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CH and first countable, countably compact spaces ☆

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Abstract

We show that it is consistent with the Continuum Hypothesis that first countable, countably compact spaces with no uncountable free sequences are compact. As a consequence, we get that CH does not imply the existence of a perfectly normal, countably compact, non-compact space, answering a question of Nyikos (Question 287 in the numbering of van Mill and Reed, Open Problems in Topology, Elsevier, Amsterdam, 1990, p. 127). © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

One of the major results in set-theoretic topology is that perfect, countably compact spaces may or may not be compact, depending on set-theoretic assumptions. Weiss [18] proved that if Martin's Axiom holds and $\mathfrak{c} > \omega_1$, then countably compact perfect regular spaces are compact, while Ostaszewski showed in [11] that $\diamondsuit$ implies the existence of a countably compact, perfectly normal non-compact space.

In [1], the author and Roitman proved that the Continuum Hypothesis is not enough to guarantee the existence of an Ostaszewski space. In this paper, we generalize the notion of forcing used there to show that CH is not enough to produce a countably compact perfect regular space that is not compact.

Our result is actually more general—in our model every first countable, countably compact space is either compact or contains an uncountable free sequence. We note that

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neither of the hypotheses on the space can be dropped, as Juhász et al. [9] have shown that CH suffices to build a first countable S-space, and Hajnal and Juhász [6] constructed a countably compact, non-compact S-space from CH.

The author and Peter Nyikos have continued the line of research of this paper in [2]. It is shown there that it is consistent with CH that every first countable closed preimage of ω_1 contains a closed copy of ω_1. This allows us (using the notion of forcing defined in this paper) to construct models of ZFC + CH in which first countable, countably compact spaces are either compact or contain closed subsets homeomorphic to ω_1.

2. Topological preliminaries

The topological ideas that we use are simple ones, but we will take a moment to review a few of the basic notions. Good references for the background material are [3] and [4]. Vaughan's article [17] is an excellent source of information for the theory of countably compact spaces, while [7] and [8] provide much more information about free sequences and their role in topology. All topological spaces we consider are assumed to be Hausdorff.

Definition 2.1. A topological space X is countably compact if every countable open cover has a finite subcover. This is equivalent to every infinite set having a cluster point.

The following proposition has a routine proof, and is well known.

Proposition 2.2. *If X is countably compact and first countable, then X is regular. If X is a regular countably compact space with countable pseudocharacter, i.e., for each $x \in X$, there is a collection $\{U_n\colon n \in \omega\}$ of open sets with $\{x\} = \bigcap\{U_n\colon n \in \omega\}$, then X is first countable.*

Our argument makes heavy use of filters of closed sets, so we take a moment to remind the reader of the relevant definitions.

Definition 2.3. A collection $\mathcal{F}$ of non-empty closed subsets of X is called a filter if $\mathcal{F}$ is closed under finite intersections and the taking of closed supersets. A filter is maximal if it is not properly contained in any larger filter. A filter $\mathcal{F}$ is fixed if there is a single point x that is a member of each set in $\mathcal{F}$. We say $\mathcal{F}$ is countably complete if the intersection of countably many members of $\mathcal{F}$ is a member of $\mathcal{F}$ (and hence nonempty).

The connection between filters and compactness is given by the following elementary fact.

Proposition 2.4. *A space X is compact if and only if every filter of closed sets is fixed. X is countably compact if and only if every maximal filter of closed sets is countably complete.*

Definition 2.5. A sequence $\{x_\alpha\colon\ \alpha < \kappa\} \subseteq X$ is said to be a free sequence if for each $\alpha < \kappa$,

$$\operatorname{cl}\{x_\beta\colon\ \beta < \alpha\} \cap \operatorname{cl}\{x_\beta\colon\ \beta \geqslant \alpha\} = \emptyset. \tag{2.1}$$

Clearly a space with an uncountable free sequence is not hereditarily separable, as any free sequence is discrete in the subspace topology.

Proposition 2.6. *Suppose $\mathcal{F}$ is a countably complete filter of closed subsets of a space X. Also assume that X contains no uncountable free sequence. If Y is a subset of X and Y meets every set in $\mathcal{F}$, then Y has a countable subset Y_0 so that* $\operatorname{cl} Y_0$ *meets every set in $\mathcal{F}$.*

Proof. If not, then we can build an uncountable free sequence in X by a straightforward inductive construction of length ω_1, as the closure of every countable subset of Y is disjoint from a set in $\mathcal{F}$. □

We need one more proposition concerning countably compact spaces. Recall that a space X is perfect if every closed set is a G_δ, i.e., if $K \subseteq X$ is closed, there is a countable collection of open sets $\{U_n\colon\ n \in \omega\}$ such that $K = \bigcap\{U_n\colon\ n \in \omega\}$. In light of Proposition 2.2, we see that perfect countably compact regular spaces are first countable.

Proposition 2.7 (Stephenson [12]). *If X is perfect and countably compact, then X has no uncountable discrete subspace.*

Proof. Left to reader, or see [12] or [17]. □

From this point on, our convention is that X is a first countable, countably compact non-compact space with no uncountable free sequences. We also use $\mathcal{U}$ exclusively as the symbol for a maximal filter of closed subsets of X that is not fixed. Note that $\mathcal{U}$ will be countably complete as X is countably compact. If we are in need of an adjective in the course of our discussion, we will simply say that X and $\mathcal{U}$ are “appropriate”.

Definition 2.8. A subset A of X is said to be large if A meets every member of $\mathcal{U}$; otherwise we say A is small.

The following proposition collects a few simple observations.

Proposition 2.9.
(1) *If A is large, then* $\operatorname{cl} A \in \mathcal{U}$.
(2) *A countable union of small sets is small.*
(3) *If A is large and $B \in \mathcal{U}$, then $A \cap B$ is large.*
(4) *Every point $x \in X$ has a neighborhood with small closure.*

Proof. Left to reader. The proof of the fourth clause is where we need the fact that X is T_3. □

3. Set-theoretic preliminaries

We assume that the reader has some experience working with proper forcing as it is presented in [14] or [5]; in particular, we assume the reader is used to working with countable elementary submodels of $H(\lambda)$—the sets whose hereditary cardinality is less than λ—where λ is some large enough regular cardinal.

Our conventions regarding iterated forcing are standard (see, for example, [5]). If $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha\colon \alpha < \kappa \rangle$ is a countable support iteration, we adopt the convention that P_0 is the trivial one element partial order for notational convenience. We will write $\Vdash_\alpha$ in lieu of $\Vdash_{P_\alpha}$.

