1. Introduction

Let $S$ denote a stationary subset of limit ordinals of $\omega_1$. A ladder system on $S$ is a sequence $\{L_\alpha : \alpha \in S\}$ such that each $L_\alpha$ is an unbounded subset of $\alpha$ of order type $\omega$.

A ladder system is uniformizable if for each sequence $\langle f_\alpha : \alpha \in S \rangle$ of functions $f_\alpha : L_\alpha \to \omega$ there is an $F : \omega_1 \to \omega$ such that $F \upharpoonright L_\alpha =^* f_\alpha$ for each $\alpha \in S$. I.e., for each $\alpha \in S$,

$$\{\beta \in L_\alpha : F(\beta) \neq f_\alpha(\beta)\}$$

is finite.

We now formulate natural weakenings of uniformizable denoted, for each $n \in \omega$, by $P_n$: A ladder system is said to satisfy $P_n$ if for each $f : S \to \omega$ there is an $F : \omega_1 \to [\omega]^{n+1}$ such that for each $\alpha \in S$,

(a) $F \upharpoonright L_\alpha$ is eventually constant with value $s_\alpha$, and

(b) $f(\alpha) \in s_\alpha$.

Note that $P_0$ is equivalent to the version of uniformizable obtained by considering only sequences of constant functions $f_\alpha$.

We will say that a ladder system satisfies $P_{<\omega}$ if for each $f : S \to \omega$ there is an $F : \omega_1 \to [\omega]^{<\omega}$ satisfying (a) and (b) above.

If we drop the requirement that the restrictions $F \upharpoonright L_\alpha$ are eventually constant we obtain uniformization properties that we denote $M_n$ and $M_{<\omega}$. E.g., a ladder system is said to satisfy $M_{<\omega}$ if for each $f : S \to \omega$ there is an $F : \omega_1 \to [\omega]^{<\omega}$ such that for each $\alpha \in S$, $f(\alpha) \in F(\beta)$ for all but finitely many $\beta \in L_\alpha$.

Most of these uniformization properties can be characterized in terms of properties of a certain topological space naturally associated to any ladder system. If $L$ is a ladder system, let $X_L$ denote the topology space $\omega_1 \times \{0\} \cup S \times \{1\}$ where every point $(\alpha, 0)$ is isolated and for each $\alpha \in S$, a basic neighborhood of $(\alpha, 1)$ consists of $\{(\alpha, 1)\}$ along with a cofinite subset of $L_\alpha \times \{0\}$. Such a space is always first countable and locally compact. The stationarity of $S$ implies that it is not collectionwise Hausdorff.

Spaces $X_L$ have been considered by many to construct examples of normal not collectionwise Hausdorff spaces (see [11] and [2]. It is folklore that a ladder system $L$ satisfies $P_0$ if and only if $X_L$ is normal. The property $M_{<\omega}$ is characterized by $X_L$ being countably metacompact. For this reason, we will say that $L$ is countably metacompact in the case that it satisfies $M_{<\omega}$.

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Figure 1

Claim 1. Let $L$ be a ladder system on a stationary set $S$. $X_L$ is countably metacompact if and only if $L$ satisfies $\mathcal{M}_{<\omega}$.

Proof. $X_L$ is countably metacompact if and only if each partition of $S$ into countably many sets has an open expansion which is point finite on the isolated points. Suppose $f : S \rightarrow \omega$. By countable metacompactness, fix a point-finite open expansion $\{U_n : n \in \omega\}$. Let $F(\beta) = \{n : \beta \in U_n\}$. Clearly $F : \omega_1 \rightarrow [\omega]^{<\omega}$ is as required. Conversely, given a partition and a corresponding function $f$ fix the $F$ given by $\mathcal{M}_{<\omega}$. Define $U_n = f^{-1}(n) \cup \{\beta \in \omega_1 : n \in F(\beta)\}$. It is straightforward to check that $\{U_n : n \in \omega\}$ is the required point finite open expansion.

The additional conclusion in $\mathcal{P}_{<\omega}$ that $F \upharpoonright L_\alpha$ is eventually constant corresponds to the open expansion being locally finite, hence, $\mathcal{P}_{<\omega}$ implies that $X_L$ is countably paracompact. We don’t know whether it is equivalent.

Question 1. If $X_L$ is countably paracompact, does $L$ satisfy $\mathcal{P}_{<\omega}$?

The diagram of known ZFC implications between all the properties is summarized in Figure 1.
It should be remarked that $MA(\omega_1)$ implies that every ladder system is uniformizable and that $2^{\aleph_0} < 2^{\aleph_1}$ implies that no ladder system is uniformizable (see [6]). However, there is a ladder system on $\omega_1$ with the property $\mathcal{M}_{<\omega}$. Indeed any ladder system $L$ with the property that the $n^{th}$ element of each ladder $L_\alpha$ is of the form $\beta+n$ with $\beta$ a limit ordinal satisfies property $\mathcal{M}_{<\omega}$. Indeed let the function $F$ on $\omega_1$ be defined by $F(\alpha) = \{0,1,...n\}$ where $n$ is of the form $\beta+n$ for some limit ordinal $\beta$. Then $F$ uniformizes in the sense of $\mathcal{M}_{<\omega}$ every function $f$. However, it is not hard to see that $V=L$ implies that no ladder system on a stationary subset of $\omega_1$ can satisfy $\mathcal{P}_{<\omega}$ (see the remark following the proof of Theorem 24).

In Section 2 of this paper we prove that for each $n \in \omega$ it is consistent with CH that for every stationary set and every ladder $L$ on $S$, $L$ satisfies $\mathcal{P}_{n+1}$ but does not satisfy $\mathcal{M}_n$. Thus, there are not other ZFC implications between the properties $\mathcal{P}_n$ and $\mathcal{M}_m$ for any $m,n < \omega$. Moreover, by taking $n = 0$, we obtain that it is consistent with CH that every ladder system space $X_L$ is countably paracompact (in a strong sense) but not normal.

This leaves a few questions open, including the following:

**Question 2.** Is it consistent that all ladder systems satisfy $\mathcal{P}_{<\omega}$ but not $\mathcal{P}_n$ for any $n$?

The next set of properties of ladder systems we will consider are in some sense anti-uniformization properties. The following is the strongest of these. A ladder system satisfying this property will also be called *thin*.

$$(G_1)$$ For each $f : \omega_1 \to \omega$ the set $\{\alpha \in S : |\alpha''| \alpha = \aleph_0\}$ is nonstationary.

By strengthening what $f$ is allowed to do on a nonstationary set, we obtain the following weakenings of $(G_1)$

$$(G_2)$$ For each $f : \omega_1 \to \omega$ the set $\{\alpha \in S : f \upharpoonright L_\alpha$ is finite-to-one} is nonstationary.

$$(G_3)$$ For each $f : \omega_1 \to \omega$ the set $\{\alpha \in S : f \upharpoonright L_\alpha$ is eventually one-to-one} is nonstationary.

By instead demanding that every $f$ fails to have certain properties on a stationary set, we obtain even weaker properties:

$$(H_1)$$ For each $f : \omega_1 \to \omega$ the set $\{\alpha \in S : |\alpha''| \alpha < \aleph_0\}$ is stationary.

$$(H_2)$$ For each $f : \omega_1 \to \omega$ the set $\{\alpha \in S : f \upharpoonright L_\alpha$ is not finite-to-one} is stationary.

$$(H_3)$$ For each $f : \omega_1 \to \omega$ the set $\{\alpha \in S : f \upharpoonright L_\alpha$ is not eventually one-to-one} is stationary.

So, from the definitions we get the diagram of implications in Figure 2.

Note that if a ladder system $L$ is uniformizable, then it fails to satisfy $H_3$. Indeed, any $F$ uniformizing a sequence of one-to-one functions $\langle f_\alpha \rangle$ will witness the failure of $H_3$. However, some of the weaker versions of uniformizable can be consistent with some of the anti-uniformization properties. Indeed, in [11], Shelah proved it consistent that there is a ladder system $L$ on a stationary set such that $X_L$ is normal and that the closed discrete set of non-isolated points is not a $G_\delta$-set. Using our terminology, Shelah’s ladder system satisfies $\mathcal{P}_0$ and $H_2$. Burke and Balogh [2] proved it consistent that there is a ladder system defined on a club subset of $\omega_1$ satisfying $\mathcal{M}_{<\omega}$ and $H_2$. 

UNIFORMIZATION AND ANTI-UNIFORMIZATION PROPERTIES OF LADDER SYSTEMS 3
In Section 3 we give two proofs establishing the consistency of the existence of a $G_1$ ladder system. We also consider the $G_1$ property on ladders on stationary subsets of cardinals greater than $\omega_1$.

In Section 4 we consider the question whether every countably paracompact subspace of $\omega_1^2$ is normal. For a ladder $L$ on a stationary set $S$, we associate a non-normal subspace $Z_L \subseteq \omega_1^2$. The main result of that section is that $Z_L$ is countably paracompact if and only if $L$ has properties $G_1$ and $M_{<\omega}$. We also consider another closely related construction of a subspace of $\omega_1$.

Section 5 is devoted to the following open problem: Is there (consistently) a countably paracompact, locally compact screenable space which is not paracompact. It is shown the existence of such a space of cardinality $\omega_1$ is equivalent to the existence of a ladder system on some stationary set that is $G_1$ and $M_{<\omega}$. In addition the existence of an example of larger cardinality is characterized by the existence of a $G_1$ and $M_{<\omega}$ ladder system on $\omega_1$. In the final section we establish some results connecting $G_1$ to another known ladder system property.

