0 & \text{otherwise.} \end{cases}
\]

For each \( i \in I \), there is an \( \alpha_i < \lambda \) such that

\[
(3.10) \quad \alpha \geq \alpha_i \implies \neg(\xi_i < f_\alpha(i) \text{ and } \zeta_i < f_\alpha(i+1)).
\]

If we let \( \alpha^* = \sup\{\alpha_i : i \in I\} < \lambda \) and choose \( \alpha > \alpha^* \) such that \( f <^* f_\alpha \), then a contradiction is immediate. \( \square \)

Corollary 10. There is a closed unbounded set of \( \delta < \lambda \) such that for all sufficiently large \( i < \kappa \), if we are given \( \xi < \mu_i \) and \( \zeta < \mu_{i+1} \), then the set of \( \alpha \in A \) for which

\[
(3.11) \quad \xi < f_\alpha(i) \text{ and } \zeta < f_\alpha(i+1)
\]

is unbounded in \( \delta \).

Proof. Let \( j^* < \kappa \) be selected so that the assertion of Lemma 9 holds for all \( i \in [j^*, \kappa) \). Given such an \( i \), for each pair \((\xi, \zeta) \in \mu_i \times \mu_{i+1} \) we let \( A'(\xi, \zeta) \) be the corresponding set of \( \alpha \in A \) satisfying (3.8). There are only \( \mu \) sets \( A'(\xi, \zeta) \), and therefore the set of \( \delta < \lambda \) that are limit points of all the \( A'(\xi, \zeta) \) simultaneously is closed and unbounded in \( \lambda \). \( \square \)

Now we let \( E \subseteq \lambda \) consist of all those limit ordinals \( \delta < \lambda \) in the closed unbounded set from Corollary 10 that are closed under the function \( h \) from (3.7); it is clear that \( E \) is closed and unbounded in \( \lambda \).

Fix \( \delta \in S \cap E \), and recall that we must show that there exist \( \gamma^* < \delta \) and \( \theta^* < \mu \) for which (3.3) hold. Start by choosing \( \beta \in A \setminus \delta \), and consider the walk \( \beta = \beta_0 > \beta_1 > \cdots > \beta_n+1 = \delta \) from \( \beta \) down to \( \delta \) along \( \bar{e} \).

Our focus is the ordinal \( \beta_n \), the penultimate stage of the walk. We note the following facts from the preceding section of the paper:

- \( \delta \in e_{\beta_n} \), and so \( C_\delta \subseteq e_{\beta_n} \),
- there is an ordinal \( \gamma^* < \delta \) such that for any \( \alpha \in (\gamma^*, \delta) \), the walk along \( \bar{e} \) from \( \beta \) to \( \alpha \) begins with the sequence \( \beta_0 > \cdots > \beta_n \), and
- there is a \( \theta^* < \mu \) such that any \( \beta^* \in \text{nacc}(C_\delta) \) with \( \text{cf}(\beta^*) > \theta^* \) is also in \( \text{nacc}(e_{\beta_n}) \).
Suppose now that $\beta^* \in nacc(C_3) \cap E$ with $\gamma^* < \beta^*$ and $cf(\beta^*) > \theta^*$; we find an $\alpha \in A$ such that $c(\alpha, \beta) = \beta^*$. We fix $i^* < \kappa$ so large that

$$\left| M_\beta^* \right| < \mu_{i^*}, \tag{3.12}$$

and such that for $\xi < \mu_{i^*}$ and $\zeta < \mu_{i^*} + 1$, the set of $\alpha \in A$ for which

$$\xi < f_\alpha(i^*) \text{ and } \zeta < f_\alpha(i^* + 1) \tag{3.14}$$

is unbounded in $\beta^*$.

Next, we define

$$N = \text{Sk}_\alpha(M_\beta \cup \mu_{i^*}). \tag{3.15}$$

Clearly $M_\beta^* \subseteq N$, and $\mu_{i^*} \subseteq N$. More importantly, however, is the following fact about the way characteristic functions of models interact with Skolem hulls.

**Claim 11.** $\chi_{\text{Sk}_\alpha}(M_\beta \cup \mu_{i^*}) = \chi_{\beta^*} \upharpoonright [i^* + 1, \kappa)$.  \[\Box\]

**Proof.** Suppose $i^* < i < \kappa$, and let $\alpha \in N \cap \mu_i$. We must find $\beta > \alpha$ in $M \cap \mu_i$. There is a Skolem term $\tau$ and parameters $\alpha_0, \ldots, \alpha_j, \beta_0, \ldots, \beta_k$ such that $\alpha = \tau(\alpha_0, \ldots, \alpha_j, \beta_0, \ldots, \beta_k)$ where each $\alpha_\ell$ is $< \mu_{i^*}$ and each $\beta_\ell$ is an element of $M_{\beta^*} \setminus \mu_{i^*}$.

