ON ITERATED FORCING FOR SUCCESSORS OF REGULAR CARDINALS

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Abstract. We investigate the problem of when $\leq\lambda$–support iterations of $<\lambda$–complete notions of forcing preserve λ^+. We isolate a property — *properness over diamonds* — that implies λ^+ is preserved and show that this property is preserved by λ–support iterations. Our condition is a relative of that presented by Rosłanowski and Shelah in [2]; it is not clear if the two conditions are equivalent. We close with an application of our technology by presenting a consistency result on uniformizing colorings of ladder systems on $\{\delta<\lambda^+ : \mathrm{cf}(\delta)=\lambda\}$ that complements a theorem of Shelah in [4].

1. Definitions

One of the mysteries of iterated forcing theory is the lack of a good solution to the following "equation" for an uncountable regular cardinal λ:

$$\frac{\text{proper forcing}}{\text{countable support iteration}}=\frac{x}{\lambda\text{–support iteration}}.$$

The goal of this paper is to present a generalization of properness to the context of larger cardinals. We make no claim that ours is the "right" generalization; however, the proof that our condition is preserved by λ–support iteration is close to the proof that properness is preserved by countable support iteration and seems quite natural.

Throughout this paper, we make the following assumptions:

- λ is a regular cardinal satisfying $\lambda=\lambda^{<\lambda}$.
- $\mathfrak{D}$ is a normal filter on λ "with diamonds", i.e., $\mathfrak{D}$ is closed under diagonal intersections, and for every $S\in\mathfrak{D}^+$, there is a sequence $\langle A_\delta:\delta\in S\rangle$ such that for every $A\subseteq\lambda$,
$$\{\delta\in S: A\cap\delta=A_\delta\}\in\mathfrak{D}^+.$$
- χ is a regular cardinal that is "large enough".

We are going to be looking at when λ^+ is preserved by $(\leq)\lambda$–support iterations of $(<)\lambda$–complete notions of forcing. Just as in the case of proper forcing, we will have to look at how our forcing notions interact with elementary submodels.

Date: October 20, 2003.

1991 *Mathematics Subject Classification.* 03E35.

Key words and phrases. uncountable support iterations, proper forcing.

The author would like to thank the Graduate College of the University of Northern Iowa for a Summer Fellowship in 2002 that provided support while this research was carried out.

Definition 1.1. Let N be an elementary submodel of $H(\chi)$. We say that N is *relevant* if

- $|N| = \lambda$
- $N^{<\lambda} \subseteq N$
- $N = \bigcup_{\alpha<\lambda} N_\alpha$, where $\langle N_\alpha : \alpha < \lambda\rangle$ is a continuous $\in$–increasing sequence of elementary submodels of $H(\chi)$ such that $\langle N_\beta : \beta \leq \alpha\rangle \in N_{\alpha+1}$ and $|N_\alpha| < \lambda$. (We say that $\langle N_\alpha : \alpha < \lambda\rangle$ is a *filtration* of N.)

The natural attempt at generalizing properness results in a definition along the following lines:

Definition 1.2. A notion of forcing P is said to be λ–proper if for all sufficiently large regular cardinals χ, there is some $x \in H(\chi)$ such that whenever M is a relevant elementary submodel of $H(\chi)$ with $\{P, x\} \in M$ and p is an element of $M \cap P$, there is a condition $q \leq p$ such that

$$q \Vdash \text{“}M[\dot{G}_P] \cap \mathrm{Ord} = M \cap \mathrm{Ord}\text{”}.$$

Such a condition q is said to be (M, P)–generic.

Some of the qualities of properness generalize in a straightforward fashion to this new context. For example, λ–proper notions of forcing do not collapse λ^+, and it is easy to prove that both λ^+–closed and λ^+–c.c. notions of forcing are λ–proper. Unfortunately, λ–properness is not in general preserved in iterations — work of Shelah and Stanley [5] on generalizing Martin's Axiom to $\aleph_1$–complete, $\aleph_2$–c.c. notions of forcing made this clear in the early 1980's (Weiss's article [6] contains a nice discussion of this phenomenon). This paper presents a strengthening of λ–properness that *is* preserved by appropriate iterations.

The following assumption is necessary for our arguments to work (although we can weaken it slightly). The most important use is in the proof of Corollary 1.6.

Assumption: All notions of forcing under consideration in this paper are assumed to be λ–closed.

Definition 1.3. Let P be a notion of forcing, and let N be a relevant elementary submodel of $H(\chi)$.

(1) A set $A \subseteq P$ is λ–linked if every $A_0 \in [A]^{<\lambda}$ has a lower bound in P.
(2) An (N, P)–diamond over $\mathfrak{D}$ is a sequence $\bar{A} = \langle A_\delta : \delta \in S\rangle$ such that
  - $S \in \mathfrak{D}^+$
  - A_δ is a subset of $N_\delta \cap P$ with a lower bound in P
  - whenever $A \subseteq N \cap P$ is λ–linked,

$$\{\delta \in S : N_\delta \cap A = A_\delta\} \in \mathfrak{D}^+. \tag{1.1}$$

(3) In the context of (2), if $N_\delta \cap A = A_\delta$ then we say that $\bar{A}$ *guesses* A *at* δ.

Our first observation is that (N, P)–diamonds are nothing mysterious – they are just regular diamond sequences that have been cosmetically altered.

Lemma 1.4. Let N be a relevant model with filtration $\langle N_\alpha : \alpha < \lambda\rangle$. Further suppose $\mathfrak{D}$ has diamonds. Then for $S \in \mathfrak{D}^+$ we can find an (N, P)–diamond $\langle A_\delta : \delta \in S\rangle$.

Proof. Let $\langle B_\delta : \delta \in S\rangle$ be a $\mathfrak{D}$–diamond sequence, and let $f : \lambda \to N \cap P$ be a bijection. Given $\delta \in S$, ask if $f[B_\delta]$ is a λ–linked subset of $N_\delta \cap P$. If so, then we let $A_\delta = f[B_\delta]$; if not, then let A_δ be $\{\mathbf{1}_P\}$, where $\mathbf{1}_P$ denotes the maximal element of P.

Now suppose A is a λ–linked subset of $N \cap P$. Since $\langle B_\delta : \delta \in S\rangle$ is a diamond sequence, we know that the set of δ for which $B_\delta = f^{-1}(A) \cap \delta$ is in $\mathfrak{D}^+$.

There is a closed unbounded set C such that $f \restriction \delta$ is a bijection between δ and $N_\delta \cap P$. If $\delta \in C$ and $B_\delta = f^{-1}(A) \cap \delta$, then $A_\delta = N_\delta \cap A$. Since $C \in \mathfrak{D}$, we see that $\langle A_\delta : \delta \in S\rangle$ is an (N, P)–diamond. □

Starting with the next lemma, we use without mention that the filter $\mathfrak{D}$ has a natural interpretation in generic extensions of the universe — in $V[G]$, we let $\mathfrak{D}$ refer to the normal filter generated by $\mathfrak{D} \cap V$.