2. CH and the uniformization properties

In this section we prove that for each $n \in \omega$ it is consistent with CH that every ladder system satisfies $P_{n+1}$ but fails to satisfy $M_n$. In the case $n = 1$ this implies the consistence of every ladder system space $X_L$ is countably paracompact in a strong sense but not normal. Our proof is based on a theorem of Shelah (Theorem 9 below). A proof of Shelah’s theorem has only been published for the case $n = 0$. The proof of Shelah’s theorem for all other $n \in \omega$ is essentially the same.
Let $\bar{C} = \{C_\delta : \delta \in S\}$ be a ladder system on a stationary set $S$ of countable limit ordinals, and let $f \in \omega^{\omega_1}$. We will define a notion of forcing $P = P_{f, \bar{C}}$ that adjoins a function $g : \omega_1 \rightarrow [\omega]^{n+2}$ such that for all $\delta \in S$,
- $g|C_\delta$ is eventually constant
- $f(\delta) \in g(\alpha)$ for all but finitely many $\alpha \in C_\delta$.

**Definition 2.** A condition $p \in P$ satisfies
- $p : \alpha + 1 \rightarrow [\omega]^{n+2}$ for some $\alpha < \omega_1$
- if $\delta \leq \alpha$ is in $S$, then $p|C_\delta$ is eventually constant, say with value $F_\delta$, and $p(\delta) \in F_\delta$.

**Lemma 3** (Extension Lemma). Let $p : \alpha + 1 \rightarrow [\omega]^{n+2}$ be a condition in $P$. Given $\beta > \alpha$ and $x \in [\omega]^{n+2}$, there is an extension $q$ of $p$ with $\text{dom}(q) = \beta + 1$ and $q(\beta) = x$.

**Proof.** We prove this by induction on $\beta$ for all sets $x \in [\omega]^{n+2}$. The case where $\beta$ is a successor ordinal is trivial. If $\beta$ is a limit ordinal, we let $\{\beta_n : n < \omega\}$ list $C_\beta \setminus \alpha + 1$ in increasing order. We apply our induction hypothesis repeatedly to obtain a sequence $\{p_n : n < \omega\}$ such that
- $p_0 \leq p$
- $p_{n+1} \leq p_n$
- $\text{dom} p_n = \beta_n + 1$
- $p_n(\beta_n) = \{f(\beta), \ldots, f(\beta + n)\}$.

Once this is done, we define
$$ q = \bigcup_{n \in \omega} p_n \cup \{\langle \beta, x \rangle\}. $$

**Corollary 4.** Given $p : \alpha + 1 \rightarrow [\omega]^{n+2}$ in $P$, a finite set $A$ of ordinals in $\omega_1 \setminus (\alpha + 1)$, and a set $x \in [\omega]^{n+2}$, there is $q \leq p$ in $P$ such that $\text{dom} q = \text{max}(A) + 1$ and $q(\beta) = x$ for all $\beta \in A$.

**Proposition 5.** Let $N \prec H(\lambda)$ be countable with $\{f, \bar{C}, P\} \in N$. Let $A \subseteq \delta = N \cap \omega_1$ be any set of order–type $\omega$ cofinal in $\delta$, and let $x \in [\omega]^{n+2}$. Given $p \in N \cap P$, there is an $(N, P)$–generic sequence $\{p_n : n \in \omega\}$ below $p$ such that the function defined by $\cup\{p_n : n \in \omega\}$ is eventually constant with value $x$ on $A$.

**Proof.** We can assume that there is a sequence $\langle N_i : i < \omega \rangle$ of countable elementary submodels of $H(\lambda)$ such that
- $N_i \in N_{i+1}$
- $N = \bigcup_{n \in \omega} N_i$
- $\{f, \bar{C}, P, p\} \in N_0$

Let $\{D_\alpha : \alpha \in \omega\}$ list the dense open subsets of $P$ that are elements of $N$, and let $\delta_i = N_i \cap \omega_1$ for $i < \omega$. By enlarging $A$, we may assume that $\{\delta_i : i < \omega\} \subseteq A$ — this will not affect the order–type of $A$ as the $\delta_i$’s are cofinal in $\delta$. We will define $\{p_n : n \in \omega\}$ by induction on $n$ so that
- $\alpha \in A \cap \text{dom} p_{n+1} \setminus \text{dom} p_n$, then $p_{n+1}(\alpha) = x$
Given \( p_n \), we first choose \( i \) large enough that \( \{ p_n, D_n \} \in N_i \). Since \( A \cap N_i \) is finite, we can apply Corollary 4 inside \( N_i \) with \( p_n \) and \( A \cap N_i \) in place of \( p \) and \( A \) to obtain a condition which we shall denote \( q_n \). Now inside \( N_i \), we extend \( q_n \) to \( p_{n+1} \in D_n \). Clearly \( p_{n+1} \) has all the properties required of it, as does the sequence \( \{ p_n : n \in \omega \} \).

**Corollary 6.** \( P \) is totally proper. More generally, if \( N \prec H(\lambda) \) is countable with \( \{ C, P, f \} \in N, p \in N \cap P, \) and \( x \in [\omega]^{n+2} \), then there is a totally generic \( q \leq p \) such that \( \text{dom} q = \delta + 1 \) (where \( \delta = N \cap \omega_1 \)) and \( q(\delta) = x \).

**Proof.** We apply Proposition 5 with \( C_\delta \) in place of \( A \), and \( \{ f(\delta), ..., f(\delta) + n + 1 \} \) in place of \( x \). The sequence \( \{ p_n : n < \omega \} \) will have a lower bound \( q \) with domain \( \delta + 1 \), so we can define \( q(\delta) = x \) as required.

The following corollary is the place where it is crucial that forcing conditions map into the set of pairs of natural numbers instead of into \( \omega \) itself.

**Corollary 7.** There is a simple \( n + 2 \)-completeness system \( \mathbb{D} \) such that \( P \) is \( \mathbb{D} \)-complete.

**Proof.** Recall that a completeness system is called simple if there is a first order formula \( \psi \) such that

\[
\text{D}(N, P, p) = \{ A_x : x \text{ a finitary relation on } N \},
\]

where

\[
A_x = \{ G \in \text{Gen}(N, P) : (N \cup \mathcal{P}(N), \in) \models \psi(G, x, p, P, N) \}.
\]

In our case, the formula \( \psi \) says that “if \( x \) is a pair \( \langle y, z \rangle \) such that \( y \) is an \( \omega \)-sequence cofinal in \( N \cap \omega_1 \) and \( z \in \omega \), then \( \bigcup G \upharpoonright y \) is eventually constant and \( z \) is an element of this limit value”.

Note that \( P \) is \( \mathbb{D} \)-complete, as if \( x = \langle C_\delta, f(\delta) \rangle \), then any member of \( A_x \) has a lower bound. Thus we need only show that \( \mathbb{D} \) is an \( n + 2 \)-completeness system. To see this, let \( \{ x_i : i < n + 2 \} \) be a set of \( n + 2 \) finitary relations on \( N \); we must show that \( \bigcap_{i<n+2} A_{x_i} \) is non-empty.

The non–trivial case is where all \( x_i \) and \( x_1 \) satisfy the hypothesis of the implication in the formula \( \psi \). Let \( x_i = \langle y_i, z_i \rangle \). Let \( A = \bigcup_{i<n+2} y_i \). We apply Proposition 5 to this \( A \) with \( \{ z_0, ..., z_n + 1 \} \) in place of \( x \). The sequence \( \{ q_n : n \in \omega \} \) that the conclusion gives us then generates a member of \( A_{x_i} \) for every \( i < n + 2 \).

**Proposition 8.** \( P \) is \( < \omega_1 \)-proper.

**Proof.** That \( P \) is \( \alpha \)-proper for every \( \alpha \) follows by induction on \( \alpha \) using Corollary 6.

**Ensuring the failure of \( \mathcal{M}_\alpha \):** We need the following result of Shelah (see [10] Chapter VIII Claim 4.10 for the the proof in the case that \( n = 0 \); the proof of the general case is similar).

**Theorem 9.** Let \( \mathbb{P} = \langle P_\alpha, Q_\alpha : \alpha < \alpha_0 \rangle \) be an iteration with countable support such that each \( Q_\alpha \) is \( < \omega_1 \)-proper and \( \mathbb{D}_\alpha \)-complete for some simple \( n + 2 \)-completeness system \( \mathbb{D}_\alpha \). Suppose that \( \langle N_i : i \leq \beta \rangle \) is a countable increasing continuous sequence of countable models such that

- \( \langle N_j : j \leq i \rangle \in N_{i+1} \)
- \( \xi \leq \zeta \in N_0 \cap \alpha_0 + 1 \)
Suppose further that \((N_i \cap \alpha_0 : i \leq \beta)\) is long for \((\xi, \zeta), (G_i : i < n + 2)\) are directed subsets of \(P_i \cap N_\beta\). \(r_i \in P_i\) is a lower bound for \(G_i\) for each \(i < n + 2\), \(G_i \cap N_0 = G_j \cap N_0\) for all \(i, j < n + 2\), and for all \(\eta \leq \beta, G_i \cap N_\eta \cap \text{Gen}(N_\eta, P_i)\).