Now we define a function $F$ with domain $[\mu_{i^*}, \kappa)$ by

$$F(x_0, \ldots, x_j) = \begin{cases} \tau(x_0, \ldots, x_j, \beta_0, \ldots, \beta_k) & \text{if this is an ordinal less than } \mu_i, \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

The function $F$ is an element of $M_{\beta^*}$, hence so is the ordinal $\beta = \sup(\text{ran}(F))$. Clearly $\beta < \mu_i$ as $\mu_i$ is a regular cardinal $> \mu_{i^*}$, and therefore

$$\alpha < \sup(\text{ran}(F)) \in M_{\beta^*} \cap \mu_i. \tag{3.17}$$

The preceding claim is just a special case of a result from the folklore; for example, Shelah uses a version of the same result in his analysis of the function $\Delta$ in [5].

**Claim 12.** We can find $\alpha \in N \cap A \cap \beta^*$ such that

1. $\max(\gamma^*, \sup(e_{i^*} \cap \beta^*)) < \alpha$,
2. $f_\alpha(i^*) > f_{\beta_m}(i^*)$ for all $m \leq n$, and

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\[\text{So } \chi_\beta(i) = \sup(M_{\beta^*} \cap \mu_i) \text{ for } i^* \leq i < \kappa.\]
Proof. First, note that $\max(\gamma^*, \sup(e_{\beta^*})) + 1 < \beta^*$, as $E$ consists of limit ordinals and $\beta^* \in nacc(e_{\beta^*})$. Let us define
\[
\xi^* := \max\{f_{\beta_0}(i^*), \ldots, f_{\beta_n}(i^*)\},
\]
and
\[
\eta^* := f_{\beta^*}(i^* + 1).
\]
We note that $\xi^* \in N$ because $\mu_i \subseteq N$, while $\eta^* \in N$ because $\beta^* \in N$. The set $D$ is in $N$ as well, and therefore
\[
N \models \beta^* \in D.
\]
Putting these facts together, we conclude that inside the model $N$, the set of $\alpha \in A$ for which
\[
f_{\alpha}(i^*) > \xi^* \quad \text{and} \quad f_{\alpha}(i^* + 1) > \eta^*
\]
is unbounded in $\beta^*$.

The set $N \cap \beta^*$ is unbounded in $\beta^*$ because $C_{\beta^*} \subseteq M_{\beta^*} \subseteq N$. Thus, we can find $\alpha \in N \cap A \cap \beta^*$ such that (3.21) and (3.22) hold, and such that
\[
\max(\gamma^*, \sup(e_{\beta_n} \cap \beta^*)) + 1 < \alpha,
\]
and this $\alpha$ is as required. \qed

Claim 13. If $\alpha$ is selected as in Claim 12, then $c(\alpha, \beta) = \beta^*$.

Proof. Because $\gamma^* < \alpha < \delta$, we know that the walk from $\beta$ down to $\alpha$ begins with the sequence $\beta = \beta_0 > \cdots > \beta_n$, i.e., it agrees with the walk from $\beta$ to $\delta$ until the last step before $\delta$.

Since $\beta^* \in C_\delta \subseteq e_{\beta^*}$ and $\sup(e_{\beta_n} \cap \beta^*) < \alpha < \beta^*$, it follows that the next step past $\beta_n$ in the walk from $\beta$ to $\alpha$ takes us to $\beta^*$, i.e., the walk along $\bar{e}$ from $\beta$ down to $\alpha$ begins with the sequence
\[
\beta = \beta_0 > \cdots > \beta_n > \beta^*.
\]
Note that since $\alpha \in N$, it follows from 3.12 that
\[
f_{\alpha}(i) < \chi_{\alpha}(i)
\]
for all $i \in [i^*, \kappa)$. From Claim 11, we conclude
\[
f_{\alpha}(i^* + 1, \kappa) < f_{\beta_m}(i^* + 1, \kappa)
\]
for all $m \leq n$, and therefore (3.21) tells us
\[
\Delta(\alpha, \beta_m) = \Delta(\alpha, \beta) = i^* \quad \text{for all} \quad m \leq n.
\]
However, (3.22) implies
\[(3.28) \quad \Delta(\alpha, \beta^*) \neq \Delta(\alpha, \beta) .\]
By definition of c, it follows that \(c(\alpha, \beta) = \beta^*\), as required. □

In summary, we have established that (3.3) holds for our choice of \(\gamma^*\) and \(\theta^*\), and therefore
\[(3.29) \quad \lambda \setminus \text{ran}(c \rest [A]^2) \in \text{id}_p(\bar{C}, \bar{I})\]
as required. □

4. Applications and generalizations

**Definition 14.** Let \(\lambda = \mu^+\), where \(\mu\) is singular, and let \(S\) be a stationary subset of \(\lambda\) consisting of ordinals of cofinality \(\text{cf}(\mu)\). A pair \((\bar{C}, \bar{I})\) is said to be \(S\)-good if
1. \(\bar{C} = \langle C_\delta : \delta \in S \rangle\) is an \(S\)-club system such that \(\text{sup}\{\|C_\delta\| : \delta \in S\} < \mu\),
2. \(I_\delta = J^{b[\mu]}_{C_\delta}\), and
3. \(\text{id}_p(\bar{C}, \bar{I})\) is a proper ideal.