Lemma 1.5. Let $\langle A_\delta : \delta \in S\rangle$ be an (N, P)–diamond, and let Q be a λ–closed notion of forcing. If $\dot{A}$ is a Q–name for a λ–linked subset of $N \cap P$, then

$$\Vdash_Q \{\delta \in S : N_\delta \cap \dot{A} = A_\delta\} \in \mathfrak{D}^+. \tag{1.2}$$

Proof. If not, then we can find a condition q as well as a Q–name $\dot{A}$ and a sequence $\langle \dot{C}_i : i < \lambda\rangle$ of Q–names such that

- $\Vdash_Q \dot{A}$ is λ–linked,
- $\Vdash_Q \dot{C}_i \in \mathfrak{D} \cap V$, and
- $q \Vdash \delta \in S \cap \triangle_{i<\lambda} \dot{C}_i \implies A_\delta \neq N_\delta \cap \dot{A}$.

Since Q is λ–closed, we can find sequences $\langle q_\alpha : \alpha < \lambda\rangle$, $\langle C_\alpha : \alpha < \lambda\rangle$, and $\langle B_\alpha : \alpha < \lambda\rangle$ such that

- $\alpha < \beta < \lambda \implies q_\beta \leq q_\alpha \leq q$ in Q
- $C_\alpha \in \mathfrak{D}$
- $B_\alpha \subseteq N_\alpha \cap P$
- $q_\alpha \Vdash \dot{C}_\alpha = C_\alpha$ and $N_\alpha \cap \dot{A} = B_\alpha$

Define $C = \triangle_{\alpha<\lambda} C_\alpha$. Since $\mathfrak{D}$ is a normal filter, we know that $C \in \mathfrak{D}$.

Note that the sequence $\langle B_\alpha : \alpha < \lambda\rangle$ increases with α. Define

$$B = \bigcup_{\alpha<\lambda} B_\alpha. \tag{1.3}$$

It is not hard to see that B is λ–linked (in the ground model), so there is a $\delta \in S \cap C$ such that $N_\delta \cap B = A_\delta$. This is a contradiction as q_δ is an extension of q, yet

$$q_\delta \Vdash \delta \in S \cap \triangle_{i<\lambda} \dot{C}_i \text{ and } N_\delta \cap \dot{A} = B_\delta = A_\delta. \tag{1.4}$$

□

Corollary 1.6. If $\bar{A}$ is an (N,P)–diamond and $G \subseteq P$ is a generic subset of P, then

$$\{\delta \in S : N_\delta \cap G = A_\delta\} \in \mathfrak{D}^+. \tag{1.5}$$

Proof. This follows because G is λ–directed, hence λ–linked. □

Definition 1.7. A sequence $\bar{R} = \langle (R_\delta, q_\delta) : \delta \in S\rangle$ is said to be an (N,P)–rule if

- $\langle R_\delta : \delta \in S\rangle$ is an (N,P)–diamond,
- q_δ is a lower bound for R_δ in $N \cap P$, and
- if $D \in N$ is a dense subset of P, then $q_\delta \in D$ for all sufficiently large $\delta \in S$.

Definition 1.8. A notion of forcing P is proper over $\mathfrak{D}$–diamonds if (it is λ–closed and) there is an $x \in H(\chi)$ such that for every relevant model N with $x \in N$, whenever we are given an (N,P)–rule $\bar{R} = \langle (R_\delta, q_\delta) : \delta \in S\rangle$, for every $p \in N \cap P$ there is $q \leq p$

$$q \Vdash \text{for some } C \in \mathfrak{D}, \text{ if } \delta \in S \cap C \text{ and } R_\delta = N_\delta \cap \dot{G}_P, \text{ then } q_\delta \in \dot{G}_P.$$

We say that q is $(N,P,\bar{R})$–generic.

In other words, q is $(N,P,\bar{R})$–generic if q forces that in the generic extension, for $\mathfrak{D}$–almost all $\delta \in S$, if R_δ guesses $N_\delta \cap G$, then $q_\delta \in G$. We say that q forces $N \cap G$ to *obey the rule* $\bar{R}$. Note as well that the set C appearing in the above definition is *not* required to be in the ground model; we only require that such a set can be found in the generic extension. This fact is crucial in the arguments we present.

Proposition 1.9. Suppose N is a relevant model containing P, $\bar{R}$ is an (N,P)–rule, and q is $(N,P,\bar{R})$–generic. Then q is (N,P)–generic, i.e.,

$$q \Vdash N[\dot{G}_P] \cap \mathrm{Ord} = N \cap \mathrm{Ord}\,. \tag{1.6}$$

In particular, if P is proper over $\mathfrak{D}$–diamonds, then forcing with P preserves the cardinal λ^+.

2. Iterations

We begin with an outline that shows how properness for $\mathfrak{D}$–diamonds is preserved in a simple two–step iteration. Thus, suppose P is proper for $\mathfrak{D}$–diamonds and $\Vdash_P$ "$\dot{Q}$ is proper for $\mathfrak{D}$–diamonds".

Fix $x \in H(\chi)$ as required in Definition 1.8 for the partial order P, and let $\dot{y}$ be a P–name for the parameter associated with $\dot{Q}$ in the generic extension. We claim that for every relevant model N containing $\{x, \dot{y}\}$, whenever we are given an $(N, P * \dot{Q})$–rule $\bar{R}$ and $p * \dot{q} \in N \cap P * \dot{Q}$, we can find an extension of $p * \dot{q}$ that forces $\dot{G}_{P*\dot{Q}}$ to obey the rule $\bar{R}$.

Let $\bar{R} = \langle (R_\delta, p_\delta * \dot{q}_\delta) : \delta \in S\rangle$ be an $(N, P * \dot{Q})$–rule. For $\delta \in S$, define

$$A_\delta := \{p : (\exists \dot{q})[p * \dot{q} \in R_\delta]\}.$$

If we define $\bar{R} \restriction P = \{(A_\delta, p_\delta) : \delta \in S\}$, it is straightforward to verify that $\bar{R} \restriction P$ is an (N,P)–rule.

Now let G be a generic subset of P containing an $(N, P, \bar{R} \restriction P)$–generic condition. In the extension $V[G]$, we know that

$$S_0 := \{\delta \in S : N_\delta \cap G = A_\delta \text{ and } p_\delta \in G\}$$

is in $\mathfrak{D}^+$.

For $\delta \in S_0$, let us define

$$B_\delta = \{\dot{b}[G] : a * \dot{b} \in R_\delta \text{ for some } a \in P\}.$$

Next, we set

$$\bar{R}/G = \{(B_\delta, \dot{q}_\delta[G]) : \delta \in S_0\}.$$

Back in the ground model, we let $\bar{R}/\dot{G}_P$ be a P–name for the object $\bar{R}/G$ in the generic extension.