Definition 3.1. Let P be a notion of forcing, let N be a countable elementary submodel of some $H(\lambda)$ with $P \in N$, and let $p \in N \cap P$. We define
(1) $N^P = \{\dot{\tau} \in N\colon \dot{\tau}$ is a P-name$\}$,
(2) $\mathrm{Gen}(N, P) = \{G \subseteq N \cap P\colon G$ is an N-generic filter on $N \cap P\}$,
(3) $\mathrm{Gen}^+(N, P) = \{G \in \mathrm{Gen}(N, P)\colon G$ has a lower bound in $P\}$,
(4) $\mathrm{Gen}(N, P, p) = \{G \in \mathrm{Gen}(N, P)\colon p \in G\}$,
(5) $\mathrm{Gen}^+(N, P, p) = \mathrm{Gen}(N, P, p) \cap \mathrm{Gen}^+(N, P)$.

Definition 3.2. Let P be a notion of forcing, and let N be a countable elementary submodel of $H(\lambda)$ for some large enough regular λ. We say a condition $q \in P$ is totally (N, P)-generic if whenever D is a dense open subset of P that is in N, we can find a condition $p \in N \cap D$ with $q \leqslant p$. Said another way, q is a lower bound for some $G \in \mathrm{Gen}(N, P)$. We say that P is totally proper if, given N as above, every $p \in N \cap P$ has a totally (N, P)-generic extension q.

It is not difficult to prove that a notion of forcing is totally proper if and only if it is proper and the forcing adds no new reals (for an explicit proof, see [1]). In the presence of properness, adding no new reals is equivalent to the forcing adding no new ω-sequences of elements of the ground model.

It is instructive to compare the definition of "q is totally (N, P)-generic" with the definition of "q is (N, P)-generic" so important to proper forcing. Recall that q is said to be (N, P)-generic if for every dense open subset D of P that is an element of N, the countable set $N \cap D$ is pre-dense below q. This means that q forces that the generic subset $\dot{G}_P$ intersects $N \cap D$. Total properness requires more: not only does q force that $\dot{G}_P$ meets D, it does so in an explicit fashion—there is a condition $r \in N \cap D$ such that $q \leqslant r$.

Our convention will be that $\dot{G}$ is the canonical P-name for the generic subset of P. If N is a countable elementary submodel of $H(\lambda)$ containing P, then by $N[\dot{G}]$ we mean a P-name forced by every condition to be the set of interpretations of P-names that are members of N, i.e., $N[\dot{G}]$ is a P-name for the image of N^P under the evaluation function in $V[G]$.

The basic theorems on forcing remain valid in the context of looking at N and $G \in \mathrm{Gen}(N, P)$. In particular, it makes sense to speak of the generic extension $N[G]$ where

$G \in \mathrm{Gen}(N, P)$. The point is that in this situation, the model $N[G]$ is in V as both N and G are. There are some technical details arising from the fact that N is not transitive that make a precise definition of $N[G]$ bothersome (they are worked out in [1]); we can avoid all this by making the following definition.

Definition 3.3. Let N be a countable elementary submodel of $H(\lambda)$ for some large regular λ, let $P \in N$ be totally proper, and let G be a subset of P. If ϕ is a formula and $\dot{\tau}_0, \ldots, \dot{\tau}_n$ are in N^P, then the formula

$$N[G] \models \phi(\dot{\tau}_0, \ldots, \dot{\tau}_n) \tag{3.1}$$

means

$$(\exists p \in N \cap G)\big[p \Vdash \phi(\dot{\tau}_0, \ldots, \dot{\tau}_n)\big]. \tag{3.2}$$

The preceding definitions are extremely important in Section 6 of the paper. We will tend to use the notation of (3.1) because of space considerations, but the reader should keep in mind (3.2) as it is more concrete.

The most important case of the above definition is when G is in $\mathrm{Gen}(N, P)$, but we will also be considering situations where G is "larger", i.e., situations where $N \cap G \in \mathrm{Gen}(N, P)$ even though G is not a subset of $N \cap P$. Note that if N, ϕ, $\dot{\tau}_0, \ldots, \dot{\tau}_n$ are as in the previous definition, and $N \cap G$ is in $\mathrm{Gen}(N, P)$, then either

$$N[G] \models \phi(\dot{\tau}_0, \ldots, \dot{\tau}_n) \tag{3.3}$$

or

$$N[G] \models \neg\phi(\dot{\tau}_0, \ldots, \tau_n). \tag{3.4}$$

This is because the set of conditions in P that decide $\phi(\dot{\tau}_0, \ldots, \dot{\tau}_n)$ is a dense set in P that is an element of N.

Proposition 3.4. *If P is totally proper then any generic subset G of P is countably closed, i.e., given $\{p_n\colon n \in \omega\} \subseteq G$, there is a $q \in G$ that extends each p_n.*

Proof. Since P is totally proper, the set $\{p_n\colon n \in \omega\}$ is in the ground model. Also, the set

$$D = \{p \in P\colon\ p \text{ extends each } p_n \text{ or } p \text{ is incompatible with some } p_n\} \tag{3.5}$$

is dense in P. Since D is in the ground model, there is some $q \in G \cap D$. Since G is directed and each p_n is in G, it must be the case that q extends each p_n. □

The following claim will be used in the proof of Proposition 3.6.

Claim 3.5. *If P is totally proper and r is (N, P)-generic, then the set of totally (N, P)-generic conditions is pre-dense below r.*

Proof. Choose $r' \leqslant r$. We must produce a totally (N, P)-generic $s \leqslant r'$. Let G be any generic subset of P that contains r'. Since r is (N, P)-generic, we know that $N \cap G$ is generic over the model N. Since $N \cap G$ is countable, Proposition 3.4 gives us a condition $q \in G$ that is a lower bound for $N \cap G$. Since G is a filter, we can find a condition $r'' \in G$ that is below both r' and q. Now back in V, we have a name for r'' forced by r' to have the required properties, i.e., there is are names $\dot{t}$ and $\dot{H}$ such that

$$r' \Vdash \dot{t} \in \dot{G}_P, \quad \dot{t} \leqslant r', \quad \dot{H} \in \mathrm{Gen}(N, P), \quad \text{and} \quad \dot{t} \text{ is a lower bound for } \dot{H}. \tag{3.6}$$

In V, we can find an $s \leqslant r'$, $t \in P$, and $H \in \mathrm{Gen}(N, P)$ such that

$$s \Vdash \dot{t} = t \quad \text{and} \quad \dot{H} = H. \tag{3.7}$$

(Note that such an H must exist as P is totally proper and $\dot{H}$ is a countable set of elements from the ground model.) Since $s \Vdash t \in \dot{G}$ and P is (without loss of generality) separative, it must be the case that $s \leqslant t$, and hence s is a lower bound for $H \in \mathrm{Gen}(N, P)$. This means $s \leqslant r'$ is totally (N, P)-generic and the proposition is established. □

Proposition 3.6. *Let P be totally proper, and let $N \in M$ be countable elementary submodels of $H(\lambda)$ with $P \in N$. If r is both (N, P)-generic and (M, P)-generic, then $r \Vdash N \cap \dot{G} \in M \cap \mathrm{Gen}^+(N, P)$.*

Proof. Let G be any generic subset of P that contains r. First, by the previous proposition, we know that $N \cap G$ has a lower bound in P. Thus we need only verify that $N \cap G$ is a member of M. To see this, define

$$D = \{p \in P\colon \quad p \text{ is totally } (N, P)\text{-generic, or } P \text{ has no totally } (N, P)\text{-generic extension}\}. \tag{3.8}$$

The D is a member of M and is dense and open in P. Since $r \in G$ is (M, P)-generic, we know that there is some q in $M \cap D \cap G$. By the preceding lemma, q has a totally (N, P)-generic extension and so q must be totally (N, P)-generic by the definition of D.