Then there is a directed \(G^* \subseteq P_i \cap N_0\) such that \(G_0 \cap N_0 \subseteq G^*, G^* \in \text{Gen}(N_0, P_i, P)\), and

\[
\tag{4}
r_i \Vdash_{P_i} \langle q | [\xi, \zeta] : q \in G^* \rangle \text{ has a lower bound in } P_i / P_i
\]

for all \(i < n + 2\).

Our iteration \(P\) is a countable support iteration where at even stages we force with \((<\omega, \omega_2)\) and at odd stages we force with \(P_{j,C}\) for some \(f\) and \(C\). Clearly a bookkeeping argument will take care that all ladders satisfy \(\mathcal{P}_{n+1}\) in the extension so the crux of the matter is to prove that none of the ladder system spaces satisfy \(\mathcal{M}_n\).

Suppose that \(p \in P_{\omega_2}\) forces that the ladder system space built from \((\hat{S}, \hat{C})\) is \(\mathcal{M}_n\). We may assume that \((\hat{S}, \hat{C})\) are in the ground model as we can find some limit stage \(\alpha < \omega_2\) with \((S, C) \in V[G_\alpha]\). The first thing we do is force with \((<\omega, \omega, \bar{2})\).

This adjoins a function \(f : \omega_1 \rightarrow \omega\). Let \(\hat{g}\) be a \(P_{\omega_2}\) name such that \(p \Vdash \hat{g} : \omega_1 \rightarrow [\omega]^n + 1\) uniformizes \(\hat{f}\) in the sense of \(\mathcal{M}_n\).

Let \(\langle N_i : i \leq \beta \rangle\) be a sequence of models as in the assumptions of Theorem 9 with \(\xi = 1\) and \(\zeta = \omega_2\), and with \(\hat{g}, f\) all in \(N_0\).

Let \(\delta = N_0 \cap \omega_1\), and let \(G_i\) be chosen as in the assumptions of Theorem 9 so that \(\cup G_i(\bar{\delta}) = i\) for each \(i < n + 2\). This is easy as \((<\omega, \omega, \bar{2})\) is countably complete.

Let \(G^*\) be as in the conclusion of the theorem. From \(G^*\), we can decode the values of \(\hat{g}(\gamma)\) for all \(\gamma < \delta\). Since \(p \in G^*\), we know that \(\hat{g}\) must uniformize \(f\).

Consider the decided sequence of values \(\langle g(\gamma) : \gamma \in \mathcal{C}_\delta \rangle\). These sets are of size \(n + 1\), so there are at most \(n + 1\) values \(k\) such that \(k \in g(\gamma)\) for all but finitely many \(\gamma \in \mathcal{C}_\delta\). This means that from \(G^*\) we can decode the value of \(f(\delta)\) up to a set of \(n + 1\) possible values. Take \(i < n + 2\) such that \(i\) is not one of these values. This is a contradiction, as \(r_i \Vdash f(\delta) = i\) and \(r_i\) can be extended to a lower bound for \(G^*\).

3. Consistency of \(G_1\).

In this section we give two proofs of the consistency of the existence of \(G_1\) ladder systems. The first proof is from \(V = L\), more specifically from Devlin’s \(\Diamond^#\), (see [5]). For our purposes we will say that a sequence \(\{N_\alpha : \alpha \in \omega_1\}\) is a \(\Diamond^#\)-sequence, if

1. Each \(N_\alpha\) is a countable transitive model of a suitable portion of ZFC,
2. \(\{N_\alpha \cap P(\alpha) : \alpha \in \omega_1\}\) forms a \(\Diamond^+\) sequence, and
3. \(\{\alpha : \alpha = (\omega_1)^{N_\alpha}\}\) is stationary.

For a more precise formulation of \(\Diamond^#\) see [5].

**Theorem 10.** \(\Diamond^#\) implies the existence of a thin ladder system on a stationary subset of \(\omega_1\).

**Remark:** Kunen has shown that \(V = L\) implies the existence of a \(G_1\) ladder system defined on all of \(\omega_1\).
Proof. Fix \( \{N_\alpha : \alpha \in \omega_1\} \) a \( \diamondsuit^\# \)-sequence and let \( S = \{\alpha : \alpha = (\omega_1)^{N_\alpha}\} \). So \( S \) is stationary. We define the ladder system on \( S \). Fix \( \alpha \in S \) and let \( \{f_k : k \in \omega\} \) be an enumeration of \( N_\alpha \cap \omega_1 \). Define \( L_\alpha \) recursively: Fix \( n_0 \) minimal such that

\[
N_\alpha \models "f_0^{-1}(n_0) \text{ is uncountable}".
\]

We can do this since \( N_\alpha \) “thinks” that \( \alpha \) is \( \omega_1 \) and hence that \( f_0 \) is a function from \( \omega_1 \) to \( \omega \). Let \( I_0 = f_0^{-1}(n_0) \). Having defined \( I_k \) so that \( N_\alpha \models "I_k \text{ is uncountable}" \), fix \( n_{k+1} \) so that

\[
N_\alpha \models "I_k \cap f_0^{-1}(n_{k+1}) \text{ is uncountable}".
\]

So \( I_0 \supseteq I_1 \supseteq \ldots \supseteq I_k \supseteq \ldots \) and each \( I_k \) is unbounded in \( \alpha \). Choose \( L_\alpha \) an increasing cofinal \( \omega \)-sequence in \( \alpha \) such that \( L_\alpha \subseteq^+ I_k \) for each \( k \in \omega \). Clearly \( f_k \upharpoonright L_\alpha \) is eventually constant for each \( k \in \omega \).

To see that \( L = \{L_\alpha : \alpha \in S\} \) satisfies \( G_1 \), fix \( f : \omega_1 \rightarrow \omega \) arbitrary. By the property of being a \( \diamondsuit^\# \) sequence, \( C = \{\alpha : f \upharpoonright \alpha \in N_\alpha\} \) is club. Clearly, by construction, \( f \upharpoonright L_\alpha \) is eventually constant for any \( \alpha \in S \cap C \).

Our next proof is a forcing construction of a ladder system satisfying \( G_1 \). We obtain the model by collapsing a Mahlo cardinal to \( \omega_1 \). First we define a single forcing that collapses \( \omega_1 \). Let \( Q \) be the set of triples \((p,F,r)\) such that

1. \( p \) is a function from some \( n \in \omega \) to \( \omega_1 \) such that \( n < m < \text{dom}(p) \) implies \( p(n) < p(m) \).
2. \( F \) is a finite subset of \( \omega^1 \omega \).
3. \( r : F \rightarrow \omega \).
4. \( \{\alpha \in \omega_1 : \forall f \in F(f(\alpha) = r(f))\} \) is uncountable.

We order \( Q \) by \( (p,F,r) \leq (q,G,s) \) if

1. \( p \) extends \( q \), \( F \supseteq G \), \( r \) extends \( s \), and
2. \( p(n) \in g^{-1}(s(g)) \) for each \( n \in \text{dom}(p) \setminus \text{dom}(q) \) and each \( g \in G \).

Since \( |Q| = 2^{\aleph_1} \) we have that

**Lemma 11.** \( Q \) is \( (2^{\aleph_1})^+ \)-cc.

This cannot be improved: Given \( \{f_\alpha : \alpha < 2^{\aleph_1}\} \subseteq \omega^1 \omega \) such that \( \{f_\alpha^{-1}(0) : \alpha < 2^{\aleph_1}\} \) forms an almost disjoint family, the conditions \((0,\{f_\alpha\},\{f_0,0\})\) are pairwise incompatible (any common extension of two such conditions would violate (4)).

Let \( g \) be a \( Q \)-name for the generic function \( \bigcup \{p : \exists F, r : (p,F,r) \in Q\} \) and let \( L \) be a \( Q \)-name for the range of \( g \). Then \( L \) is an \( \omega \)-sequence cofinal in \( (\omega_1)^V \), hence \( \omega_1 \) is collapsed by \( Q \).

It is easy to verify that for each \( f \in \omega^1 \omega \), the set \( D_f = \{(p,F,r) : (p,F,r) \in Q : f \in F\} \) is dense in \( Q \). Thus, the following lemma holds:

**Lemma 12.** For any \( f \in \omega^1 \omega \), \( \models_Q "f \text{ is eventually constant on } L." \)

Let \( \kappa \) be Mahlo. So, the set of inaccessible ordinals less than \( \kappa \) is stationary in \( \kappa \). Let \( C = \{\kappa_i : i \in \kappa\} \) be an increasing enumeration of all cardinals in \( \kappa \). So \( C \) is club. And let \( S \) be the set of inaccessibles. We fix an iteration \( (P_i,Q_i : i \in \kappa) \) as follows. For each \( i \in \kappa \) let \( Q_i \) be defined recursively as follows.

1. If \( i \) is a successor and \( \models_p "\kappa_i \text{ is uncountable}" \), let \( Q_i \) be \( \text{Fn} (\omega, \kappa_i) \). So \( Q_i \) collapses \( \kappa_i \) to \( \omega \).
2. If \( \kappa_i \) is inaccessible, let \( Q_i \) be a \( P_i \) name for \( Q \).
3. Otherwise let \( Q \) be the trivial poset.
Let $P_\kappa$ be the finite support iteration of the $Q_i$'s (countable also works). It easily follows that $P_\kappa$ has the $\kappa$-cc. It is also follows that for each inaccessible $\kappa_i$, $P_\kappa$ is $\kappa_i$-cc and collapses all $\kappa_j$ for $j < i$ to countable ordinals. Hence $\Vdash_{P_\kappa} \kappa_i = \omega_1$. So $Q_i$ adds an $\omega$-sequence cofinal in $\kappa_i$. For each $\delta \in S$ let $L_\delta$ be the $P_{\delta+1}$-name for the ladder added by $Q_\delta$. We work with the ladder system $L = \{L_\delta : \delta \in S\}$.