Proposition 2.1. If p is $(N, P, \bar{R} \restriction P)$–generic, then

$$p \Vdash \text{“}\bar{R}/\dot{G}_P \text{ is an } (N[\dot{G}_P], \dot{Q})\text{–rule”}.$$

Proof. Let G be a generic subset of P that contains p. We will prove in detail that $\{B_\delta : \delta \in S_0\}$ is an $(N[G], \dot{Q}[G])$–diamond; the rest of the proposition can be verified by similar means.

Let E be a λ–linked subset of $N[G] \cap \dot{Q}[G]$. Define H to be the set of all terms $a * \dot{b}$ satisfying

- $a \in N \cap G$,
- $\dot{b}$ is a P–name from N, and
- $\dot{b}[G] \in E$.

Claim 2.2. $V[G] \models$ “ H is a λ–linked subset of $N \cap P * \dot{Q}$”.

Proof. It should be clear that H is a subset of $N \cap (P * \dot{Q})$, so we concentrate on proving that H is λ–linked. Let $H_0 \subseteq H$ have cardinality $< \lambda$. Since P is λ–closed, we know that H_0 is in the ground model. Furthermore, N is closed under sequences of length $< \lambda$ so we also know that $H_0 \in N$ and so there is an ordinal $\delta \in S_0$ such that $H_0 \in N_\delta$.

Let us define

$$H_0^P = \{a : a * \dot{b} \in H_0 \text{ for some } \dot{b}\},$$

and

$$H_0^Q = \{\dot{b} : a * \dot{b} \in H_0 \text{ for some } a\}.$$

Clearly H_0^P and H_0^Q are elements of N_δ as they are definable from H_0. Our decision to choose $\delta \in S_0$ guarantees that p_δ is a lower bound for $N_\delta \cap G$. By our definition of H, we may conclude that p_δ is a lower bound for H_0^P.

Now in $V[G]$, we know that E is λ–linked, so that $\{\dot{b}[G] : \dot{b} \in H_0^Q\}$ has a lower bound in Q. Since $N_\delta[G] \prec V[G]$ there must be a P–name $\dot{q} \in N_\delta$ such that

$$N_\delta[G] \models \text{“}\dot{q}[G] \text{ is a lower bound for } \{\dot{b}[G] : \dot{b} \in H_0^Q\}\text{”}.$$

We will finish upon verifying that $p_\delta * \dot{q}$ is a lower bound for H_0. Let $a * \dot{b} \in H_0$ be given. We know immediately that $p_\delta \leq a$, so we need only see

$$p_\delta \Vdash \dot{q} \leq \dot{b}.$$

This follows because there must exist $r \in N_\delta \cap G$ such that $r \Vdash \dot{q} \leq \dot{b}$ and $p_\delta \leq r$. Thus H is a λ–linked subset of $N \cap (P * \dot{Q})$ in $V[G]$. □

To finish our proof that $\{B_\delta : \delta \in S_0\}$ is an $(N[G], \dot{Q}[G])$–diamond, we take advantage of Lemma 1.5:

$$V[G] \models \{\delta \in S : N_\delta \cap H = R_\delta\} \in \mathfrak{D}^+.$$

Given such a δ, Any such δ must be in S_0 by our definition of H, and by our definition of H it follows that $N_\delta[G] \cap E = B_\delta$, as required. □

The statement of the following theorem now makes sense; we leave the proof to the reader as we do not need it in the proof of Theorem 2. (The proof consists of arguments like those given above.)

Theorem 1. *A condition $p * \dot{q} \in P * \dot{Q}$ is $(N, P * \dot{Q}, \bar{R})$–generic if and only if p is $(N, P, \bar{R} \restriction P)$–generic and*

$$p \Vdash \dot{q} \text{ is } (N[G], \dot{Q}, \bar{R}/\dot{G}_P)\text{–generic.}$$

Now what happens with longer iterations? Assume now that $\mathbb{P} = \langle P_i, \dot{Q}_i : i < \kappa \rangle$ is λ–support iteration of λ–closed notions of forcing such that

$$\Vdash_{P_i} \dot{Q}_i \text{ is proper for } \mathfrak{D}\text{–diamonds.} \tag{2.1}$$

We will show that P_κ, the limit of $\mathbb{P}$, is proper for $\mathfrak{D}$–diamonds, so in particular forcing with P_κ preserves λ^+.

Theorem 2 (Preservation Theorem)**.**
Let $\langle P_i, \dot{Q}_i : i < \kappa \rangle$ be a λ–support iteration such that

$$\Vdash_{P_i} \dot{Q}_i \text{ is proper over } \mathfrak{D}\text{–diamonds.}$$

Then P_κ is proper over $\mathfrak{D}$–diamonds.

Definition 2.3. Let N be a relevant model with $\mathbb{P} \in N$, and suppose $i < j$ in $N \cap (\kappa + 1)$. Let $\bar{A} = \langle A_\delta : \delta \in S \rangle$ be an (N, P_j)–diamond; without loss of generality S consists entirely of limit ordinals. Given $\delta \in S$, we define

$$A_\delta \restriction i = \{p \restriction i : p \in A_\delta\}, \tag{2.2}$$

and

$$\bar{A} \restriction i = \langle A_\delta \restriction i : \delta \in S \rangle. \tag{2.3}$$

Similarly, if $\bar{R} = \langle (R_\delta, q_\delta) : \delta \in S \rangle$ is an (N, P)–rule, we define

$$\bar{R} \restriction i = \langle (R_\delta \restriction i, q_\delta \restriction i) : \delta \in S \rangle. \tag{2.4}$$

Lemma 2.4. Let N be a relevant model containing $\mathbb{P}$, and let $i < j$ in $N \cap (\kappa+1)$. If $\bar{A}$ is an (N, P_j)–diamond, then $\bar{A} \restriction i$ is an (N, P_i)–diamond. If $\bar{R}$ is an (N, P_j)–rule, then $\bar{R} \restriction i$ is an (N, P_i)–rule.

Proof of the Iteration Theorem.

We prove by induction on $j \in N \cap \kappa + 1$ that whenever we are given objects i, $\dot{p}$, and r such that

- $i < j$
- $r \in P_i$
- $\Vdash_{P_i} \dot{p} \in P_\kappa$
- $\dot{p} \in N$
- $r \Vdash \dot{p} \restriction i \in \dot{G}_{P_i}$
- r is $(N, P_i, \bar{R} \restriction i)$–generic

we can find a condition $s \in P_j$ such that

- $s \restriction i = r$
- s is $(N, P_j, \bar{R} \restriction j)$–generic
- $s \Vdash \dot{p} \restriction j \in \dot{G}_j$

CASE 1: j is a successor ordinal

Let $j = j_0 + 1$. Since j_0 must be in $N \cap (\kappa + 1)$, we may apply our induction hypothesis to obtain a condition $s_0 \in P_{j_0}$ such that

- $s_0 \restriction i = r$
- s_0 is $(N, P_{j_0}, \bar{R} \restriction j_0)$–generic, and
- s_0 forces that $\dot{p} \restriction j_0$ is in $\dot{G}_{j_0}$.