Thus $N \cap G = \{p \in N \cap P\colon\ q \leqslant p\}$ is definable from parameters in M. (Note that the “hard direction” of the equality follows by an argument using the fact that a member of $\mathrm{Gen}(N, P)$ is a maximal filter on $N \cap P$. This follows by an easy genericity argument, as for $r \in N \cap P$, the set of conditions that either extend r or are incompatible with r is dense in P and an element of N.) □

We also need a couple of propositions dealing with how a countable elementary submodel interacts with the spaces in which we are interested. Recall that X is assumed to be a first countable, countably compact non-compact space with no uncountable free sequences.

Proposition 3.7. *Let N is a countable elementary submodel of $H(\lambda)$ with $X \in N$. If x is in $N \cap X$, and U is any open neighborhood of x, then we can find an open neighborhood V of x that is a member of N and that is contained in U.*

Proof. This follows immediately from the fact that X is first countable, as N contains every member of a neighborhood base for x. □

Proposition 3.8. *Let N be a countable elementary submodel of $H(\lambda)$ that contains X and $\mathcal{U}$ (the filter of closed sets from the last section). Suppose x is an element of each set in $N \cap \mathcal{U}$ (the set of such x is in $\mathcal{U}$ by the countable completeness of $\mathcal{U}$). If $A \in N \cap \mathcal{U}$, then x is in the closure of $N \cap A$.*

Proof. By Proposition 2.6, there is a countable $B_0 \subseteq A$ such that $B = \mathrm{cl}\, B_0 \in \mathcal{U}$. By elementarity, there is such a B_0 in N (so $B \in N \cap \mathcal{U}$ as well) and since B_0 is countable, we know that $B_0 \subseteq N$. Since x is also a member of B, we are done. □

4. Promises and a notion of forcing

The following definition is the most important ingredient in our argument.

Definition 4.1. A function f is called a promise if dom f is large and for each x in dom f, $f(x)$ is a neighborhood of x. If f is a promise, we say that $y \in X$ is banned by f if $\{x \in \mathrm{dom}\, f\colon\ y \in f(x)\}$ is small. We let

$$\mathrm{Ban}(f) = \big\{y \in X\colon\ y \text{ is banned by } f\big\}. \tag{4.1}$$

Proposition 4.2. *If f is a promise and y is not banned by f, then y has a neighborhood U so that $\{x \in \mathrm{dom}\, f\colon\ U \subseteq f(x)\}$ is large. In particular,* Ban f *is closed.*

Proof. Let $\{U_n\colon\ n \in \omega\}$ be a neighborhood base at y. If $y \in f(x)$, then there is some n so that $U_n \subseteq f(x)$. Thus we can write

$$\big\{x \in \mathrm{dom}\, f\colon\ y \in f(x)\big\} = \bigcup_{n\in\omega} \big\{x \in \mathrm{dom}\, f\colon\ U_n \subseteq f(x)\big\}. \tag{4.2}$$

Since a countable union of small sets is small and the set on the left is large, it must be the case that for some n that $\{x \in \mathrm{dom}\, f\colon\ U_n \subseteq f(x)\}$ is large. □

Proposition 4.3. Ban(f) *is small for every promise f.*

Proof. Assume f is a promise with Ban(f) large. Since Ban(f) is closed, it must be the case that Ban$(f) \in \mathcal{U}$. By Proposition 2.6, there is a countable subset $\{a_n\colon\ n \in \omega\}$ of Ban(f) such that

$$A = \mathrm{cl}\{a_n\colon\ n \in \omega\} \in \mathcal{U}. \tag{4.3}$$

Notice that A is a subset of Ban(f). Now let $B = A \cap \mathrm{dom}\, f$; B is large because $A \in \mathcal{U}$ and dom f is large. Since every member of A is banned by f, for every $x \in B$ there is an n such that $a_n \in f(x)$, i.e.,

$$B = \bigcup_{n\in\omega} B_n, \quad \text{where } B_n = \big\{x \in B\colon\ a_n \in f(x)\big\}. \tag{4.4}$$

Since B is large, it must be the case that B_n is large for some n. This is a contradiction, as $a_n \in \mathrm{Ban}(f)$. □

Definition 4.4. Define a notion of forcing $P = P_X$ by putting p into P if and only if $p = ([p], \Phi_p)$, where

(1) $[p]$ is a countable subset of X and $\mathrm{cl}[p]$ is not in $\mathcal{U}$.

(2) Φ_p is a countable collection of promises.

A condition q extends p if $[q] \supseteq [p]$, $\Phi_q \supseteq \Phi_p$, for each promise $f \in \Phi_p$, the set

$$Y(f, q, p) = \{x \in \mathrm{dom}\, f\colon\ [q] \setminus [p] \subseteq f(x)\} \tag{4.5}$$

is large, and $f \restriction Y(f, q, p) \in \Phi_q$.

The reader should verify that P is in fact a partially ordered set—we use (4.5) to ensure that $\leqslant$ is transitive.

Definition 4.5. Given $p \in P$ and $D \subseteq P$ dense and open, let

$$\mathrm{Bad}(p, D) = \{x \in X\colon\ x \text{ has a neighborhood } U_x \text{ such that there is no } q \leqslant p \text{ with } q \in D \text{ and } [q] \setminus [p] \subseteq U_x\}. \tag{4.6}$$

Proposition 4.6. *If $p \in P$ and $D \subseteq P$ is dense open, then* $\mathrm{Bad}(p, D)$ *is small.*

Proof. Suppose not. Then the function f with domain $\mathrm{Bad}(p, D)$ that sends x to the neighborhood U_x that witnesses $x \in \mathrm{Bad}(p, D)$ is a promise. Let $q = ([p], \Phi_p \cup \{f\})$; clearly q extends p in P. Since D is dense, there is an extension $r \leqslant q$ with $r \in D$. By definition, this means

$$Y(f, r, q) = \{x \in \mathrm{dom}\, f\colon\ [r] \setminus [q] \subseteq f(x)\} \tag{4.7}$$

is large. In particular, it is non-empty. This is a contradiction, for if x is in $Y(f, r, q)$ then $[r] \setminus [q] = [r] \setminus [p]$ is a subset of in $f(x) = U_x$. □

Theorem 1. *If X is a first countable, countably compact, non-compact space with no uncountable free sequences, then P is totally proper.*

Proof. Let $N \prec H(\lambda)$ be countable with $X, \mathcal{U}$, and P in N, and let $p \in N \cap P$ be arbitrary. Let $\{D_n\colon n \in \omega\}$ list all dense open subsets of P that are members of N.