Given any $P_\kappa$-name $f$ for a function $f : \kappa \rightarrow \omega$, by $\kappa$-cc we have that the set of $\delta$ for which there is a $P_\delta$ name $f_\delta$ such that $\Vdash_{P_\kappa} f_\delta = f \upharpoonright \delta$ is club. Thus, by the lemma above, for any $P_\kappa$-name $f$ for a function $\kappa \rightarrow \omega$, the set of $\delta$ for which $\Vdash_{P_\kappa} \text{ "} \tau \upharpoonright \delta \text{ is eventually constant" is club on } S$.

Finally, by $\kappa$-cc it follows that $S$ remains stationary in $V^{P_\kappa}$. Thus $G_1$ holds in $V^{P_\kappa}$.

Thus the existence of $G_1$ ladders is consistent. However, such ladder systems are very unstable. If $P$ is a finite support iteration of length $\omega_1$, then $P$ adds a function $g : \omega_1 \rightarrow \omega$ that is $\text{Fn}(\omega_1, \omega, < \omega)$-generic over the universe. For a ladder system $L$ in the ground model, this $g$ has the property that $g \upharpoonright L_\alpha$ has infinite range for every $\alpha \in \omega_1$. Thus, the ladder $L$ fails to have property $H_1$ in the extension. On the other hand, if $P$ is a countable support iteration of length at least $\omega_1$, the $P$ adds a function $g$ that is $\text{Fn}(\omega_1, \omega, < \omega)$-generic over the universe. This $g$ has the property that $g \upharpoonright L_\alpha$ is eventually one-to-one for stationary many $\alpha$. Thus, the ladder $L$ fails to have property $G_3$ in the extension.

Thus we have the following

**Theorem 13.** Suppose that $L$ is a ladder system and that $P$ is a finite or countable support iteration of length at least $\omega_1$. Then $\Vdash_{P} L$ does not satisfy $G_1$.

Next we consider ladder systems on stationary subsets of cardinals $\kappa > \omega_1$. For these cardinals, relatively weak assumptions imply no such ladder is $G_1$.

**Theorem 14.** Let $\kappa$ be a regular cardinal. Suppose there is a cardinal $\lambda$ such that $\lambda^\omega < \kappa \leq 2^\lambda$. Then there is no thin ladder system on any stationary subset $S$ of $\{\alpha < \kappa : \text{cf}(\alpha) = \omega\}$.

**Proof.** Let $L = \{L_\alpha : \alpha \in S\}$ be a ladder system. Identify $\kappa$ with a subset of the space $2^\omega$, and let $B$ be any base for $2^\omega$ of cardinality $\lambda$. Given any countable subset $A = \{a_n\}_{n \in \omega}$ of $\kappa$, find $B_n \in B$ with $a_n \in B_n$ and $B_n \cap \{a_i : i < n\} = \emptyset$. Then

$$P = \{\kappa \cap B_n \cup \bigcup_{i < n} B_i : n \in \omega\} \cup \{\kappa \setminus \bigcup_{i \in \omega} B_i\}$$

is a partition of $\kappa$ each element of which contains at most one member of $A$. Since the hypothesis implies $\kappa > \omega$, we see that there is a set $F$ of $\text{cf} \cdot \text{c}$-many, in particular less than $\kappa$-many, functions from $\kappa$ into $\omega$ such that every countable function from $\kappa$ into $\omega$ is extended by some function from $F$.

Now, for each $\alpha \in S$, there is $f_\alpha \in S$ such that the range of $f_\alpha \upharpoonright L_\alpha$ is unbounded in $\omega$. Since $|F| < \kappa$ and $\kappa$ is regular, there is an $f \in F$ and a stationary $S' \subset S$ such that $f_\alpha = f$ for each $\alpha \in S'$. I.e., the coloring $f$ is unbounded on a stationary set of ladders, so $L$ does not satisfy $G_1$. \hfill \Box

**Corollary 15.**  

(1) Assume the Continuum Hypothesis. Then there is no thin ladder system on any stationary subset of $\{\alpha < \omega_2 : \text{cf}(\alpha) = \omega\}$.

(2) Assume the Singular Cardinal Hypothesis. If $\kappa > \omega$ is regular, and not strongly inaccessible or the successor of a singular strong limit of countable cofinality, then there is no $G_1$ ladder system on any stationary subset of \{\(\alpha < \kappa : cf(\alpha) = \omega\)\}.

Proof. For (1), CH implies that the hypothesis of the theorem is satisfied with $\kappa = \omega_2$ and $\lambda = \omega_1$. For (2), it is not difficult to show that under the Singular Cardinal Hypothesis, the hypothesis on $\kappa$ in (2) is equivalent to the hypothesis on $\kappa$ in the theorem. \(\dashv\)

4. Subspaces of $\omega_1^2$

Recently N. Kemoto and others have been systematically studying separation properties of products of ordinals and their subspaces. One of the more interesting questions left open by this investigation is the following:

**Question 3.** Is every countably paracompact subspace of $\omega_1^2$ normal?

In [8] the following characterization theorem was proven.

**Theorem 16.** For each $X \subseteq \omega_1^2$, $X$ is normal if and only if $X$ is countably paracompact and strongly collectionwise Hausdorff.

Hence every normal subspace of $\omega_1^2$ is countably paracompact. In addition it was shown in the same paper that $\omega_1^2$ is hereditarily collectionwise Hausdorff. Therefore any countably paracompact non-normal subspace of $\omega_1^2$ would be an example of a countably paracompact first countable collectionwise Hausdorff space. In any model of ZFC where first countable countably paracompact spaces are strongly collectionwise Hausdorff, we have that a positive answer to Question 3. Although it is an open question whether $V=L$ implies that countably paracompact first countable spaces are strongly collectionwise Hausdorff (see [9]) it was shown in [8] that Question 3 still has a positive answer assuming $V=L$.

In this section we consider a natural non-normal subspace of $\omega_1^2$ constructed from a ladder system on a stationary subset of $\omega_1$ and prove that this space is countably paracompact if and only if the ladder system has properties $G_1$ and $M_{<\omega}$. We are left with the question whether it is consistent to assume the existence of a ladder system which has properties $M_{<\omega}$ and $G_1$. See Section 6 for further discussion of this question.

Fix a ladder system $L$ on a stationary $S \subseteq \omega_1$ with the property that each $L_\alpha$ consists of successor ordinals. We denote the following subspace of $\omega_1^2$ by $Z_L$:

\[
\{(\alpha, \alpha + 1) : \alpha \in S\} \cup \{(\beta, \gamma) : \exists \alpha \in S, \beta \in L_\alpha \text{ and } \gamma = \alpha + 1 \text{ or } \gamma > \alpha \text{ is a limit}\}.
\]

Stationarity of $S$ easily implies that $Z_L$ is not strongly collectionwise Hausdorff. Notice that by assumption on the ladder system, for each $(\beta, \gamma) \in Z_L$, if $\beta \notin S$ then $\beta$ is a successor.

**Theorem 17.** $Z_L$ is countably paracompact if and only if $L$ has properties $G_1$ and $M_{<\omega}$.
Proof. Assume that \( L \) has properties \( G_1 \) and \( \mathcal{M}_{<\omega} \). Fix a decreasing sequence \( (D_n)_{n\in\omega} \) of closed subsets of \( Z_L \) such that \( \bigcap D_n = \emptyset \). Let \( E_n = \{ \alpha \in S : (\alpha, \alpha + 1) \in D_n \} \) and define \( f_0 : S \to \omega \) by \( f_0(\alpha) = \max\{ n : \alpha \in E_n \} \). Fix a corresponding \( g_0 : \omega_1 \to [\omega]^{<\omega_0} \) uniformizing \( f_0 \) in the sense of property \( \mathcal{M}_{<\omega} \). Let

\[
W'_n = \{(\beta, \alpha + 1) : \beta \in L_\alpha, f_0(\alpha) = n \text{ and } n \in g_0(\beta)\}.
\]

And let

\[
W_n = \{(\alpha, \alpha + 1) : \alpha \in S \} \cap D_n \cup \bigcup_{i \geq n} W'_i.
\]

Notice that \( W_n \) is an open neighborhood of the sets \( D_n \cap \{(\alpha, \alpha + 1) : \alpha \in S\} \).

To define neighborhoods at the rest of the points of \( D_n \) first note that since \( \omega_1 \) is countably compact, for each \( \beta \) a successor, there is an \( n_\beta \) such that \( \{\beta\} \times \omega_1 \cap D_n = \emptyset \) for each \( n \geq n_\beta \). Letting \( f_1(\beta) = n_\beta \) and by applying \( G_1 \), we may fix a a club \( C \) consisting of limit ordinals such that \( f_1^n L_\alpha \) is finite for each \( \alpha \in C \cap S \).