At this point, we are essentially in the case where we are doing a two–step iteration – if we view P_j as a two–step iteration $P_{j_0} * \dot{Q}_{j_0}$, then the arguments presented at the beginning of this section show how to extend s_0 to the required $(N, P_j, \bar{R} \restriction j)$–generic condition s.

CASE 2: j is a limit ordinal of cofinality $< \lambda$

Lemma 2.5. Suppose $\epsilon \in N \cap (\kappa+1)$ satisfies $\mathrm{cf}(\epsilon) < \lambda$, and we are given sequences $\langle i_\alpha : \alpha < \mathrm{cf}(\epsilon) \rangle$ and $\langle r_\alpha : \alpha < \mathrm{cf}(\epsilon) \rangle$ such that

- $\langle i_\alpha : \alpha < \mathrm{cf}(\epsilon) \rangle$ is a strictly increasing sequence of ordinals in $N \cap \epsilon$
- r_α is $(N, P_{i_\alpha}, \bar{R} \restriction i_\alpha)$–generic
- $\alpha < \beta < \kappa \Longrightarrow r_\beta \restriction i_\alpha = r_\alpha$.

Then the condition $s := \bigcup_{\alpha < \mathrm{cf}(\epsilon)} r_\alpha$ is $(N, P_\epsilon, \bar{R} \restriction \epsilon)$–generic.

Proof. Clearly $s \in P_\epsilon$ as we are using λ–support iteration. Let G be any generic subset of P_ϵ that contains s; we will work in the generic extension $V[G]$.

For $\alpha < \mathrm{cf}(\epsilon)$, let $G_\alpha = G \restriction P_{i_\alpha}$. Clearly $r_\alpha \in G_\alpha$ and G_α is a generic subset of P_{i_α}, so there is a set $C_\alpha \in \mathfrak{D}$ such that

$$\delta \in S \cap C_\alpha \text{ and } N_\delta \cap G_\alpha = A_\alpha \restriction i_\alpha \Longrightarrow q_\delta \restriction i_\alpha \in G_\alpha. \tag{2.5}$$

Let $C = \bigcap_{\alpha<\mathrm{cf}(\epsilon)} C_\alpha \in \mathfrak{D}$. Given $\delta \in S \cap C$ if $\bar{A}$ guesses G at δ, then (2.5) implies that $q_\delta \restriction i_\alpha \in G_\alpha$ for all $\alpha < \mathrm{cf}(\epsilon)$. Since G is a generic subset of P_ϵ, it follows that q_δ is in G, as required. □

Now we return to the case where $\mathrm{cf}(j) < \lambda$. Let $\langle i_\alpha : \alpha < \mathrm{cf}(j)\rangle$ be increasing, continuous, and cofinal in $N \cap j$ — note that we can achieve continuity because N is closed under sequences of length $< \lambda$. Without loss of generality we assume $i_0 = i$.

By induction on $\alpha < \mathrm{cf}(j)$, we choose conditions $r_\alpha \in P_{i_\alpha}$ such that

- $r_0 = r$
- $r_\alpha \Vdash \dot{p} \restriction i_\alpha \in \dot{G}_{P_{i_\alpha}}$
- if $\beta < \alpha$ then $r_\alpha \restriction i_\beta = r_\beta$
- if α is a limit ordinal, then $r_\alpha = \bigcup_{\beta<\alpha} r_\beta$
- r_α is $(N, P_{i_\alpha}, \bar{R} \restriction i_\alpha)$–generic

The construction of $\langle r_\alpha : \alpha < \mathrm{cf}(j)\rangle$ is straightforward — at successor stages we apply our induction hypothesis, while at limit stages we invoke Lemma 2.5 to show that the construction continues.

Another application of Lemma 2.5 shows us that s is $(N, P_j, \bar{R} \restriction j)$–generic; the other requirements for s are also easily verified.

CASE 3: $\mathrm{cf}(j) = \lambda$

Let $\langle i_\alpha : \alpha < \lambda\rangle$ be increasing, continuous, and cofinal in $N \cap j$ with $i_0 = i$. Let $\langle D_\alpha : \alpha < \lambda\rangle$ list all dense open subsets of P_j that are elements of N.

By induction on $\alpha < \lambda$, we will define objects $\dot{p}_\alpha$ and r_α such that

(1) $r_0 = r$, $\dot{p}_0 = \dot{p} \restriction j$
(2) r_α is $(N, P_{i_\alpha}, \bar{R} \restriction i_\alpha)$–generic
(3) $r_\alpha \restriction i_\beta = r_\beta$ for $\beta < \alpha$
(4) $r_\alpha \Vdash \dot{p}_\alpha \in N \cap P_j$ and $\dot{p}_\alpha \restriction i_\alpha \in \dot{G}_{P_{i_\alpha}}$
(5) $r_{\alpha+1} \Vdash \dot{p}_{\alpha+1} \in D_\alpha$
(6) for $\beta < \alpha$, $r_\alpha \Vdash \dot{p}_\alpha \leq \dot{p}_\beta$
(7) for $\alpha \in S$, r_α forces the statement

(⊗) if $q_\alpha \restriction i_\alpha \in \dot{G}_{i_\alpha}$ and q_α is a lower bound for $\langle \dot{p}_\beta : \beta < \alpha\rangle$, then $\dot{p}_\alpha = q_\alpha \restriction j$.

Construction of $\langle \dot{p}_\alpha : \alpha < \lambda\rangle$ and $\langle r_\alpha : \alpha < \lambda\rangle$:

Initial stage:

We have already defined r_0 and $\dot{p}_0$.

Successor stages:

Assume now that α is a successor ordinal, say $\alpha = \beta + 1$. Our construction will give us objects r_β and $\dot{p}_\beta$ satisfying the appropriate conditions. We apply our induction hypothesis with i_α, i_β, $\dot{p}_\beta \restriction i_\alpha$, r_β, and $\bar{R} \restriction i_\alpha$ standing for the objects j, i, $\dot{p}$, r, and $\bar{R}$ appearing there. This gives us an object r_α such that

- r_α is $(N, P_{i_\alpha}, \bar{R} \restriction i_\alpha)$–generic,
- $r_\alpha \restriction i_\beta = r_\beta$, and
- $r_\alpha \Vdash \dot{p}_\beta \restriction i_\alpha \in \dot{G}_{i_\alpha}$.

Now let G be any generic subset of P_{i_α} that contains r_α. We know that $N \cap G$ is P_{i_α}–generic over N because r_α is (N, P_{i_α})–generic. Since $D_\beta \in N$, a standard genericity argument tells us that there is a condition $p_\alpha \in N[G] \cap P_j = N \cap P_j$ such that

- $p_\alpha \restriction i_\alpha \in G$,
- $p_\alpha \leq \dot{p}_\beta[G]$, and
- $p_\alpha \in D_\beta$.