We want to build a sequence of conditions $\{p_n\colon n \in \omega\}$ so that $p_0 = p$, $p_{n+1} \leqslant p_n$, $p_{n+1} \in N \cap D_n$, and most importantly

$$\{p_n\colon n \in \omega\} \text{ has a lower bound in } P. \tag{4.8}$$

To ensure (4.8), we need

$$\mathrm{cl} \bigcup_{n\in\omega} [p_n] \notin \mathcal{U} \tag{4.9}$$

and if f is a promise from some Φ_{p_m}, then

$$\left\{y \in \operatorname{dom} f\colon \bigcup_{n\in\omega}[p_n] \setminus [p_m] \subseteq f(y)\right\} \text{ is large.} \tag{4.10}$$

If we are successful in doing this, then $q = ([q], \Phi)$ will be a lower bound for the sequence, where

$$[q] = \bigcup_{n\in\omega}[p_n] \tag{4.11}$$

and

$$\Phi = \bigcup_{n\in\omega}\Phi_{p_n} \cup \bigl\{f \restriction \bigl\{y \in \operatorname{dom}\colon [q] \setminus [p_n] \subseteq f(y)\bigr\}\colon f \in \Phi_{p_n},\ n \in \omega\bigr\}. \tag{4.12}$$

To start, let $\{A_n\colon n \in \omega\}$ enumerate $N \cap \mathcal{U}$, and choose for each n a point $x_n \in N \cap A_0 \cap \cdots \cap A_n$. Since X is first countable and countably compact, by passing to a subsequence we can assume

$$\{x_n\colon n \in \omega\} \text{ converges to a point } x. \tag{4.13}$$

Notice that if $A \in N \cap \mathcal{U}$, all but finitely many x_n's are in A, hence the point x is in A as well. This implies that if $f \in N$ is a promise, then $x \notin \operatorname{Ban}(f)$.

Let $\{U_n\colon n \in \omega\}$ be a neighborhood base for x. Since X is regular, we can assume that U_0 has small closure and $\operatorname{cl} U_{n+1} \subseteq U_n$.

By passing to a subsequence and reindexing, we can assume that x_n is in U_n. By Proposition 3.7, we can fix for each $n \in \omega$ an open neighborhood V_n for x_n such that $V_n \in N$ and $V_n \subseteq U_n$.

We now construct our sequence $\{p_n\colon n \in \omega\}$ in ω stages. As we proceed, we will also be defining a function g, choosing a value for $g(n)$ at stage $n+1$.

At stage $n+1$ of our construction, we will be handed p_n and D_n as well as a promise $f_n \in \Phi_i = \Phi_{p_i}$ for some $i \leqslant n$. Let

$$B = Y(f_n, p_n, p_i) = \bigl\{y \in \operatorname{dom} f_n\colon [p_n] \setminus [p_i] \subseteq f_n(y)\bigr\}. \tag{4.14}$$

By definition of extension, B is large and $f_n \restriction B \in \Phi_n$. Since x is not banned by $f \restriction B \in N$, we can find a neighborhood V of x such that

$$K(V, f_n) = \bigl\{y \in B\colon V \subseteq f_n(y)\bigr\} \text{ is large.} \tag{4.15}$$

Since $\operatorname{Bad}(p_n, D_n) \cap \{x_n\colon n \in \omega\}$ is finite, we can choose $g(n)$ large enough so that $U_{g(n)} \subseteq V$, $x_{g(n)} \notin \operatorname{Bad}(p_n, D_n)$, and $g(n) > g(i)$ for all $i < n$.

Now $V_{g(n)} \in N$ is a neighborhood of $x_{g(n)} \notin \operatorname{Bad}(p_n, D_n)$, so in N we can find an extension p_{n+1} of p_n that is in D_n and such that $[p_{n+1}] \setminus [p_n] \subseteq V_{g(n)}$. Notice that it is important for $V_{g(n)}$ to be an element N, as otherwise we could not guarantee the existence of a suitable p_{n+1} in N.

Clearly we can arrange our construction so that every promise f that appears in some Φ_i along the way gets handled at some stage $n_f + 1$ in the construction.

Our construction guarantees that

$$\bigcup_{n\in\omega}[p_n]\setminus[p]\subseteq U_0 \tag{4.16}$$

and so condition (4.9) holds. Given a promise $f\in\Phi_i$ for some $i<\omega$, let n_f+1 be the stage where we took care of f. Let

$$B=\bigl\{y\in\operatorname{dom} f\colon [p_{n_f}]\setminus[p_i]\subseteq f(y)\bigr\}. \tag{4.17}$$

We found at stage n_f+1 a neighborhood $U_{g(n_f)}$ of x for which $K(U_{g(n_f)},f)$ (see (4.15)) is large. Since

$$[q]\setminus[p_{n_f}]\subseteq U_{g(n_f)} \tag{4.18}$$

we have (4.10). Thus the sequence $\{p_n\colon n\in\omega\}$ has a lower bound. □

Now that we know the forcing is totally proper, we need to verify that in fact it adds an uncountable free sequence in X.

Proposition 4.7. *If A is large, then then set of conditions q with $[q]\cap A\neq\emptyset$ is dense in P.*

Proof. Let $p=([p],\Phi)$ be an arbitrary condition. If $\{f_n\colon n\in\omega\}$ is a list of all promises from Φ, then we know that

$$B=\bigcup_{n\in\omega}\operatorname{Ban}(f_n) \tag{4.19}$$

is a small set. Fix $y\in A\setminus B$. Then clearly

$$q=\Bigl([p]\cup\{y\},\Phi\cup\bigl\{f_n\restriction\{x\in\operatorname{dom} f_n\colon y\in f_n(x)\}\colon n\in\omega\bigr\}\Bigr) \tag{4.20}$$

is an extension of p with the required property. □

Corollary 4.8. *In the generic extension, X contains an uncountable free sequence.*

Proof. Let G be a generic subset of P, and let Y be the union of the first components of conditions in G.

The filter $\mathcal{U}$ a countably complete filter of closed sets in $V[G]$, though it is no longer maximal. By the preceding proposition, Y meets every set in $\mathcal{U}$. Let $Y_0\subseteq Y$ be countable. Since G is countably closed (see Proposition 3.4), there is a $p\in G$ for which $Y_0\subseteq[p]$, and so $\operatorname{cl}(Y_0)$ is small. Now we can apply Proposition 2.6 (in $V[G]$) to Y and $\mathcal{U}$ and conclude that X has an uncountable free sequence. □

5. $<\omega_1$-properness

In this section, we verify that the notion of forcing introduced in the last section satisfies one of the technical conditions necessary to prove that an iteration of such forcings does not add any new reals. We will invest time in some easy preliminary results, and gradually build to the required proof.

One more strengthening of properness is relevant to our proof. Shelah's books [14] and [15] are the primary references for this material.

