UNIFORMIZATION AND ANTI-UNIFORMIZATION PROPERTIES OF LADDER SYSTEMS

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1. Introduction

Let S denote a stationary subset of limit ordinals of ω_1. A ladder system on S is a sequence $\{L_\alpha : \alpha \in S\}$ such that each L_α is an unbounded subset of α of order type ω.

A ladder system is *uniformizable* if for each sequence $\langle f_\alpha : \alpha \in S\rangle$ of functions $f_\alpha : L_\alpha \to \omega$ there is an $F : \omega_1 \to \omega$ such that $F \restriction L_\alpha =^* f_\alpha$ for each $\alpha \in S$. I.e., for each $\alpha \in S$,

$$\{\beta \in L_\alpha : F(\beta) \neq f_\alpha(\beta)\} \text{ is finite.}$$

We now formulate natural weakenings of uniformizable denoted, for each $n \in \omega$, by $\mathcal{P}_n$: A ladder system is said to satisfy $\mathcal{P}_n$ if for each $f : S \to \omega$ there is an $F : \omega_1 \to [\omega]^{n+1}$ such that for each $\alpha \in S$

(a) $F \restriction L_\alpha$ is eventually constant with value s_α, and
(b) $f(\alpha) \in s_\alpha$.

Note that $\mathcal{P}_0$ is equivalent to the version of uniformizable obtained by considering only sequences of constant functions f_α.

We will say that a ladder system satisfies $\mathcal{P}_{<\omega}$ if for each $f : S \to \omega$ there is an $F : \omega_1 \to [\omega]^{<\omega}$ satisfying (a) and (b) above.

If we drop the requirement that the restrictions $F \restriction L_\alpha$ are eventually constant we obtain uniformization properties that we denote $\mathcal{M}_n$ and $\mathcal{M}_{<\omega}$. E.g., a ladder system is said to satisfy $\mathcal{M}_{<\omega}$ if for each $f : S \to \omega$ there is an $F : \omega_1 \to [\omega]^{<\omega}$ such that for each $\alpha \in S$, $f(\alpha) \in F(\beta)$ for all but finitely many $\beta \in L_\alpha$.

Most of these uniformization properties can be characterized in terms of properties of a certain topological space naturally associated to any ladder system. If L is a ladder system, let X_L denote the topology space $\omega_1 \times \{0\} \cup S \times \{1\}$ where every point $(\alpha, 0)$ is isolated and for each $\alpha \in S$, a basic neighborhood of $(\alpha, 1)$ consists of $\{(\alpha, 1)\}$ along with a cofinite subset of $L_\alpha \times \{0\}$. Such a space is always first countable and locally compact. The stationarity of S implies that it is not collectionwise Hausdorf.

Spaces X_L have been considered by many to construct examples of normal not collectionwise Hausdorff spaces (see [11] and [2]. It is folklore that a ladder system L satisfies $\mathcal{P}_0$ if and only if X_L is normal. The property $\mathcal{M}_{<\omega}$ is characterized by X_L being countably metacompact. For this reason, we will say that L is countably metacompact in the case that it satisfies $\mathcal{M}_{<\omega}$.

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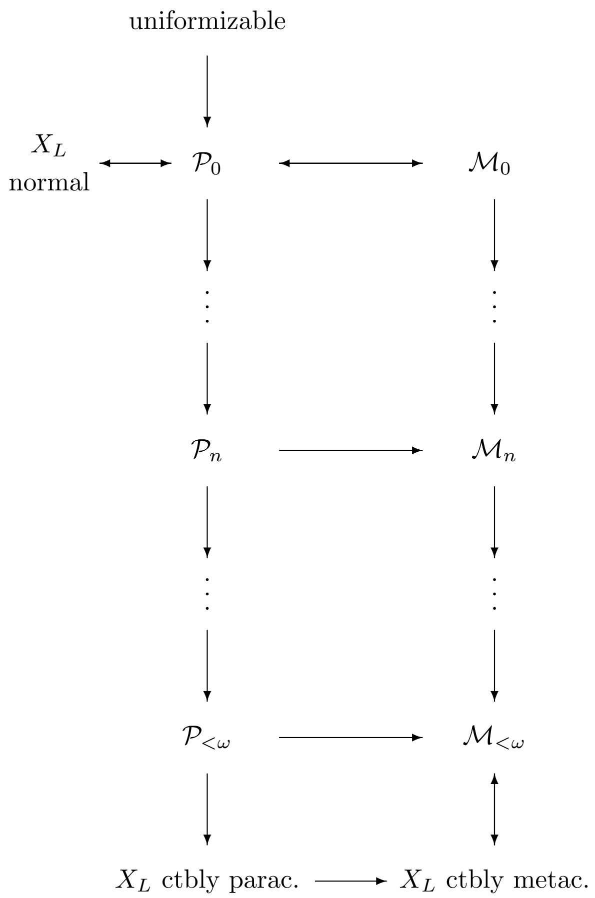