Back in V, we let $\dot{p}_\alpha$ be a name for this p_α; it should be clear that $\dot{p}_\alpha$ is as required.

Limit stages:

If α is a limit ordinal, we know

$$r_\alpha = \bigcup_{\beta<\alpha} r_\beta.$$

Since $\mathrm{cf}(\alpha) < \lambda$, Lemma 2.5 implies that r_α is $(N, P_{i_\alpha}, \bar{R} \restriction i_\alpha)$–generic. Also, our inductive assumptions imply that for all $\beta < \alpha$,

$$r_\alpha \Vdash \dot{p}_\beta \restriction i_\alpha \in \dot{G}_{i_\alpha}.$$

Let G be any generic subset of P_{i_α} with $r_\alpha \in G$. In the extension $V[G]$, each name $\dot{p}_\beta$ is interpreted as a condition p_β in $N \cap P_j$, and we know

- $\forall \beta < \alpha$, $p_\beta \restriction i_\alpha \in G$, and
- $\langle p_\beta : \beta < \alpha \rangle$ is decreasing.

Now we ask the question

Is it the case that

- $\alpha \in S$
- $q_\alpha \restriction i_\alpha \in G$, *and*
- $q_\alpha \restriction j$ *is a lower bound for* $\langle p_\beta : \beta < \alpha \rangle$ *in* P_j*?*

If the answer is yes, then we let $p_\alpha = q_\alpha \restriction j$. If the answer is no, then we let $p_\alpha \in N \cap P_j$ be a lower bound for $\langle p_\beta : \beta < \alpha \rangle$ in $N \cap P_j$ with $p_\alpha \restriction i_\alpha \in G$.

Now back in the ground model, we let $\dot{p}_\alpha$ be a name forced by r_α to be as above. Note that $\dot{p}_\alpha$ is as required in $(\otimes)$, and our construction continues.

Once we have defined r_α and $\dot{p}_\alpha$ for every $\alpha < \lambda$, we define

$$s := \bigcup_{\alpha<\lambda} r_\alpha.$$

Clearly $s \upharpoonright i = r$ and $s \Vdash \dot{p} \upharpoonright j \in \dot{G}_j$, so we need only verify that s is $(N, P_j, \bar{R} \upharpoonright j)$–generic.

Let G be any generic subset of P_j that contains s, and step into the model $V[G]$. Each $\dot{p}_\alpha$ is interpreted as some $p_\alpha \in N \cap P_j$ and our construction guarantees that the filter generated by $\langle p_\alpha : \alpha < \lambda \rangle$ is generic over N and hence equal to $N \cap G$. This tells us that s is (N, P_j)–generic.

For each $\alpha < \lambda$, the condition r_α is $(N, P_{i_\alpha}, \bar{R} \upharpoonright i_\alpha)$–generic so in $V[G]$ we can find a set $C_\alpha \in \mathfrak{D}$ that witnesses this, i.e., if $\delta \in C_\alpha \cap S$ and $q_\delta \upharpoonright i_\alpha$ guesses $N_\delta \cap G \upharpoonright i_\alpha$, then $q_\delta \upharpoonright i_\alpha \in G \upharpoonright i_\alpha$.

Since $\langle p_\alpha : \alpha < \lambda \rangle$ generates $N \cap G$ and $N \cap G$ is generic over N, there is a closed unbounded set $E \subseteq \lambda$ such that

$$\delta \in E \Longrightarrow \langle p_\alpha : \alpha < \delta \rangle \text{ generates a generic subset of } N_\alpha \cap P. \tag{2.6}$$

Let $C = E \cap \triangle_{\alpha<\lambda} C_\alpha$; since $\mathfrak{D}$ is normal, we know that $C \in \mathfrak{D}$.

Claim 2.6. If $\delta \in C \cap S$ and $\bar{A} \upharpoonright j$ guesses $N_\delta \cap G$, then $q_\delta \upharpoonright j \in G$.

Proof. Suppose we are given such a δ. It suffices to show that $q_\delta \upharpoonright i_\delta \in G_{i_\delta}$ and $q_\delta \upharpoonright j$ is a lower bound for $\langle p_\beta : \beta < \delta \rangle$ — if this happens, then our construction guarantees $p_\delta = q_\delta \upharpoonright j$ and $p_\delta \in G$.

Our definition of C implies that $\delta \in C_\beta$ for all $\beta < \delta$. Since $\bar{A} \upharpoonright j$ guesses $N_\delta \cap G$, we know that $\bar{A} \upharpoonright i_\alpha$ guesses $N \cap G_{i_\alpha}$ for all $\alpha < \lambda$. Given $\beta < \delta$, we know that $r_\beta \in G_{i_\beta}$ and r_β is $(N, P_{i_\beta}, \bar{R} \upharpoonright i_\beta)$–generic. Putting all this together, we may conclude that for all $\beta < \delta$, $q_\delta \upharpoonright i_\beta \in G_{i_\beta}$, hence $q_\delta \upharpoonright i_\delta \in G_{i_\delta}$.

Now why is $q_\delta \upharpoonright j$ a lower bound for $\langle p_\beta : \beta < \delta \rangle$? This follows because $\delta \in C$ — the sequence $\langle p_\beta : \beta < \delta \rangle$ generates $N_\delta \cap G$, and we have assumed that $\bar{A} \upharpoonright j$ guesses $N_\delta \cap G$.

Since r_δ forces $(\otimes)$ to hold, we know that $\dot{p}_\delta[G] = q_\delta \upharpoonright j$, hence $q_\delta \upharpoonright j \in G$.

□

We have therefore shown that s is $(N, P_j, \bar{R} \upharpoonright j)$–generic. Since $s \upharpoonright i = r$ and our construction guarantees that $s \Vdash \dot{p} \upharpoonright j \in \dot{G}_{P_j}$, so s is as required.

CASE 4: $\mathrm{cf}(j) > \lambda$

The construction in this case is essentially the same as that of the previous case (in fact, these two cases can easily be handled together by cosmetically altering the argument). Let $k = \sup(N \cap j)$; since N is closed under sequences of length $< \lambda$, it follows that $\mathrm{cf}(k) = \lambda$ and we can fix a continuous increasing sequence $\langle i_\alpha : \alpha < \lambda \rangle$ of elements of $N \cap j$ cofinal in k.

The idea now is to mimic the construction given for the case where $\mathrm{cf}(j) = \lambda$. Let $\langle D_\alpha : \alpha < \lambda \rangle$ list all dense open subsets of P_j that are elements of N. By induction on $\alpha < \lambda$, define objects $\dot{p}_\alpha$ and r_α satisfying exactly the same requirements as in the previous case — that construction did not require that j was an element of $N \cap \kappa$, only that a sequence along the lines of $\langle i_\alpha : \alpha < \lambda \rangle$ exists. One then checks that the resulting condition s defined as there has all the required properties.