Definition 5.1. We say that $\mathfrak{N} = \{N_\xi\colon \xi \leqslant \alpha\}$ is an α-tower if for some large enough regular cardinal λ, each N_ξ is a countable elementary submodel of $H(\lambda)$, the sequence $\mathfrak{N}$ is increasing and continuous at limits, and $\{N_\zeta\colon \zeta \leqslant \xi\} \in N_{\xi+1}$. A notion of forcing P is said to be α-proper if for every α-tower $\mathfrak{N}$ such that $P \in N_0$, and for every $p \in N_0 \cap P$, there is a $q \leqslant p$ that is (N_ξ, P)-generic for each $\xi \leqslant \alpha$. Such a condition q is said to be $(\mathfrak{N}, P)$-generic. If in addition we have that q is totally (N_ξ, P)-generic for all $\xi \leqslant \alpha$, then we say that q is *totally* $(\mathfrak{N}, P)$*-generic*. Finally, we say P is $< \omega_1$-proper if P is α-proper for each $\alpha < \omega_1$.

Let X, $\mathcal{U}$, P, and N be as in the last section, i.e., $N \prec H(\lambda)$ is countable with $\{X, \mathcal{U}, P\} \in N$.

Definition 5.2. Let $\{A_n\colon n \in \omega\}$ enumerate $N \cap \mathcal{U}$. We define

$$A(N) = \bigcap_{n\in\omega} A_n. \tag{5.1}$$

Definition 5.3. We say a sequence $S = \{(x_n, V_n)\colon n \in \omega\}$ is suitable for N if

(1) $x_n \in N \cap X$, $V_n \in N$,
(2) V_n is an open neighborhood of x_n,
(3) if $A \in N \cap \mathcal{U}$, then $\{x_n\colon n \in \omega\} \setminus A$ is finite,
(4) $\{V_n\colon n \in \omega\}$ converges to a point $x = \mathrm{Top}(S)$, i.e., every open neighborhood of x contains all but finitely many V_n.

Notice that if S is suitable for N then so is any infinite subset of S, and since each member of $\mathcal{U}$ is closed, we have $\mathrm{Top}(S) \in A(N)$.

Proposition 5.4. *If $x \in A(N)$ and U is any open neighborhood of x, then we can find $S = \{(x_n, V_n)\colon n \in \omega\}$ that is suitable for N, and in addition satisfies $x = \mathrm{Top}(S)$ and $V_n \subseteq U$ for each n.*

Proof. Let $\{A_n\colon n \in \omega\}$ enumerate $N \cap \mathcal{U}$, and let $\{U_n\colon n \in \omega\}$ be a decreasing neighborhood base for x with $U_0 \subseteq U$. By Proposition 3.8, for each n we can choose a point x_n such that

$$x_n \in N \cap U_n \cap A_0 \cap \cdots \cap A_n. \tag{5.2}$$

By Proposition 3.7, we can find an open neighborhood V_n of x_n that is in N and a subset of U_n. Then $\{(x_n, V_n)\colon n \in \omega\}$ is as desired. $\square$

Proposition 5.5. *Let p be an arbitrary condition in $N \cap P$, and suppose $S = \{(x_n, V_n)\colon n \in \omega\}$ is suitable for N. Then p has a totally (N, P)-generic extension q that satisfies*

$$[q] \setminus [p] \subseteq \bigcup_{n\in\omega} V_n. \tag{5.3}$$

Proof. This follows immediately from the proof of Theorem 1, using these x_n's for the x_n's occurring in that proof. □

Theorem 2. *P is $< \omega_1$-proper.*

Proof. We prove by induction of $\alpha < \omega_1$ that whenever $\mathfrak{N} = \{N_\xi\colon \xi \leqslant \alpha\}$ is an α-tower with X, P, and $\mathcal{U}$ all in N_0, $p \in N_0 \cap P$, $x \in A(N_\alpha)$, and U is an open neighborhood of x, there is a totally $(\mathfrak{N}, P)$-generic $q \leqslant p$ with $[q] \setminus [p] \subseteq U$.

Propositions 5.4 and 5.5 together handle the base and successor cases, so assume that α is a countable limit ordinal, and let $\mathfrak{N}$, p, $x \in A(N_\alpha)$, and U be given. Let $\{\alpha_n\colon n \in \omega\}$ be an increasing ω-sequence cofinal in α.

Observe that if $\xi < \alpha$, then $A(N_\xi) \in N_{\xi+1}$. Also note that if V is any open neighborhood of x and $\xi < \alpha$, then Proposition 3.8 tells us that

$$N_{\xi+1} \cap A(N_\xi) \cap V \neq \emptyset. \tag{5.4}$$

Let $\{U_n\colon n \in \omega\}$ be a neighborhood base for x that satisfies $U_0 \subseteq U$, $U_{n+1} \subseteq U_n$, and $\operatorname{cl} U_0 \notin \mathcal{U}$.

We construct a decreasing sequence $\{p_n\colon n \in \omega\}$ of conditions in P such that

(1) $p_{n+1} \in N_{\alpha_n+1}$,
(2) p_{n+1} is totally (N_ξ, P)-generic for $\xi \leqslant \alpha_n$,
(3) $[p_{n+1}] \setminus [p_n] \subseteq U_0$,
(4) $\{p_n\colon n \in \omega\}$ has a lower bound.

To ensure (4), we take care of each promise that appears in some p_n in much the same way as in the proof of Theorem 1. This means that we will be defining a function $g \in \omega^\omega$ as well, with $g(n)$ being defined at stage $n + 1$.

At stage $n + 1$, we will be given $p_n \in N_{\alpha_{n-1}+1}$ as well as a promise $f \in \Phi_{p_i}$ for some $i \leqslant n$. Since $x \in A(N_{\alpha_{n-1}+1})$, we know x is not banned by f. By Proposition 4.2, we can choose $g(n) > g(i)$ for $i < n$ so that

$$\bigl\{y \in \operatorname{dom} f\colon U_{g(n)} \subseteq f(y)\bigr\} \text{ is large.} \tag{5.5}$$

By Proposition 3.8, we can find

$$x_n \in N_{\alpha_n+1} \cap U_{g(n)} \cap A(N_{\alpha_n}). \tag{5.6}$$

Let $V_{g(n)} \in N_{\alpha_n+1}$ be an open neighborhood of x that is a subset of $U_{g(n)}$ and apply the induction hypothesis inside N_{α_n+1} with p_n in place of p, $N_{\alpha_{n-1}+1}$ in place of N_0, and N_{α_n} in place of N_α to get p_{n+1}.

The argument that shows $\{p_n\colon n \in \omega\}$ has a lower bound q is just as in the proof of Theorem 1. □

6. Iteration

In this section, we finish the verification that the forcing notions of interest to us can be iterated without adding new reals. The technology we use comes from the last section of [1], but we also will take time to point out the relationship between the iteration theorem presented in that paper and the iteration theorems of Shelah presented in [13–15].