Shelah has shown (Claim 2.6 of [6]) in the case where \(\lambda = \mu^+\) for \(\mu\) of uncountable cofinality that \(S\)-good pairs exist for any stationary \(S \subseteq S^\lambda_{\text{cf}(\mu)}\). The question of whether a similar result holds in the countable cofinality case is still open.

**Theorem 2.** Suppose \(\lambda = \mu^+\) for \(\mu\) singular, and let \((\bar{C}, \bar{I})\) be an \(S\)-good pair for \(S\) a stationary subset of \(S^\lambda_{\text{cf}(\mu)}\). If \(\text{id}_p(\bar{C}, \bar{I})\) is not weakly \(\sigma\)-saturated (i.e., \(\lambda\) can be partitioned into disjoint sets \(\langle S_i : i < \sigma \rangle\) such that \(S_i \notin \text{id}_p(\bar{C}, \bar{I})\) for all \(i < \sigma\)), then
\[(4.1) \quad \lambda \rightarrow [\lambda]^2_\sigma .\]

**Proof.** We use a standard construction. Let \(p : \lambda \rightarrow \sigma\) be such that \(p^{-1}(i) \notin \text{id}_p(\bar{S}, \bar{I})\) for all \(i < \sigma\), and let \(c : [\lambda]^2 \rightarrow \lambda\) be the coloring from our main theorem. Define \(F : [\lambda]^2 \rightarrow \sigma\) by
\[(4.2) \quad F(\alpha, \beta) = p(c(\alpha, \beta)) .\]
Given \(i < \sigma\) and \(A \subseteq \lambda\) unbounded, we can find \(\alpha < \beta\) in \(A\) such that \(F(\alpha, \beta) = i\) because Theorem 1 implies that
\[(4.3) \quad \text{ran}(c \rest [A]^2) \cap p^{-1}(i) \neq \emptyset .\] □
The preceding theorem shows us that $\lambda \nrightarrow [\lambda]^2_\lambda$ is related to the weak saturation of ideals of the form $\text{id}_p(\bar{C}, \bar{I})$. This was discovered by Shelah in [6] using a much more complicated coloring, and in that paper he also deals with sufficient conditions under which $\text{id}_p(\bar{C}, \bar{I})$ fails to be weakly $\lambda$–saturated.

Our argument also generalizes to give a stronger version of $\lambda \nrightarrow [\lambda]^2_\lambda$ using arguments mapped out by Shelah in Section 4 of [6]. We mention this generalization without proof because it uses the exact same technique as Shelah.

**Theorem 3.** In the circumstances of Theorem 1, the coloring $c$ has the stronger property that given a sequence $\langle t_\alpha : \alpha < \lambda \rangle$ of pairwise disjoint subsets of $\lambda$, each of cardinality $\theta < \text{cf}(\mu)$, there is a set $R$ with $\lambda \setminus R \in \text{id}_p(\bar{C}, \bar{I})$ such that for each $i \in R$, there are $\alpha < \beta$ such that

$$c \upharpoonright t_\alpha \times t_\beta \text{ is constant with value } i. \tag{4.4}$$

The following definition is ubiquitous in Shelah’s work in this area; we mention it solely because it allows us to state a nice corollary of the above theorem.

**Definition 15.** Let $\lambda$ be an infinite cardinal, and suppose $\kappa + \theta \leq \mu \leq \lambda$. $\text{Pr}_1(\lambda, \mu, \kappa, \theta)$ means that there is a symmetric two–place function $c$ from $\lambda$ to $\kappa$ such that if $\xi < \theta$ and for $i < \mu$, $\langle \alpha_{i, \zeta} : \zeta < \xi \rangle$ is a strictly increasing sequence of ordinals $< \lambda$ with all $\alpha_{i, \zeta}$’s distinct, then for every $\gamma < \kappa$ there are $i < j < \mu$ such that

$$\zeta_1 < \xi \text{ and } \zeta_2 < \xi \implies c(\alpha_{i, \zeta_1}, \alpha_{i, \zeta_2}) = \gamma. \tag{4.5}$$

**Corollary 16.** In the situation of Theorem 3, if $\text{id}_p(\bar{C}, \bar{I})$ is not weakly $\sigma$–saturated, then $\text{Pr}_1(\lambda, \lambda, \sigma, \text{cf}(\mu))$ holds.

**Proof.** The proof is easy — the same coloring and proof used in Theorem 2 works. \hfill \Box

Finally, we remark that Corollary 16 is originally due to Shelah — it is one of the main points of Section 4 of [6]. The coloring he uses is much more complicated in large part because he starts with the partition of $\lambda$ into sets not in $\text{id}_p(\bar{C}, \bar{I})$, and uses it to define his function. The advantage of our proof (in addition to being much simpler) is that we have a single coloring defined independently of any decomposition of $\lambda$ witnessing the failure of weak saturation of $\text{id}_p(\bar{C}, \bar{I})$.

\footnote{In fact, he warns the reader that his proof is “not so short.”}
References


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