□

3. On the λ^{++} chain condition

In this section, we give a fairly easy generalization of the fact that CH implies that the limit of a countable support iteration of proper posets, each of cardinality $\leq \aleph_1$, has the $\aleph_2$–c.c. — the proof of Proposition 3.1 follows from the proof of Theorem 2.7 of [1] *mutatis mutandis.* We include a proof of this result for completeness and for the convenience of future citations. We use the standard notation $S^{\lambda^{++}}_{\lambda^+}$ to denote $\{\delta < \lambda^{++} : \mathrm{cf}(\delta) = \lambda^+\}$.

Proposition 3.1. Assume $\lambda^{<\lambda} = \lambda$, $2^\lambda = \lambda^+$, and let $\mathbb{P} = \langle P_i, \dot{Q}_i : i < \lambda^{++}\rangle$ be a λ–support iteration such that

(1) P_i is λ–proper for $i \leq \lambda^{++}$
(2) $\Vdash_{P_i}$ "$|\dot{Q}_i| \leq \lambda^+$"

Then $P_{\lambda^{++}}$ satisfies the λ^{++}–chain condition.

Proof. Let $\{p_\xi : \xi < \lambda^{++}\}$ be given. For each $\xi < \lambda^{++}$, let us fix a model $M_\xi \prec H(\chi)$ such that

- $\{\mathbb{P}, \xi, p_\xi\} \in M_\xi$
- $|M_\xi| = \lambda$
- M_ξ is closed under sequences of length $< \lambda$.

By an application of the $\triangle$–system Lemma, without loss of generality there is a set $H \subseteq \lambda^{++}$ such that

$$\xi \neq \zeta \Longrightarrow M_\xi \cap M_\zeta \cap \lambda^{++} = H.$$

For each $\xi < \lambda^{++}$, let $\bar{M}_\xi$ be the transitive collapse of M_ξ. Each $\bar{M}_\xi$ is an element of $H(\lambda^+)$ and since $|H(\lambda^+)| = 2^\lambda = \lambda^+$, without loss of generality there is a structure $\bar{M} \in H(\lambda^+)$ such that $\bar{M}_\xi = \bar{M}$ for all $\xi < \lambda^{++}$. Let $\pi_\xi : M_\xi \to \bar{M}$ be the Mostowski isomorphism between M_ξ and $\bar{M}$. Since $|\bar{M}| = \lambda$, without loss of generality

$$\xi \neq \zeta \Longrightarrow \pi_\xi(p_\xi) = \pi_\zeta(p_\zeta).$$

Putting all this together, we see that without loss of generality we may assume that for $\xi \neq \zeta$, there is an isomorphism $h_{\xi,\zeta} : M_\xi \to M_\zeta$ such that $h_{\xi,\zeta}(p_\xi) = p_\zeta$.

Claim 3.2. There is a stationary set $S \subseteq S^{\lambda^{++}}_{\lambda^+}$ such that $M_\xi \cap \xi = H$ for all $\xi \in S$ (where H is the root of our $\triangle$–system).

Proof. For $\xi < \lambda^{++}$, let us define $f(\xi) = \sup(M_\xi \cap \xi)$. Note that $\mathrm{cf}(\xi) = \lambda^+ \Longrightarrow f(\xi) < \xi$, so by Fodor's Lemma there is a stationary set $S_0 \subseteq S \subseteq S^{\lambda^{++}}_{\lambda^+}$ and a $\gamma < \lambda^{++}$ such that

$$\xi \in S_0 \Longrightarrow \gamma = \sup(M_\xi \cap \xi).$$

Now $|[\gamma]^\lambda| = |\gamma|^\lambda \leq (\lambda^+)^\lambda = \lambda^+$, so there must be a set $A \subseteq \gamma$ and a stationary $S \subseteq S_0$ such that

$$\xi \in S \Longrightarrow A = M_\xi \cap \xi.$$

Note it must be the case that $A \subseteq H$ because $A \subseteq M_\xi \cap M_\zeta \cap \lambda^{++}$ for $\xi \neq \zeta$ in S. If we choose $\xi \in S \setminus \sup(H)$, then $H \subseteq M_\xi \cap \xi$ and hence $H \subseteq A$. Thus $H = A$, and we have that

$$\xi \in S \Longrightarrow M_\xi \cap \xi = H$$

as required. □

Now let $C \subseteq \lambda^{++}$ be the set of ordinals closed under the function $\xi \mapsto \sup(M_\xi \cap \lambda^{++})$. The set $I := S \cap C$ is stationary in λ^{++}, hence of size λ^{++}. Furthermore, if $\xi < \zeta$ in I, then

- $H = M_\xi \cap M_\zeta \cap \lambda^{++}$
- $H \subseteq \min(M_\xi \cap \lambda^{++} \setminus H)$
- $M_\xi \cap \lambda^{++} \subseteq \min(M_\zeta \cap \lambda^{++} \setminus H)$.

In other words, the $\triangle$–system $\{M_\xi \cap \lambda^{++} : \xi \in I\}$ is "not entangled", and H is an initial segment of $M_\xi \cap \lambda^{++}$ for all $\xi \in I$.

Claim 3.3. $H \cap \lambda^+$ is an initial segment of λ^+. Furthermore if $\xi \in I$ then $M_\xi \cap \lambda^+ = H \cap \lambda^+$.

Proof. The second assertion is obvious. The first assertion follows because we assumed that each M_ξ is closed under sequences of length $< \lambda$. □

Let us now fix once and for all $\xi < \zeta$ in I, and let $h : M_\xi \to M_\zeta$ be the isomorphism that carries p_ξ to p_ζ.

Claim 3.4. Suppose $\theta \in H$. Let p be (M_ξ, P_θ)–generic. If $r \in M_\xi \cap P_\theta$ and p extends r, then p also extends $h(r)$.

Proof. The proof is by induction on $\theta \in H$.

Case 1: θ is a limit ordinal

Let $r \in M_\xi \cap P_\theta$ be given. The support of r is a subset of θ of size at most λ that is an element of M_ξ. Since $\lambda \subseteq M_\xi$, we know that $\mathrm{supp}(r) \subseteq M_\xi \cap \theta$ and hence $\mathrm{supp}(r) \subseteq H$. This implies that $\mathrm{supp}(r) = \mathrm{supp}(h(r))$ because $h \restriction H$ is the identity.

Thus it suffices to prove that for all $\mu \in M_\xi \cap \theta$ that $p \restriction \mu$ is an extension $h(r) \restriction \mu = h(r \restriction \mu)$, and this follows from the induction hypothesis.