Definition 6.1. Assume the following:
- P is a totally proper notion of forcing,
- $\Vdash_P \dot{Q}$ is totally proper,
- N is a countable elementary submodel of $H(\lambda)$ and $\{P, \dot{Q}\} \in N$,
- $\overline{G} \in \mathrm{Gen}(N, P)$,
- $\dot{q} \in N^P$ is a name for a condition in $\dot{Q}$.

We say that $\{\dot{q}_n\colon n < \omega\}$ is an $(N[\overline{G}], \dot{Q}, \dot{q})$-generic sequence if
(1) $\dot{q}_0 = \dot{q}$,
(2) $\{\dot{q}_n\colon n \in \omega\} \subseteq N^P$,
(3) $N[\overline{G}] \models \dot{q}_{n+1} \leqslant \dot{q}_n$,
(4) if $\dot{D} \in N^P$ is a P-name for a dense subset of $\dot{Q}$, then for some n, $N[\overline{G}] \models \dot{q}_n \in \dot{D}$.

More generally, we define the set $\mathrm{Gen}(N[\overline{G}], \dot{Q}, \dot{q})$ by $\overline{H} \in \mathrm{Gen}(N[\overline{G}], \dot{Q}, \dot{q})$ if
(i) $\overline{H} \subseteq N^P$,
(ii) $\dot{q} \in \overline{H}$,
(iii) if $\dot{r} \in \overline{H}$, $\dot{s} \in N^P$, and $N[\overline{G}] \models \dot{r} \leqslant \dot{s}$ in $\dot{Q}$, then $\dot{s} \in \overline{H}$,
(iv) if $\dot{r}$ and $\dot{s}$ are in $\overline{H}$, then there is $\dot{t} \in \overline{H}$ such that $N[\overline{G}] \models \dot{t} \leqslant \dot{r} \wedge \dot{t} \leqslant \dot{s}$,
(v) if $\dot{D} \in N^P$ is a name for a dense subset of $\dot{Q}$, then there is an $\dot{r} \in \overline{H}$ for which $N[\overline{G}] \models \dot{r} \in \dot{D}$.

The connection between $(N[\overline{G}], \dot{Q}, \dot{q})$-generic sequences and members of $\mathrm{Gen}(N[\overline{G}], \dot{Q}, \dot{q})$ is straightforward—the generic sequences generate members of $\mathrm{Gen}(N[\overline{G}], \dot{Q}, \dot{q})$ in a natural fashion.

Turning to the specific case of interest to us, let us fix a totally proper notion of forcing P, and let $\dot{X}$ and $\dot{\mathcal{U}}$ be P-names for a space and filter of closed sets as in Section 2. Let $\dot{Q}$ be a P-name for the notion of forcing $P_{\dot{X}}$ that we have been investigating, and $N_0 \in N_1$ be countable elementary submodels of $H(\lambda)$, with N_0 containing P, $\dot{X}$, $\dot{\mathcal{U}}$, and $\dot{Q}$.

Proposition 6.2. *Given $\overline{G} \in \mathrm{Gen}^+(N_0, P) \cap N_1$, $\dot{q} \in N_0^P$ a P-name for a condition in $\dot{Q}$, and a countable set $\{G_n\colon n \in \omega\}$ of members of $\mathrm{Gen}(N_1, P)$ that extend $\overline{G}$, we can find an $(N_0[\overline{G}], \dot{Q}, \dot{q})$-generic sequence sequence $\{\dot{q}_n\colon n \in \omega\}$ so that if*

$$r \Vdash (\exists n \in \omega)[N_1 \cap \dot{G} = G_n], \tag{6.1}$$

then

$$r \Vdash \{\dot{q}_n\colon n \in \omega\} \text{ has a lower bound in } \dot{Q}. \tag{6.2}$$

The proof of this proposition is just a diagonalization argument. Each G_n should be thought of as a guess at what the P-generic object $\dot{G}$ looks like when restricted to N_1. All of these guesses agree that $\dot{G} \cap N_0$ will turn out to be $\overline{G}$, so this means they will agree on what happens to objects with names in N_0. Each of the models $N_1[G_n]$ has its own ideas about what the space named by $\dot{X}$ looks like (although they agree on information about $\dot{X}$ that comes from $N_0[\overline{G}]$), and the following argument shows that all of these different possibilities are somewhat compatible—we can build an $(N_0[\overline{G}], \dot{Q})$-generic sequence that will be guaranteed to have a lower bound as long as one of the G_n's is a correct guess at $N_1 \cap \dot{G}$.

Proof. Let $\{\dot{A}_n\colon n \in \omega\}$ be a listing in N_1 of all P-names $\dot{A}$ from N_0 such that

$$\exists p \in N_0 \cap \overline{G}\ p \Vdash \dot{A} \in \dot{\mathcal{U}} \tag{6.3}$$

and then choose for each n a P-name $\dot{x}_n$ from N_0 that satisfies

$$\exists p \in N_0 \cap \overline{G}\ p \Vdash \dot{x}_n \in \dot{A}_0 \cap \cdots \cap \dot{A}_n. \tag{6.4}$$

From now on, we will be using the notation of (3.2) instead of that of (3.1) in order to save a bit of space.

Our first goal is to thin out the sequence $\{\dot{x}_n\colon n \in \omega\}$ so that it will converge no matter which of the G_n's turns out to be $N_1 \cap \dot{G}$. We define in ω stages a sequence $\{I_n\colon n \in \omega\}$ of subsets of ω so that

(1) $I_0 = \omega$, $I_{n+1} \subseteq I_n$,
(2) $I_n \in N_1$ for each $n \in \omega$,
(3) $N_1[G_n] \models \{\dot{x}_i\colon i \in I_{n+1}\}$ converges in $\dot{X}$.

This is easily done because every condition forces that $\dot{X}$ is sequentially compact—at stage $n+1$, since $I_n \in N_1$, the set $\{\dot{x}_n\colon n \in I_n\}$ is in $N_1[G_n]$, and so there is a set I_{n+1} in $N_1[G_n]$ such that

$$N_1[G_n] \models \{\dot{x}_i\colon i \in I_{n+1}\} \text{ converges.} \tag{6.5}$$

Since P is totally proper, I_{n+1} is in $N_1[G_n] \cap V = N_1$ as required. Let $I \subseteq \omega$ be such that $I \setminus I_n$ is finite for each n—even though I is not going to be in N_1, we know that if $r \in P$ forces that there is an $n \in \omega$ such that $N_1 \cap \dot{G} = G_n$, then r forces that $\{\dot{x}_i\colon i \in I\}$ converges. Our next goal is to find for each $i \in I$ a name $\dot{V}_i \in N_0^P$ for an open neighborhood of $\dot{x}_i$ so that if G is a generic subset of P that extends some G_n, then the sequence of open sets $\{\dot{V}_i\colon i \in I\}$ converges in $\dot{X}$, i.e., if x is the limit of the $\dot{x}_i$'s, then every neighborhood of x contains all but finitely many of the $\dot{V}_i$'s.