For each \( \gamma \in C \) let \( \gamma^+ \) be the minimal element of \( C \) above \( \gamma \). For each \( \gamma \in C \) the space \( Z_L \cap (\gamma, \gamma^+]^2 \) is a clopen metrizable subspace of \( Z_L \). Thus, there are open sets \( O_\alpha \supseteq D_n \cap (\gamma, \gamma^+]^2 \) such that

\[
\bigcap_{n \in \omega} \overline{O_\alpha} = \emptyset.
\]

For each \( (\beta, \eta) \in Z_L \cap D_n \) for which \( \beta \not\in S \) fix an open set

\[
U_\alpha(\beta, \eta) \subseteq \{\beta\} \times (0, \eta] \cap Z_L
\]

so that

(a) if \( (\beta, \eta) \in (\gamma, \gamma^+]^2 \) for some \( \gamma \in C \) then \( U_\alpha(\beta, \eta) \subseteq O_\alpha \cap (\gamma, \gamma^+]^2 \).

(b) if \( \beta \in (\gamma, \gamma^+] \) for some \( \gamma \in C \) and \( \eta > \gamma^+ \) then \( U_\alpha(\beta, \eta) \subseteq \{\beta\} \times (\gamma^+, \eta] \).

Now let \( O_n = \bigcup\{ U_\alpha(\beta, \eta) : (\beta, \eta) \in D_n, \ \beta \not\in S \} \) and let

\[
G_n = W_n \cup O_n.
\]

Claim 18. \( \bigcap_{n \in \omega} \overline{G_n} = \emptyset. \)

Proof. First fix \( \beta \not\in S \). Fix \( n \) large enough so that for every \( m \geq n \) both \( \{\beta\} \times \omega_1 \cap D_m = \emptyset \) and \( m \not\in g_0(\beta) \). Then \( \{\beta\} \times \omega_1 \cap G_m = \emptyset \) for each \( m \geq n \).

Next, fix \( \alpha \in S \) and consider the point \( (\alpha, \alpha + 1) \). For each \( m \not\in f_0(\alpha) \) it is clear that \( \alpha + 1 \times \{\alpha + 1\} \cap W'_m = \emptyset \). So

\[
(L_\alpha \cup \{\alpha\}) \times \{\alpha + 1\} \cap W_m = \emptyset
\]

for each \( m > f_0(\alpha) \). Thus for \( m > f_0(\alpha) \), if \( (\alpha, \alpha + 1) \in \overline{G_m} \) then \( (\alpha, \alpha + 1) \in \overline{m} \).

Now consider two cases:

Case 1: \( \alpha \in C \). In this case, \( f_1^n L_\alpha \) is finite, so we can fix \( n_1 \in \omega \) such that \( \{\beta\} \times \omega_1 \cap D_m = \emptyset \) for all \( \beta \in L_\alpha \) and all \( m \geq n_1 \). Thus

\[
(L_\alpha \times \omega_1) \cap O_m = \emptyset \text{ for each } m \geq n_1.
\]

Case 2: \( \alpha \not\in C \). In this case, fix \( \gamma \in C \) such that \( \gamma < \alpha < \gamma^+ \). Fix \( n_1 \) such that \( (\alpha, \alpha + 1) \not\in \overline{O_m} \) for each \( m \geq n_1 \). Thus, by choice of the open sets \( U_\alpha(\beta, \eta) \) for those \( \beta < \gamma \) and by choice of the open sets \( U_\alpha(\beta, \eta) \) for those \( \eta > \gamma^+ \) we have \( (\alpha, \alpha + 1) \not\in \overline{O_m} \) for all \( m \geq n_1 \).

In either case, \( (\alpha, \alpha + 1) \not\in \overline{G_m} \) for all \( m \geq n_1 \) so \( Z_L \) is countably paracompact.

For the converse, first suppose that that \( L \) does not satisfy the weak uniformizability property \( \mathcal{M}_{<\omega} \) and fix a partition \( f : S \to \omega \) such that for any \( g : \omega_1 \to \mathcal{P}(\omega) \)
if for each \( \alpha \in S \) \( f(\alpha) \in g(\beta) \) for all but finitely many \( \beta \in L_\alpha \), then there is a \( \beta \) such that \( g(\beta) \) is infinite. Consider the closed sets \( E_n = \{(\alpha, \alpha + 1) : f(\alpha) = n \} \). Fix open sets \( W_n \supseteq E_n \). For each \( \beta \notin S \) let \( g(\beta) = \{n : (\beta, \alpha + 1) \in W_n \text{ for some } \alpha \in f^{-1}(n)\} \). Each \( W_n \) is open so \( f(\alpha) \in g(\beta) \) for all but finitely many \( \beta \in L_\alpha \). Thus we may fix \( \beta \) such that \( g(\beta) \) is infinite. Fix \( \alpha_n \in f^{-1}(n) \) for each \( n \in g(\beta) \). Let \( \gamma \) be a limit of the set \( \{\alpha_n : n \in g(\beta)\} \). Then the family \( \{W_n : n \in \omega\} \) is not locally finite at \( (\beta, \gamma) \). So if \( L \) does not satisfy \( M_{<\omega} \), then \( Z_L \) is not countably paracompact.

On the other hand, assume that \( L \) doesn’t satisfy \( G_1 \) and fix a function \( f : \omega_1 \to \omega \) such that \( T = \{\alpha : |f'|^{\omega_1}| = 80\} \) is stationary. Let

\[
D_n = \{(\beta, \gamma) \in Z_L : f(\beta) \geq n, \beta < \gamma \text{ and } \gamma \text{ is a limit}\}.
\]

Notice that each \( D_n \) is closed and that \( \bigcap_{n \in \omega} D_n = \emptyset \). Suppose that for each \( n \), \( U_n \) is an open set containing \( D_n \) such that \( U_0 \supseteq U_1 \supseteq \ldots \). Fix a countable elementary submodel \( \mathcal{M} \) such that everything relevant, e.g., \( T \), \( \{D_n : n \in \omega\} \), \( f \), \( \{U_n : n \in \omega\} \), lies in \( \mathcal{M} \). Then \( \mathcal{M} \cap \omega_1 \in T \). Clearly, the following claim will complete the proof.

**Claim 19.** Let \( \alpha_0 = \mathcal{M} \cap \omega_1 \). Then \( (\alpha_0, \alpha_0 + 1) \in \bigcap \{U_n : n \in \omega\} \).

**Proof.** For each \( \beta \in L_{\alpha_0} \), let \( Z_\beta = \{(\beta, \gamma) : \gamma > \beta \text{ is a limit}\} \cap Z_L \). Notice that \( Z_\beta \subseteq D_{f(\beta)} \subseteq U_{f(\beta)} \). By the pressing down lemma, there is a \( \gamma_\beta \) such that \( \{\beta \times \omega_1 \setminus \gamma_\beta\} \cap Z_\beta \subseteq U_\beta \) for each \( \beta \not\in f(\beta) \). Since all objects under consideration are in \( \mathcal{M} \) we may assume that \( \gamma_\beta \in \mathcal{M} \) for each \( \beta \in L_{\alpha_0} \). Now fix \( n \in \omega \) arbitrary and fix \( \eta < \alpha_0 \). Fix \( \beta \in L_{\alpha_0} \) such that \( \beta > \eta \) and such that \( f(\beta) \geq n \). This can be done since \( \alpha_0 \in T \). Thus, \( (\beta, \alpha + 1) \in U_{f(\beta)} \cap (\eta, \alpha_0) \times (\alpha + 1) \). Thus, \( (\alpha_0, \alpha_0 + 1) \in U_n \) for every \( n \in \omega \). Thus \( Z_L \) is not countably paracompact.

We now present an alternate construction of a subspace of \( \omega_1^2 \) that is a natural candidate for an anti-dowker subspace. Let LIM and SUCC denote respectively the set of limit ordinals in \( \omega_1 \) and the set of successor ordinals in \( \omega_1 \). Let \( S \subseteq \text{LIM} \) be stationary and let \( B \subseteq S \) be such that \( B \) is discrete in itself (hence \( B \) is nonstationary). Let \( L \) be a ladder system on \( S \). Without loss of generality, suppose that for \( \alpha < \beta \in B \), all ordinals of \( I_\alpha \) are below all ordinals of \( I_\beta \). Also, assume that \( \bigcup L \subseteq \text{SUCC} \). \( H_L \) denotes the following subspace of \( \omega_1^2 \):

\[
\{(\beta, \alpha + 1) : \alpha \in S \setminus B \text{ and } \beta \in I_\alpha \cup \{\alpha\}\} \cup \{(\beta, \gamma) : \beta \in B \text{ and } \gamma \in S \setminus \beta + 1\}
\]

Let \( X_L \) be the associated ladder system topology on \( \omega_1 \times \{0\} \cup S \times \{1\} \) described above. While it is possible to characterize when \( H_L \) is an anti-dowker completely in terms of combinatorial properties of \( L \), it is simpler to consider the corresponding properties of \( X_L \):

**Theorem 20.** \( H_L \) is antidowker if and only if \( X_L \) satisfies all of the following conditions:

1. Sets \( S \setminus B \) and \( B \) have no disjoint open neighborhoods in \( X_L \).
2. Every countable partition of \( B \) can be extended to a countable family of open sets of \( X_L \) which is locally finite at every point of \( S \setminus B \).
3. Every countable partition of \( S \setminus B \) can be extended to a countable family of open sets of \( X_L \) which is locally finite at every point of \( B \).