Case 2: $\theta = \mu + 1$ for some μ

Our induction hypothesis implies that $p \restriction \mu$ extends $h(r) \restriction \mu$, so we need to establish that

$$p \restriction \mu \Vdash p(\mu) \text{ extends } h(r)(\mu). \tag{3.1}$$

We do this by showing that every extension t of $p \restriction \mu$ has an extension t' such that

$$t' \Vdash r(\mu) = h(r)(\mu).$$

How can we do this? This is where we take advantage of the assumption that $\Vdash_{P_\mu}$ "$|\dot{Q}_\mu| \leq \lambda^+$". Without loss of generality, we can assume that every condition in P_μ forces that the underlying set of $\dot{Q}_\mu$ is a subset of λ^+, and hence conditions in $\dot{Q}_\mu$ are forced to be ordinals $< \lambda^+$. Now we know that $M_\xi \cap \lambda^+ = M_\zeta \cap \lambda^+ = H \cap \lambda^+$, and so ordinals in $M_\xi \cap \lambda^+$ are fixed by h.

So suppose t is an extension of $p \restriction \mu$ in P_μ. Since t is (M_ξ, P_μ)–generic and $r(\mu)$ is forced to be an ordinal $< \lambda^+$, we can find $s \in M_\xi \cap P_\mu$ and $\alpha \in M_\xi \cap \lambda^+$ such that

- s is compatible with t, and
- $s \Vdash r(\mu) = \alpha$.

By applying the isomorphism h, we see that $h(s) \Vdash h(r)[h(\mu)] = h(\alpha)$. However, μ and α are fixed by h, so in fact we achieve

$$h(s) \Vdash h(r)(\mu) = \alpha.$$

Let t' be a condition witnessing that s and t are compatible in P_μ. Clearly t' is (M_ξ, P_μ)–generic, so our induction hypothesis (applied to t' and s) implies that t' extends $h(s)$ as well. Thus

$$t' \Vdash r(\mu) = h(r)(\mu) = \alpha.$$

Since t was an arbitrary extension of $p \restriction \mu$, we see

$$p \restriction \mu \Vdash r(\mu) = h(r)(\mu).$$

We know that $p \restriction \mu$ forces $p(\mu)$ to extend $r(\mu)$, and therefore

$$p \restriction \mu \Vdash p(\mu) \text{ extends } h(r)(\mu)$$

as required.

□

Corollary 3.5. If $\xi < \zeta$ in I, then p_ξ and p_ζ are compatible.

Proof. Let $\theta = \sup(M_\xi \cap \lambda^{++} \setminus H)$. It suffices to prove that $p_\xi \restriction \theta$ and $p_\zeta \restriction \theta$ have a common extension r in P_θ, as

$$q := r \cup p_\xi \restriction (M_\xi \cap \lambda^{++} \setminus H) \cup p_\zeta \restriction (M_\zeta \cap \lambda^{++} \setminus H)$$

gives a common extension of p_ξ and p_ζ.

This follows easily from what we just proved — let r be an (M_ξ, P_θ)–generic condition stronger than $p_\xi \restriction \theta$. We know that $\mathrm{supp}(p_\xi) \cap \theta = \mathrm{supp}(p_\zeta) \cap \theta \subseteq H$ and for all $\mu \in H$, $r \restriction \mu$ extends both $p_\xi \restriction \mu$ and $p_\zeta \restriction \mu$. □

This finishes the proof that $P_{\lambda^{++}}$ has the λ^{++}–chain condition. □

4. An Example

Let $S \subseteq S^{\omega_2}_{\omega_1} := \{\delta < \omega_2 : \mathrm{cf}(\delta) = \omega_1\}$ be stationary. Recall that a continuous ladder system on S is a family of functions $\bar\eta = \langle \eta_\delta : \delta \in S\rangle$ such that η_δ is a strictly increasing and continuous from ω_1 onto a cofinal subset of δ.

A continuous ladder system $\bar\eta$ has the club uniformization property if whenever $\bar c = \langle c_\delta : \delta \in S\rangle$ is a family of functions from ω_1 to $\{0, 1\}$, there is a function h such that for all $\delta \in S$, the set $\{i < \omega_1\ c_\delta(i) = h(\eta_\delta(i))\}$ contains a closed unbounded subset of ω_1.

Shelah [4] has shown that if the Continuum Hypothesis is true, then no continuous ladder system on (all of) $S^{\omega_2}_{\omega_1}$ has the club uniformization property. If we are looking at a stationary $S \subseteq S^{\omega_2}_{\omega_1}$ such that $S^{\omega_2}_{\omega_1} \setminus S$ is stationary as well, then the techniques of [3] show how to build a model where the Continuum Hypothesis holds and continuous ladder systems on S have the club uniformization property.

Let us fix a stationary, co–stationary $E_0 \subseteq \omega_1$ and let $\mathfrak{D}$ be the club filter restricted to $\omega_1 \setminus E_0$. Further assume that $\mathfrak{D}$ has diamonds — this follows if $V = L$ or if, e.g., $\diamondsuit^*(\omega_1 \setminus E_0)$ holds.

We will force a weak version of the club uniformization property to hold for a continuous ladder system $\bar{\eta} = \langle \eta_\delta : \delta \in S\rangle$ on $S := S^{\omega_2}_{\omega_1}$; what we achieve is that for every family $\bar{c} = \langle c_\delta : \delta \in S\rangle$ of functions mapping ω_1 to $\{0, 1\}$, there is a function $h : \omega_2 \to 2$ such that for each $\delta \in S$,

$$\{i \in E_0 : h(\eta_\delta(i)) \neq c_\delta(i)\} \text{ is non–stationary.} \tag{4.1}$$

Said another way, for each $\delta \in S$ there is a closed unbounded $C_\delta \subseteq \omega_1$ such that

$$i \in C_\delta \cap E_0 \Longrightarrow h(\eta_\delta(i)) = c_\delta(i); \tag{4.2}$$

i.e., h achieves success at almost every point in $\eta_\delta[E_0]$.

Let us fix a continuous ladder system $\bar{\eta} = \langle \eta_\delta : \delta \in S\rangle$. Suppose $\langle c_\delta : \delta \in S\rangle$ is a family of functions each mapping ω_1 to $\{0, 1\}$. Our first goal is to define a notion of forcing that will adjoin a function h such that (4.1) is satisfied for all $\delta \in S$.

A condition p is simply an approximation to the desired h of size $\leq \omega_1$, i.e., $p \in P$ if p is a function satisfying

- $\mathrm{dom}(p) \in [\omega_2]^{\leq \omega_1}$
- $\mathrm{ran}(p) \subseteq \{0, 1\}$
- for all $\delta \in S$,

$$\{i \in E_0 : p(\eta_\delta(i)) \neq c_\delta(i)\} \text{ is non–stationary.}$$

Clearly P is $< \omega_1$–closed and for each $\alpha < \omega_2$, the set of conditions with α in their domain is dense in P. Thus forcing with P adds no new countable sequences to the ground model and adjoins a function from ω_2 to $\{0, 1\}$.

Claim 4.1. P is proper for $\mathfrak{D}$–diamonds.