We start by fixing for each $n \in \omega$ a name $\dot{y}_n$ from N_1^P such that

$$N_1[G_n] \models \dot{y}_n \text{ is the limit of } \{\dot{x}_i\colon i \in I_{n+1}\}. \tag{6.6}$$

Also fix for each $n \in \omega$ a set of P-names $\{\dot{U}_{m,n}\colon m \in \omega\}$ from N_1 such that

$$N_1[G_n] \models \{\dot{U}_{m,n}\colon m \in \omega\} \text{ is a decreasing neighborhood base for } \dot{y}_n \text{ in } \dot{X}, \text{ and } \operatorname{cl} \dot{U}_{0,n} \notin \dot{\mathcal{U}}. \tag{6.7}$$

For each $n \in \omega$ and $i \in I_{n+1}$, choose a name $\dot{V}_{i,n}$ from N_0 such that

$$N_0[\overline{G}] \models \dot{V}_{i,n} \text{ is a neighborhood of } \dot{x}_i \tag{6.8}$$

and

$$N_1[G_n] \models \{\dot{V}_{i,n}\colon\ i \in I_{n+1}\} \text{ converges to } \dot{y}_n. \tag{6.9}$$

The existence of such open neighborhoods follows because $N_1[G_n]$ knows that $\dot{X}$ is first countable, and Proposition 3.7 applied in $N_1[G_n]$ to the model $N_0[G_n]$ means that we can find a suitable name from N_0.

For each $i \in I$, let $\dot{V}_i$ be a P-name such that

$$N_0[\overline{G}] \models \dot{V}_i = \dot{V}_{i,0} \cap \cdots \cap \dot{V}_{i,i}. \tag{6.10}$$

The sequence $\{\dot{V}_i\colon\ i \in I\}$ is as desired, because for each $n \in \omega$, $\dot{V}_i[G_n]$ is a subset of $\dot{V}_{i,n}[G_n]$ for all but finitely many $i \in I$.

We are now ready to build our sequence $\{\dot{q}_n\colon\ n \in \omega\}$. Fix an enumeration $\{\dot{D}_n\colon\ n \in \omega\}$ of all names from N_0 for dense open subsets of $\dot{Q}$. In ω stages, we will define $\{\dot{q}_n\colon\ n \in \omega\}$ satisfying conditions (1)–(4), as well as an auxiliary function $g \in \omega^\omega$—at stage $n+1$ we define $\dot{q}_{n+1}$ and $g(n)$.

To start, we set $\dot{q}_0 = \dot{q}$. At stage $n+1$, we will be handed $\dot{q}_n$, $\dot{D}_n$, as well as a name $\dot{f} \in N_0^P$ for a promise appearing in $\Phi_{\dot{q}_i}$ for some $i \leqslant n$ that we must "take care of" (to be defined shortly) with respect to G_m for some $m \in \omega$.

Since

$$N_0[\overline{G}] \models \dot{q}_n \text{ extends } \dot{q}_i \text{ in } \dot{Q}, \tag{6.11}$$

there is an $\dot{f}' \in N_0^P$ such that $\dot{f}' \in \Phi_{\dot{q}_n}$ and

$$N_0[\overline{G}] \models \dot{f}' \quad \text{is the restriction of } \dot{f} \text{ to the set of } x \in \operatorname{dom} \dot{f}, \\ \text{such that } [\dot{q}_n] \setminus [\dot{q}_i] \subseteq \dot{f}(x). \tag{6.12}$$

Work for a moment in the model $N_1[G_m]$. Since G_m contains a lower bound for $\overline{G}$ (by Proposition 3.4), everything in the preceding paragraph remains true if we replace $\overline{G}$ by G_m. Since

$$N_1[G_m] \models \dot{y}_m \notin \operatorname{Ban}(\dot{f}'), \tag{6.13}$$

there is a $j \in \omega$ so that

$$N_1[G_m] \models \bigl\{x \in \operatorname{dom}(\dot{f}')\colon\ \dot{U}_{j,m} \subseteq \dot{f}'(x)\bigr\} \text{ is large.} \tag{6.14}$$

Thus we can define $g(n)$ large enough so that $g(n) > g(i)$ for $i < n$,

$$N_0[\overline{G}] \models \dot{x}_{g(n)} \notin \operatorname{Bad}(\dot{q}_n, \dot{D}_n), \tag{6.15}$$

and for all $k \geqslant g(n)$ in I,

$$N_1[G_m] \models \dot{V}_k \subseteq \dot{U}_{j,m}. \tag{6.16}$$

Once we do this, choose $\dot{q}_{n+1} \in N_0^P$ such that

$$N_0[\overline{G}] \models \dot{q}_{n+1} \leqslant \dot{q}_n,\ \dot{q}_{n+1} \in \dot{D}_n, \quad \text{and} \quad [\dot{q}_{n+1}] \setminus [\dot{q}_n] \subseteq \dot{V}_{g(n)}. \tag{6.17}$$

Clearly we can arrange that every promise that appears in some $\Phi_{\dot{q}_n}$ along the way gets handled with respect to each G_m at some stage in our construction.

Also, it is clear that $\{\dot{q}_n\colon n \in \omega\}$ is an $(N_0[\overline{G}], \dot{Q}, \dot{q})$-generic sequence.

Suppose $r \in P$ is such that

$$r \Vdash \exists n \in \omega \text{ such that } N_1 \cap \dot{G} = G_n, \tag{6.18}$$

and let G be any generic subset of P that contains r. It suffices to show that in $V[G]$, the sequence of conditions $\{\dot{q}_n[G]\colon n \in \omega\}$ has a lower bound. Working in the model $V[G]$, let $m \in \omega$ be such that $N_1 \cap \dot{G} = G_m$.

Let $n_0 + 1$ be the first place where we took care of some promise with respect to G_m. Our construction made sure that $g(n_0)$ was large enough so that if $i \in I \setminus g(n_0)$, then

$$N_1[G_m] \models \dot{V}_i \subseteq \dot{U}_{0,m}. \tag{6.19}$$

If $n \geqslant n_0$, then we made sure

$$N_0[\overline{G}] \models [\dot{q}_{n+1}] \setminus [\dot{q}_n] \subseteq \dot{V}_{g(n)}. \tag{6.20}$$

Since $\overline{G} \subseteq G_m \subseteq G$, we know that in $V[G]$,

$$\bigcup_{n \geqslant n_0} [\dot{q}_{n+1}] \setminus [\dot{q}_{n_0}] \subseteq \dot{U}_{0,m} \tag{6.21}$$

and thus

$$\operatorname{cl} \bigcup_{n \in \omega} [\dot{q}_n] \notin \dot{\mathcal{U}}. \tag{6.22}$$