**Proof.** To see that \( H_L \) is not normal, consider the closed subsets \( H_1 = \{(\alpha, \alpha + 1) : \alpha \in S \setminus B\} \) and \( H_2 = \{(\beta, \gamma) : \beta \in B \) and \( \gamma \in S \setminus \beta + 1\} \). Suppose that \( \mathcal{U} \) is an
Proof. Let \( \delta \) be the choice of the sets \( \mathcal{O} \) determined by the \( \mathcal{D} \) from Theorem 20. Clearly, items (2) and (3) can be similarly characterized by \( \mathcal{O} \) and \( \mathcal{W} \). Note that if \( \mathcal{O} \) be separated in \( \mathcal{D} \) such that \( \mathcal{O} \subseteq \mathcal{W} \). Choose \( \gamma \in \mathcal{D} \) a limit of \( \mathcal{D} \). Then \( (\beta, \gamma) \in \mathcal{U} \) as required. This is also a direct proof that \( H_L \) is not strongly collectionwise Hausdorr.

To see that \( H_L \) is countably paracompact, suppose that \( (D_n : n \in \omega) \) is a decreasing sequence of closed subsets of \( H_L \) with empty intersection. For each \( n \) let \( S_n = \{ \alpha \in S \setminus B : (\alpha, \alpha + 1) \in D_n \} \). Let \( B_n = \{ \beta \in \mathcal{B} : \{ \gamma \in S : (\beta, \gamma) \in D_n \} \) is uncountable. Then both \( (S_n) \) and \( (B_n) \) are decreasing sequences of closed subsets of \( \mathcal{X}_L \) with empty intersection. By (3) we may fix an open expansion \( (U_n) \) of \( (S_n) \) in \( \mathcal{X}_L \) such that \( B \cap \bigcap U_n = \emptyset \). Similarly, by (3) we may fix an open expansion \( (V_n) \) of \( (B_n) \) in \( (S \setminus B') \cap \bigcap V_n = \emptyset \). For each \( \alpha \in S_n \) let \( \alpha_n < \alpha \) such that \( l_n \setminus \alpha \subseteq U_n \), similarly define \( \beta_n < \beta \) for each \( \beta \in B_n \). Let \( C \) be club in \( \omega_1 \) such that for each \( \beta \in B \) each \( n \in \omega \) and each \( \delta > \beta \) with \( \delta \in C \), if \( \beta \notin B_n \), then the countable set \( \{ \gamma : (\beta, \gamma) \in D_n \} \subseteq \delta \). As in the proof that \( Z_L \) is countably paracompact, let \( O^2 \) be an open expansion of \( \mathcal{D} \cap (\gamma, \gamma^+) \) for each \( \gamma \in C \) such that the corresponding sequence of open sets \( Q_n = \bigcup Q_n^\gamma : \gamma \in C \) is locally finite. For \( \beta \in B_n \), fix \( \gamma \in C \) such that \( \gamma < \beta < \gamma^+ \) and let \( W_n(\beta) = (\beta_n, \beta] \times (\gamma^+, \omega_1) \cap \mathcal{H}_L \). Let \( S_n' \) be the set of \( \alpha \in S_n \) such that \( (\alpha, \alpha + 1) \notin O_n \), let \( W_n(\alpha) = (\alpha, \alpha] \times (\alpha + 1) \cap \mathcal{H}_L \). Finally let

\[
W_n = O_n \cup \bigcup \{ W_n(\alpha) : \alpha \in S_n' \} \cup \bigcup \{ W_n(\beta) : \beta \in B_n \}.
\]

By the choice of the sets \( O_n, U_n \) and \( V_n \) it follows that \( \bigcap W_n = \emptyset \). \( \dashv \)

Note that if \( X_L \) is countably paracompact, then it satisfies items (2) and (3) from Theorem 20. Clearly, items (2) and (3) can be similarly characterized by appropriately uniformizing functions \( f : B \to \omega \) and \( g : S \setminus B \to \omega \).

We now have two sets of sufficient conditions on a ladder system for the existence of anti-dowker subspaces of \( \omega^2_1 \). We conjecture that one set of these properties or something closely related will characterize the existence of an anti-dowker subspaces of a finite power of \( \omega_1 \). We do have the following theorem relating our two sufficient conditions:

**Theorem 21.** There is a ladder system \( \mathcal{L} \) satisfying \( M_{<\omega} \) and \( G_1 \) if there is a ladder system \( L = \{ l_\alpha : \alpha \in S \} \) with both of the following properties.

1. \( X_L \) is countably paracompact (e.g., this happens if \( \mathcal{L} \) satisfies \( P_{<\omega} \)).
2. \( \mathcal{Succ} \setminus \mathcal{U}^{X_L} \) is not stationary in \( \omega_1 \).

Moreover, if \( X_L \) satisfies (4) and (5), then \( X_L \) satisfies (1) - (3) of Theorem 20.

**Proof.** Let \( L = \{ l_\alpha : \alpha \in S \} \) be such that \( X_L \) satisfy (4) and (5). Without loss of generality, assume that \( \mathcal{Succ} = \mathcal{U}L \). Because \( B \) is not stationary, we can choose a cofinite subset \( l_\alpha' \subseteq l_\alpha \) for every \( \alpha \in B \) so that \( \{ l_\alpha' : \alpha \in B \} \) is a disjoint family. Denote the family \( \{ l_\alpha' : \alpha \in B \} \) by \( \mathcal{Succ} \) (for the reason described below). We can assume that \( \beta < \gamma \) whenever \( \alpha_1, \alpha_2 \in B \) with \( \alpha_1 < \alpha_2 \) and \( \beta \in l_\alpha' \) and \( \gamma \in l_\alpha' \). Finally, we assume that \( \mathcal{U} \{ l_\alpha' : \alpha \in B \} \) is \( \mathcal{Succ} \) (if not, remove all the points of \( \mathcal{Succ} \setminus (\cup \{ l_\alpha' : \alpha \in B \}) \) from \( \omega_1 \)). Denote a natural quotient map from \( \mathcal{Succ} \cup S \) onto \( \mathcal{Succ} \cup (S \setminus B) \) by \( q \). (So that (i) for every \( \alpha \in B \), \( q \) collapses \( l_\alpha' \cup \{ \alpha \} \) into
a singleton, and (ii) for every $\beta \in S \setminus B$, $q(\beta) = \beta$. Then $q$ preserves order in an obvious sense so that $\mathcal{S}_{\text{Succ}} \cup (S \setminus B)$ is order-isomorphic to a subset of $\omega_1$ with $\mathcal{S}_{\text{Succ}}$ being the set of successor ordinals and $S \setminus B$ being a stationary set. Define a ladder system $L$ on the stationary set $S \setminus B$ by the following rule. For every $\alpha \in S \setminus B$ and every $\beta \in \mathcal{S}_{\text{Succ}}$, $\beta \in L_\alpha$ if and only if $l_\alpha \cap l_\beta' \neq \emptyset$. It then follows that $X_L$ satisfies (1).

Now we show that $\bar{L}$ satisfies $G_1$. Let $\{\mathcal{S}_{\text{Succ}}^n : n \in \omega\}$ be a countable partition of $\mathcal{S}_{\text{Succ}}$. Then $B = \{B^n : n \in \omega\}$ is a partition of $B$ if we set $B^n = B \cap q^{-1}(\mathcal{S}_{\text{Succ}}^n)$. Because $X_L$ is countably paracompact, $B$ can be extended to a countable family $\mathcal{U} = \{U^n \subseteq X_L : n \in \omega\}$ of open subsets of $X_L$ which is locally finite in $X_L$. Because $B$ is collectionwise Hausdorff in $X_L$, we can assume that $U^n \cap U^m = \emptyset$ whenever $k \neq n$. Denote the set $\mathcal{S}_{\text{Succ}} \cup \mathcal{U}^{-1} \cap S$ by $F$, then $F$ is not stationary by (5). Pick up $\beta \in S \setminus (B \cup F)$. Consider the ladder $L_\beta$ from $\bar{L}$.

Because $\beta \notin F$, there are only finitely many $k \in \omega$ such that $L_\beta \cap (\cup \{L^n_k \setminus \mathcal{U} : \gamma \in B^n_k\}) \neq \emptyset$.

Because $\mathcal{U}$ is locally finite in $X_L$, there are only finitely many $k \in \omega$ such that $L_\beta \cap U^k \neq \emptyset$. (Remember that $\mathcal{U}$ is a disjoint family.) Hence if $\beta \in S \setminus (B \cup F)$, then $\{k \in \omega : L_\beta \cap (\cup \{L^n_k : \gamma \in B^n_k\}) \neq \emptyset\}$ is a finite set.

Because $\{q(L^n_\gamma) : \gamma \in B^n\} = \mathcal{S}_{\text{Succ}}^k$, we have that $\{k \in \omega : L_\beta \cap \mathcal{S}_{\text{Succ}}^k \neq \emptyset\}$ is a finite set as required.

For the rest of the proof, it is easy to check that (4) implies (2) and (3), and that (5) implies (1).
The following are equivalent:

(1) There is a countably paracompact, locally compact screenable space which is not paracompact;

(2) There is an uncountably regular cardinal \( \kappa \) and a thin, countably metacompact ladder system \( L \) on a stationary subset \( S \) of \( \{ \alpha < \kappa : cf(\alpha) = \omega \} \), such that the ladder space restricted to any \( \alpha < \kappa \) is CWH.

**Proof.** (2) \( \Rightarrow \) (1). Let \( \{ L_\alpha : \alpha \in S \} \) be a ladder system satisfying the conditions of (2). The set for our space \( X \) is \( (\kappa \times 2) \cup E \), where \( E = \{ (\alpha, \beta) : \beta \in L_\alpha \text{ or } \alpha \in L_\beta \} \).