Proof. Let N be a relevant model with filtration $\langle N_i : i < \omega_1\rangle$ and let $p \in N \cap P$ be arbitrary. Suppose $E_1 \in \mathfrak{D}^+$ and let $\bar{R} = \langle (R_\delta, q_\delta) : \delta \in E_1\rangle$ be an (N, P)–rule. Note that we may assume that $E_0 \cap E_1 = \emptyset$ because of our definition of $\mathfrak{D}$. We will construct a decreasing sequence $\langle p_\alpha : \alpha < \omega_1\rangle$ of conditions in $N \cap P$ in such a way that $q := \bigcup_{\alpha<\omega_1} p_\alpha$ is an $(N, P, \bar{R})$–generic extension of p.

Let $\gamma = N \cap \omega_2$, and for $\alpha < \omega_1$ let $\gamma_\alpha = N_\alpha \cap \omega_2$. The sequence $\langle \gamma_\alpha : \alpha < \omega_1\rangle$ is strictly increasing, continuous, and cofinal in γ.

As we build the sequence $\langle p_\alpha : \alpha < \omega_1\rangle$, we will also be defining a strictly increasing and continuous sequence of countable ordinals $\langle i_\alpha : \alpha < \omega_1\rangle$.

We begin by letting i_0 be the least $i < \omega_1$ such that $p \in N_i$, and let $p_0 \in N \cap P$ be some totally (N_{i_0}, P)–generic extension of p.

Given $\langle p_\beta : \beta \leq \alpha\rangle$ and $\langle i_\beta : \beta \leq \alpha\rangle$, we let $i_{\alpha+1}$ be the least ordinal i such that both $\langle p_\beta : \beta \leq \alpha\rangle$ and $\langle i_\beta : \beta \leq \alpha\rangle$ are elements of N_i. Note that such an i exists because $N^{<\omega_1} \subseteq N$. We let $p_{\alpha+1}$ be a totally $(N_{i_{\alpha+1}}, P)$–generic extension of p_α in $N \cap P$.

Now what happens at limit stages of the construction? If α is a limit ordinal, we will be handed $\langle p_\beta : \beta < \alpha \rangle$ and $\langle i_\beta : \beta < \alpha \rangle$. We are committed to the continuity of $\langle i_\alpha : \alpha < \omega_1 \rangle$, so this means that we are forced to choose

$$i_\alpha = \bigcup_{\beta<\alpha} i_\beta.$$

Let us define

$$r_\alpha = \bigcup_{\beta<\alpha} p_\beta.$$

Since α is a countable ordinal, we know that r_α is a condition in P, and the relevance of the model N implies that $r_\alpha \in N \cap P$. By our construction, we know that r_α is totally (N_{i_α}, P)–generic — this follows because

$$N_{i_\alpha} = \bigcup_{\beta<\alpha} N_{i_\beta}.$$

Now we ask:
Is it the case that

- $i_\alpha = \alpha$,
- $\gamma_\alpha = \eta_\gamma(i_\alpha)$, *and*
- $\alpha \in E_0 \cup E_1$?

If not, we let $p_\alpha = r_\alpha$ and the construction continues. If the answer is yes, then we have two cases to consider — the case $\alpha \in E_0$ and the case $\alpha \in E_1$

If $\alpha \in E_0$, we note first that $\mathrm{dom}(r_\alpha) \subseteq \gamma_\alpha$ — this is because $p_\beta \in N_\alpha$ for all $\beta < \alpha$ and $\mathrm{dom}(r_\alpha) = \cup_{\beta<\alpha} \mathrm{dom}(p_\beta)$. Thus we may define

$$p_\alpha = r_\alpha \cup \{\langle \gamma_\alpha, c_\gamma(\alpha) \rangle\},$$

and conclude that $p_\alpha \in N \cap P$.

If $\alpha \in E_1$, then we ask if A_α is equal to the filter on $N_\alpha \cap P$ generated by $\langle p_\beta : \beta < \alpha \rangle$. If yes, then we let $p_\alpha = q_\alpha$ (note that $q_\alpha \leq r_\alpha$ if this happens); if not, we let $p_\alpha = r_\alpha$.

In either case, the condition p_α will be in $N \cap P$ and the construction can continue.

Claim 4.2. The sequence $\langle p_\alpha : \alpha < \omega_1 \rangle$ has a lower bound in P.

Proof. Let $q = \bigcup_{\alpha<\omega_1} p_\alpha$. It is clear that q is a partial function from ω_2 to $\{0, 1\}$ with domain a set of cardinality $\aleph_1$. Since each p_α is an element of N, we know that $\mathrm{dom}(q) \subseteq \gamma$.

What we need to show is that for every $\delta \in S$, (4.1) holds. If $\delta > \gamma$, then (4.1) holds because $\mathrm{dom}(q) \subseteq \gamma$. If $\delta < \gamma$, we note that $\delta \in N$ (as $N^{<\omega_1} \subseteq N$ implies $N \cap \omega_2$ is an initial segment of ω_2), and the set of conditions whose domain includes $\delta \cup \{\eta_\delta(i) : i < \omega_1\}$ is dense in P and an element of N. Thus there is a stage α such that

$$\delta \cup \{\eta_\delta(i) : i < \omega_1\} \subseteq \mathrm{dom}(p_\alpha).$$

Since $p_\alpha \in P$, the definition of q implies (4.1) holds for δ.

The last case to consider is when $\delta = \gamma$. Note that there is a closed unbounded set of $\alpha < \omega_1$ for which $i_\alpha = \alpha$ and $\eta_\gamma(\alpha) = \gamma_\alpha$. If $\alpha \in E_0$ has these properties, then at stage α we ensured that $q(\eta_\gamma(\alpha)) = c_\gamma(\alpha)$. Thus (4.1) holds for $\gamma = \delta$, and we have established that q is a condition in P. □

Claim 4.3. The condition q is $(N, P, \bar{R})$–generic.

Proof. Again, there is a closed unbounded set of α for which $i_\alpha = \alpha$ and $\eta_\gamma(\alpha) = \gamma_\alpha$. Note that for such an α, we automatically achieve that $\langle p_\beta : \beta < \alpha \rangle$ generates an (N_α, P)–generic filter G_α — this follows because $N_\alpha = \bigcup_{\beta<\alpha} N_{i_\beta}$. If for such an α it happens that $G_\alpha = A_\alpha$, then we made sure that $p_\alpha = q_\alpha$. Since

$$q \Vdash N \cap \dot{G}_P \text{ is generated by } \langle p_\alpha : \alpha < \omega_1 \rangle,$$

we have ensured that q is $(N, P, \bar{R})$–generic. □

□

The actual construction of a model where our weak club uniformization principle holds now follows standard lines — we can use an $\aleph_1$–support iteration of length $\aleph_3$ (with iterands corresponding to the notion of forcing above) to destroy any potential counterexample. The limit of this iteration (i.e., P_{ω_3}) is countably complete (so $\aleph_1$ is preserved), proper for $\mathfrak{D}$–diamonds (so $\aleph_2$ is preserved), and ω_3–c.c. (so all other cardinals are preserved).

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