Similarly, let $\dot{f}$ be a promise appearing in $\Phi_{\dot{q}_n}$ for some n. There is a stage $n_1 + 1$ where we took care of $\dot{f}$ with respect to G_m. At that stage, we found a $j \in \omega$ such that

$$N_1[G_m] \models \bigl\{x \in \dot{f}\colon \dot{U}_{j,m} \subseteq \dot{f}(x)\bigr\} \text{ is large}, \tag{6.23}$$

and chose $g(n_1)$ large enough so that for all $i \in I \setminus g(n+1)$,

$$N_1[G_m] \models \dot{V}_i \subseteq \dot{U}_{j,m}. \tag{6.24}$$

Since g is strictly increasing, this ensures that in $V[G]$ that

$$\bigcup_{n \geqslant n_1} [\dot{q}_{n+1}] \setminus [\dot{q}_{n_1}] \subseteq \dot{U}_{j,m}. \tag{6.25}$$

Thus an argument analogous to that used to prove that P_X is totally proper shows that $\{\dot{q}_n\colon n \in \omega\}$ has a lower bound in $V[G]$. Since G was an arbitrary generic subset of P that contains r, we have that

$$r \Vdash \{\dot{q}_n\colon n \in \omega\} \text{ has a lower bound in } \dot{Q}. \qquad \square \tag{6.26}$$

Now what does the preceding proposition have to do with the iteration theorems of [1] and [14]? If we replace the sequence $\{G_n\colon n \in \omega\}$ of members of $\operatorname{Gen}(N_1, P)$ by a pair $\{G_n\colon n < 2\}$, then we get a condition that appears implicitly in Shelah's proof of [14, XVII, Claim 4.10]—this is exactly the condition you need for his proof to work—and explicitly in the forthcoming paper [13]. (In the terminology of [13], our Proposition 6.2 is "medicine against the weak diamond".)

To connect things to the iteration theorem presented in [1] (see Theorem 3 below), we need the following definition.

Definition 6.3. Let P be totally proper and let $\dot{Q}$ be a P-name for a totally proper notion of forcing. We say $\dot{Q}$ is 2-complete for P if whenever
(1) $N_0 \in N_1 \in N_2$ are countable elementary submodels of $H(\lambda)$ with $P, \dot{Q} \in N_0$,
(2) $\overline{G} \in \mathrm{Gen}^+(N_0, P) \cap N_1$,
(3) $\dot{q} \in N_0$ is a P-name for a condition in $\dot{Q}$,
there is an $\overline{H} \in \mathrm{Gen}(N_0[\overline{G}], \dot{Q}, \dot{q})$ so that whenever r is a lower bound for $\overline{G}$ that is (N_i, P)-generic for $i = 1, 2$, we have

$$r \Vdash \overline{H} \text{ has a lower bound in } \dot{Q}.$$

Our formulation of Proposition 6.2 allows us to deduce the following corollary.

Corollary 6.4. *If P and $\dot{Q}$ are as in Proposition* 6.2*, then $\dot{Q}$ is* 2*-complete for P.*

Proof. Let $N_0 \in N_1 \in N_2$, $\overline{G} \in \mathrm{Gen}^+(N_0, P) \cap N_1$, and $\dot{q} \in N_0^P$ be as in the previous definition. Let $\{G_n\colon n \in \omega\}$ enumerate those members of $\mathrm{Gen}^+(N_1, P) \cap N_2$ that contain a lower bound for $\overline{G}$. Then Proposition 6.2 gives us a sequence

$$\{\dot{q}_n\colon n \in \omega\} \subseteq N_0^P \tag{6.27}$$

such that whenever r is a lower bound for $\overline{G}$ such that

$$r \Vdash \exists n \in \omega \text{ such that } N_1 \cap \dot{G} = G_n, \tag{6.28}$$

we have that r forces the sequence of $\dot{q}_n$'s to have a lower bound. If r is a lower bound for $\overline{G}$ that is (N_i, P)-generic for $i < 3$, then (6.28) holds by Proposition 3.6. We define

$$\overline{H} = \left\{\dot{s} \in N_0^P\colon \text{ for some } n < \omega,\ N_0[\overline{G}] \models \dot{q}_n \leqslant \dot{s}\right\} \tag{6.29}$$

and it is routine to verify that $\overline{H}$ witnesses that $\dot{Q}$ is 2-complete for P. □

We now quote the iteration theorem from [1]; this will tell us that the particular iteration of interest to us results in a totally proper notion of forcing.

Theorem 3. *Let $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha\colon \alpha < \kappa\rangle$ be a countable support iteration such that*
- *$\Vdash_\alpha \dot{Q}_\alpha$ is $< \omega_1$-proper,*
- *$\dot{Q}_\alpha$ is* 2*-complete for P_α.*

Then P_κ is totally proper.

Proof. See [1]. □

Armed with the results of the previous sections, we can now give a proof of our main theorem.

Theorem 4. *It is consistent with the Continuum Hypothesis that every first countable, countably compact, non-compact space contains an uncountable free sequence.*

Proof. Assume that GCH holds in the ground model, and construct a countable support iteration

$$\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha \colon \alpha < \omega_2 \rangle \tag{6.30}$$

so that for each $\alpha < \omega_2$,

$$\Vdash_\alpha \dot{Q} = P_{\dot{X}} \text{ for some relevant space } \dot{X}. \tag{6.31}$$

Since each iterand is $< \omega_1$-proper, and for $\alpha < \omega_2$

$$\dot{Q}_\alpha \text{ is 2-complete for } P_\alpha, \tag{6.32}$$

Theorem 3 guarantees that P_{ω_2} is totally proper, and so CH holds in $V[G_{\omega_2}]$. It is enough for us to deal only with spaces of size $\aleph_1$—if CH holds and X is a regular first countable, countably compact non-compact space with no uncountable free sequences, then X contains a closed separable subspace Y that enjoys the same properties (as our filter $\mathcal{U}$ has a base of separable sets). Since Y is Fréchet, we know $|Y| \leqslant 2^{\aleph_0} = \aleph_1$, and if we shoot an uncountable free sequence through Y then we have shot one through X as well.

Since CH holds in each $V[G_\alpha]$ we have that

$$\Vdash_\alpha \dot{Q}_\alpha \text{ is } \omega_1\text{-centered} \tag{6.33}$$

(as the space we are dealing with has size $\aleph_1$ and any two conditions with the same first component are compatible) and this is enough to guarantee that P_{ω_2} has the ω_2-chain condition. Since we need only be concerned only with spaces of size and weight of at most $\aleph_1$, standard arguments allow us to construct the iteration so that in $V[G_{\omega_2}]$, every first countable, countably compact, non-compact space has an uncountable free sequence.

7. Comments

The notion of forcing we use does not require X to be countably compact—all we need is the existence of a suitable filter of closed sets $\mathcal{U}$. However, it is not clear to me that we can iterate the forcing in this situation, because the diagonal argument of the Proposition 6.2 required that our space is sequentially compact.

Our consistency result also puts some limitations on the types of S-spaces that can be constructed from CH alone. The first countable S-spaces that exist under CH are obtained by refining a Lindelöf topology and so they are all realcompact, so a natural question is if CH implies the existence of a first countable S-space that is not realcompact.

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