The set \( E \) is a set of isolated points. Let \( F \) be a finite set. A neighborhood of a point of the form \((\alpha, 0)\), is

\[
N(\alpha, 0, F) = \{ (\alpha, 0) \} \cup \{ (\beta, \alpha) : \beta \in E : \alpha \in L_\beta \} \setminus F,
\]

and a neighborhood of \((\alpha, 1)\) is

\[
N(\alpha, 1, F) = \{ (\alpha, 1) \} \cup \{ (\beta, \alpha) : \beta \in L_\alpha \} \setminus F.
\]

Note the following:

(1) \( \kappa \times 2 \) is a closed discrete set in \( X \);

(2) \( N(\alpha, e, F) \) is the one-point compactification of a subset of \( E \);

(3) For fixed \( e < 2 \), the sets \( N(\alpha, e, \emptyset), \alpha < \omega_2 \), are pairwise disjoint;

(4) \( N(\alpha, 0, \emptyset) \cap N(\beta, 1, \emptyset) \) equals \( \{ \alpha, \beta \} \) if \( \alpha \in L_\beta \), and is empty otherwise.

It follows that \( X \) is locally compact, screenable, and 2-boundedly metacompact. It is not collectionwise Hausdorff, so not paracompact, by the pressing down lemma.

Note that we have not yet used any of the special properties of the ladder system. We will use them in proving that \( X \) is countably paracompact. Let \( c : \kappa \times 2 \to \omega \) code a countable partition of \( \kappa \times 2 \). It suffices to show that there is a locally finite expansion.

Let \( C \) be a club witnessing thinness of the ladder system with respect to the coloring \( c_0(\alpha) = c(\alpha, 0) \). Let \( F_\alpha \in [S_\alpha]^{<\omega} \) witness countable metacompactness of the ladder system for the partition \( c_1(\alpha) = c(\alpha, 1) \); i.e., if \( \beta \in \kappa \), then the set \( \{ c_1(\alpha) : \beta \in L_\alpha \cap F_\alpha \} \) is finite. By the CWH property, we may assume that for \( \alpha, \alpha' \not\in C, L_\alpha \setminus F_\alpha \cap L_{\alpha'} \setminus F_{\alpha'} = \emptyset \).

Now let \( F_{0 \alpha} = \{ (\beta, \alpha) : \beta \in L_\alpha \cap F_\alpha, \alpha \not\in C \} \), and let \( F_{0 \alpha 1} = \{ (\beta, \alpha) : \beta \in F_\alpha \} \). Note that these are finite sets. Let \( U_\alpha = \bigcup_{(\alpha, 1) = n} N(\alpha, 1, F_{0 \alpha 1}) \) and \( V_\alpha = \bigcup_{(\beta, 0) = n} N(\beta, 0, F_{0 \alpha 0}) \).

It remains to prove that \( \{ U_\alpha : n \in \omega \} \) and \( \{ V_\alpha : n \in \omega \} \) are locally finite. Any limit point of \( \{ V_\alpha : n \in \omega \} \) would have the form \((\alpha, 1)\). Suppose \( N(\alpha, 1, F_{0 \alpha 1}) \cap V_\alpha \neq \emptyset \), then there is a \( \beta \) with \( c(\beta, 0) = n \) and \( N(\alpha, 1, F_{0 \alpha 1}) \cap N(\beta, 0, F_{0 \beta 0}) \neq \emptyset \). Thus this \( \beta \in L_\alpha \setminus F_\alpha \) and \( \alpha \in C \). But for \( \alpha \in C \), the set \( \{ c(\beta, 0) : \beta \in L_\alpha \} \) is finite. It follows that \( N(\alpha, 1, F_{0 \alpha 1}) \) meets only finitely many \( V_\alpha \)’s.

Finally let us see that \( \{ U_\alpha : n \in \omega \} \) is locally finite. If not, there would be a limit point of the form \((\beta, 0)\). If \( N(\beta, 0, \emptyset) \cap U_\alpha \neq \emptyset \), then there is a \( \alpha \in S \) with \( N(\beta, 0, \emptyset) \cap N(\alpha, 1, F_{0 \alpha 1}) \neq \emptyset \) and \( c(\alpha, 1) = n \). Then \( \beta \in L_\alpha \setminus F_\alpha \) and since the \( F_\alpha \)’s witness countable metacompactness of the ladder space, the set of possible “colors” \( n = c(\alpha, 1) \) is finite. So, \( N(\beta, 0, \emptyset) \) meets only finitely many \( U_\alpha \)’s.
(1) ⇒ (2): Assume that there is a countably paracompact, locally compact, screenable space which is not paracompact. We will be using the following results of P. Daniels [3]. Let $Z(\kappa)$ denote the space $\kappa \cup [\kappa]^2$, where $[\kappa]^2$ is the set of isolated points, and a neighborhood of $\alpha \in \kappa$ has the form $N(\alpha, F) = \{\alpha\} \cup \{\alpha, \beta\} \in [\kappa]^2 : \beta \in \kappa \setminus F\}$, where $F$ is finite. Note that $\kappa$ is a closed discrete set of points in $Z(\kappa)$.

(i) If there is a space as in (1), then there is a countably paracompact subspace $X$ of $Z(\kappa)$ for some uncountable regular cardinal $\kappa$ such that $X$ contains $\kappa$ and and $\kappa$ is unseparated in $X$, but every initial segment of $\kappa$ is separated in $X$;

(ii) Let $A_\alpha = \{\beta : (\alpha, \beta) \in X\}$, and let $\Gamma = \{\alpha < \kappa : \exists \beta \geq \alpha(|A_\beta \cap \alpha| \geq \omega)\}$. Then $\Gamma$ is stationary.

(iii) For each $\alpha \in \Gamma$, choose $\theta(\alpha) \geq \alpha$ such that $A_{\theta(\alpha)} \cap \alpha$ is infinite. Then there is a stationary $\Delta \subset \Gamma$ such that $\theta \restriction \Delta$ is one-to-one. Further, the set $\Omega = \{\alpha \in \Delta : cf(\alpha) = \omega$ and $sup(A_{\theta(\alpha)} \cap \alpha) = \alpha\}$ is also stationary.

Let $\Omega$ be as in (iii) above. By passing to the intersection of $\Omega$ with a club if necessary, we may assume that $\alpha < \beta \in \Omega$ implies $\theta(\alpha) < \beta$. For each $\alpha \in \Omega$, let $L_\alpha$ be a subset of $A_{\theta(\alpha)} \cap \alpha$ which is cofinal in $\alpha$ and has order type $\omega$. We claim that $L = \{L_\alpha : \alpha \in \omega\}$ is a ladder system having the desired properties.

That the ladder space of $L$ restricted to any $\alpha < \kappa$ is $CWH$ follows easily from the fact (see (i) above) that $X$ is $< \kappa - CWH$.

**Claim 1** $L$ is thin.

To see this, suppose $g : \kappa \to \omega$. Since $X$ is countably paracompact, for each $\alpha$ we can find $F_\alpha \in [\kappa]^{<\omega}$ such that, if $U_\alpha = \bigcup_{\beta(\alpha) = \alpha} N(\alpha, F_\alpha)$, then $\{U_\alpha \}_{\alpha \in \omega}$ is locally finite, and further, local finiteness at $\alpha$ is witnessed by $N(\alpha, F_\alpha)$. Suppose $g$ has infinite range on $L_\alpha$ for all $\alpha$ in a stationary set $T$. Find $\alpha \in T$ such that $F_\beta \subset \alpha$ for each $\beta < \alpha$. Then for all sufficiently large $\beta \in L_\alpha$, the point $\{\beta, \theta(\alpha)\}$ is in $N(\beta, F_\beta) \cap N(\theta(\alpha), F_{\theta(\alpha)})$. It follows that $N(\theta(\alpha), F_{\theta(\alpha)})$ meets $U_\alpha$ for infinitely many $\alpha$, contradiction.

It remains to prove:

**Claim 2** $L$ is countably metacompact.

Let $h : \Omega \to \omega$. We need to find finite sets $H_\alpha$, $\alpha \in \Omega$, such that for each $\beta$, the set $\{h(\alpha) : \beta \in L_\alpha \setminus H_\alpha\}$ is finite.

To end this, first extend $h$ to $g : \kappa \to \omega$ so that $g(\theta(\alpha)) = h(\alpha)$. Apply countable paracompactness of $X$ to obtain the finite sets $F_\alpha$, $\alpha \in \kappa$, with the same properties as in Claim 1. We claim that $H_\alpha = F_\alpha \cup F_{\theta(\alpha)}$ works. Suppose on the contrary that $\beta$ is such that $\{h(\alpha) : \beta \in L_\alpha \setminus H_\alpha\}$ is infinite. Choose $k_0 \in \omega$ such that $g(F_\beta) \subset k_0$. Then find $k_0 < k_1 < \ldots$ such that $\beta \in L_{\alpha_n} \setminus H_{\alpha_n}$ for some $\alpha_n \in \Omega$ with $h(\alpha_n) = k_n$. Then the point $\{\beta, \theta(\alpha_n)\}$ is in $N(\beta, F_\beta) \cap N(\theta(\alpha_n), F_{\theta(\alpha_n)})$ for all $n$, whence $N(\beta, F_\beta)$ meets $U_{k_n}$ for all $n$, contradiction.

6. **Consistency of $G_1 + M_{<\omega}$**

In the last two sections we have shown two different topological statements to be equivalent to $G_1 + M_{<\omega}$. In this section we discuss the consistency of this conjunction.
We do not know if this conjunction is in fact consistent. We do know various models in which it is inconsistent. For example, we have already seen that uncountable finite-support or countable-support iterations, or uniformizability of ladder systems (and hence MA_{\omega_1}), kills \(G_1\) by itself. We have also seen that \(\diamondsuit\#\) implies \(G_1\). However, a weaker diamond principle, \(\diamondsuit(S)\), kills the conjunction \(G_1 + \mathcal{M}_{<\omega}\) on \(S\).

**Theorem 24.** Let \(S\) be a stationary subset of the ordinals of countable cofinality in the regular uncountable cardinal \(\kappa\). Then \(\diamondsuit(S)\) implies there is no thin countably metacompact ladder system on \(S\).

**Proof.** \(\diamondsuit(S)\) implies that there are \(F_\alpha : \alpha \to [\omega]^{<\omega}\) for \(\alpha \in S\) such that, given any \(F : \kappa \to [\omega]^{<\omega}\), there are stationarily many \(\alpha \in S\) with \(F \upharpoonright \alpha = F_\alpha\).

Suppose \(L\) is a thin, countably metacompact ladder system on \(S\). For each \(\alpha \in S\), choose, if possible, \(f(\alpha) \in \omega\) such that \(f(\alpha) \notin F_\alpha(\beta)\) for cofinally many \(\beta \in L_\alpha\); otherwise, let \(f(\alpha) = 0\). By countable metacompactness, there is \(F : \kappa \to [\omega]^{<\omega}\) such that for each \(\alpha \in S\), \(f(\alpha) \in F(\beta)\) for all but finitely many \(\beta \in L_\alpha\). By thinness, \(F\) is eventually constant on a club \(C\) of ladders. Considering \(\alpha \in C \cap S\) such that \(F \upharpoonright \alpha = F_\alpha\) yields a contradiction. \(\dashv\)

**Remark.** It is a corollary to the above proof that a ladder system satisfying \(G_1 + \mathcal{M}_{<\omega}\) also satisfies \(\mathcal{P}_{<\omega}\) and that \(\diamondsuit(S)\) implies no ladder on \(S\) satisfies \(\mathcal{P}_{<\omega}\).

As far as higher cardinal versions are concerned, we do not know if it consistent for there to be any thin ladder system on any stationary subset of a regular cardinal greater than \(\omega_1\). Under Fleissner’s Axiom R [7], there is no need to consider cardinals higher than \(\omega_1\) for the topological application of Section 5 (see Theorem 23).

**Theorem 25.** [7] (Axiom R) If \(X\) is \(\omega_1\)-cwH, has local density \(\leq \omega_1\), and has countable tightness, then \(X\) is cwH.

**Corollary 26.** (Axiom R) Let \(\kappa > \omega_1\) be regular. Then there is no ladder system on a stationary subset of \(\kappa\) such that the ladder space restricted to any \(\alpha < \kappa\) is cwH.

**Proof.** A ladder space with the stated properties easily satisfies the hypotheses of Fleissner’s theorem but not the conclusion. \(\dashv\)

**Corollary 27.** (MA_{\omega_1} + Axiom R) There is no ladder system satisfying the conditions of Theorem 23 (2).

Finally, we mention that we also don’t know if \(G_1 + \mathcal{M}_0\) is consistent, i.e., if there could be a thin, normal ladder system.

7. \(\clubsuit^+_\text{NS}\) AND \(G_1\).

In [12] the following \(\clubsuit\) principles are introduced.

**Definition 28.** \(\clubsuit^+_\text{NS}\) is the statement: there is a ladder system \(L = \{L_\alpha : \alpha \in \omega_1\}\) such that

1. for each club \(C \subseteq \omega_1\), the set of \(\alpha\) such that \(L_\alpha \subseteq^* C\) contains a club, and
2. for each \(A \subseteq \omega_1\) there is a club \(C\) such that for each \(\alpha \in C\) either \(L_\alpha \subseteq^* A\) or \(L_\alpha \cap A\) is finite.
Lemma 29. If $L$ satisfies $\clubsuit_{NS}$, then for every stationary $S \subseteq \omega_1$ and every $\{A_n : n \in \omega\}$ a partition of $\omega_1$,

$$\{\alpha \in S : (\exists n(L_\alpha \subseteq^* A_n)) \lor (\forall n(|L_\alpha \cap A_n| < \aleph_0))\}$$

is stationary.

Proof. If not, there is $n_0$ such that the set of $\alpha \in S$ such that

$$|L_\alpha \cap A_{n_0}| = |L_\alpha \cap (\bigcup_{n>n_0} A_n)| = \aleph_0$$

is stationary. This contradicts $\clubsuit_{NS}$.

Corollary 30. If $L$ is a $\clubsuit_{NS}$ sequence, and $L$ does not satisfy $G_1$ on any stationary set, then for every stationary set there is $\{A_n : n \in \omega\}$ a partition of $\omega_1$ such that

$$\{\alpha \in S : \forall n(|A_n \cap L_\alpha| < \aleph_0)\}$$

is stationary.

For the next definition we need some notation. Let $T$ be stationary and $h : [\omega_1]^{<\omega} \rightarrow P(\omega_1)$,

$$\mathcal{F}_{T,L} = \{A \subseteq \omega_1 : \{\alpha : L_\alpha \subseteq^* A\} \text{ is club on } T\}$$

$$Z_{h,L} = \{\alpha : \exists \beta \in L_\alpha \text{ such that } \forall \eta \in L_\alpha \cap \eta \in h(L_\alpha \cap \eta)\}$$

Definition 31. $\clubsuit^+_{NS}$ is the statement: there is a ladder system $L = \{L_\alpha : \alpha \in \omega_1\}$ satisfying $\clubsuit_{NS}$ and such that, for all $X \in [P(X)]^{\omega_1}$, for all $S$ stationary, there is a $T \subseteq S$ stationary and an ultrafilter $u$ on $\omega_1$ such that

1. $\mathcal{F}_{T,L} \cap X = u \cap X$
2. for all $h : [\omega_1]^{<\omega} \rightarrow X \cap u$, $T \setminus Z_{h,L}$ is nonstationary.

Theorem 32. If $L$ satisfies $\clubsuit^+_{NS}$ and does not satisfy $G_1$, then $\mathcal{U} = \omega_1$.

Proof. Fix $\{A_n : n \in \omega\}$ a partition of $\omega_1$ such that

$$S = \{\alpha : \forall n(|A_n \cap L_\alpha| < \aleph_0)\}$$

is stationary.

Let $B_n = \bigcup_{m > n} A_n$ and let $X \subseteq P(\omega_1)$ be any family of sets such that $B_n \in X$ for all $n$. Let $T \subseteq S$ be stationary and $u$ be an ultrafilter given by the definition of $\clubsuit^+_{NS}$. Note that since $T \subseteq S$, $B_n \in \mathcal{F}_{T,L} \cap X$ for each $n$. So $B_n \in u \cap X$ for each $n \in \omega$.

For each $\alpha \in \omega_1$, let $\{\alpha_n : n \in \omega\}$ be the increasing enumeration of $L_\alpha$ and define $g_\alpha : \omega \rightarrow \omega$ by

$$g_\alpha(n) = \min\{m : B_m \cap \{\alpha_i : i \leq n\} = \emptyset\}.$$ 

We claim that the family of $g_\alpha$ is dominating. Fix $f : \omega \rightarrow \omega$ arbitrary. Define $h : [\omega]^{<\omega} \rightarrow u \cap X$ by $h(\alpha) = B_f(\alpha)$. By definition of $\clubsuit^+_{NS}$, we may fix $\alpha \in Z_{h,L} \cap T$. By definition of $Z_{h,L}$ we may fix $\beta \in L_\alpha$ such that $\eta \in h(L_\alpha \cap \eta)$ for all $\eta \in L_\alpha \setminus \beta$. Fix $n_0$ such that $\alpha(n_0) = \beta$ and fix $n > n_0$. Then

$$\alpha(n) \in h(L_\alpha \cap \alpha(n)) = B_f((L_\alpha \cap \alpha(n))) = B_f(n).$$

Since $\alpha(n) \notin B_{g_\alpha(n)}$ it follows that $g_\alpha(n) > f(n)$. It follows that $f <^* g_\alpha$ and hence that $\{g_\alpha : \alpha < \omega_1\}$ is dominating.

$\square$
In [12] chapter 8, the consistency of $\clubsuit^{++}_{\aleph_1}$ with saturation of the nonstationary ideal in a variant of a $P_{\text{max}}$ extension. The continuum is $\aleph_2$ in this model. Unfortunately, $\diamondsuit = \omega_1$ also holds in this model [13]. Nonetheless, we conjecture that consistency of a $G_1$-ladder system satisfying weak uniformization properties should be obtainable using some $P_{\text{max}}$ variation. Indeed, the statement that there is a stationary set carrying a ladder system satisfying $G_1$ is the negation of a $\Pi_2^1$ sentence, and the statement that every ladder system satisfies $M_{<\omega}$ is also $H_2$.

References

[6] Devlin and Shelah A weak version of $\diamondsuit$ which follows from $2^{\aleph_0} < 2^{\omega_1}$ Israel J. Math. 29 (1978), no. 2-3, 239-247.
[9] P. Nyikos