FIGURE 1

Claim 1. *Let L be a ladder system on a stationary set S. X_L is countably metacompact if and only if L satisfies $\mathcal{M}_{<\omega}$.*

Proof. X_L is countably metacompact if and only if each partition of S into countably many sets has an open expansion which is point finite on the isolated points. Suppose $f : S \to \omega$. By countable metacompactness, fix a point-finite open expansion $\{U_n : n \in \omega\}$. Let $F(\beta) = \{n : \beta \in U_n\}$. Clearly $F : \omega_1 \to [\omega]^{<\aleph_0}$ is as required. Conversely, given a partition and a corresponding function f fix the F given by $\mathcal{M}_{<\omega}$. Define $U_n = f^{-1}(n) \cup \{\beta \in \omega_1 : n \in F(\beta)\}$. It is straightforward to check that $\{U_n : n \in \omega\}$ is the required point finite open expansion. ⊣

The additional conclusion in $\mathcal{P}_{<\omega}$ that $F \restriction L_\alpha$ is eventually constant corresponds to the open expansion being locally finite, hence, $\mathcal{P}_{<\omega}$ implies that X_L is countably paracompact. We don't know whether it is equivalent.

Question 1. *If X_L is countably paracompact, does L satisfy $\mathcal{P}_{<\omega}$?*

The diagram of known ZFC implications between all the properties is summarized in Figure 1.

It should be remarked that $MA(\omega_1)$ implies that every ladder system is uniformizable and that $2^{\aleph_0} < 2^{\aleph_1}$ implies that no ladder system is uniformizable (see [6]). However, there is a ladder system on ω_1 with the property $\mathcal{M}_{<\omega}$. Indeed any ladder system L with the property that the n^{th} element of each ladder L_α is of the form $\beta+n$ with β a limit ordinal satisfies property $\mathcal{M}_{<\omega}$. Indeed let the function F on ω_1 be defined by $F(\alpha) = \{0, 1, ...n\}$ where α is of the form $\beta + n$ for some limit ordinal β. Then F uniformizes in the sense of $\mathcal{M}_{<\omega}$ every function f. However, it is not hard to see that V=L implies that no ladder system on a stationary subset of ω_1 can satisfy $\mathcal{P}_{<\omega}$ (see the remark following the proof of Theorem 24.

In Section 2 of this paper we prove that for each $n \in \omega$ it is consistent with CH that for every stationary set and every ladder L on S, L satisfies $\mathcal{P}_{n+1}$ but does not satisfy $\mathcal{M}_n$. Thus, there are not other ZFC implications between the properties $\mathcal{P}_n$ and $\mathcal{M}_m$ for any $m, n < \omega$. Moreover, by taking $n = 0$, we obtain that it is consistent with CH that every ladder system space X_L is countably paracompact (in a strong sense) but not normal.

This leaves a few questions open, including the following:

Question 2. *Is it consistent that all ladder systems satisfy $\mathcal{P}_{<\omega}$ but not $\mathcal{P}_n$ for any n?*

The next set of properties of ladder systems we will consider are in some sense anti-uniformization properties. The following is the strongest of these. A ladder system satisfying this property will also be called *thin*.

(G_1) For each $f : \omega_1 \to \omega$ the set $\{\alpha \in S : |f''L_\alpha| = \aleph_0\}$ is nonstationary.

By strengthening what f is allowed to do on a nonstationary set, we obtain the following weakenings of (G_1)

(G_2) For each $f : \omega_1 \to \omega$ the set $\{\alpha \in S : f \restriction L_\alpha \text{ is finite-to-one}\}$ is nonstationary.

(G_3) For each $f : \omega_1 \to \omega$ the set $\{\alpha \in S : f \restriction L_\alpha \text{ is eventually one-to-one}\}$ is nonstationary.

By instead demanding that every f fails to have certain properties on a stationary set, we obtain even weaker properties:

(H_1) For each $f : \omega_1 \to \omega$ the set $\{\alpha \in S : |f''L_\alpha| < \aleph_0\}$ is stationary.

(H_2) For each $f : \omega_1 \to \omega$ the set $\{\alpha \in S : f \restriction L_\alpha \text{ is not finite-to-one}\}$ is stationary.

(H_3) For each $f : \omega_1 \to \omega$ the set $\{\alpha \in S : f \restriction L_\alpha \text{ is not eventually one-to-one}\}$ is stationary.

So, from the definitions we get the diagram of implications in Figure 2.

Note that if a ladder system L is uniformizable, then it fails to satisfy H_3. Indeed, any F uniformizing a sequence of one-to-one functions $\langle f_\alpha \rangle$ will witness the failure of H_3. However, some of the weaker versions of uniformizable can be consistent with some of the anti-uniformization properties. Indeed, in [11], Shelah proved it consistent that there is a ladder system L on a stationary set such that X_L is normal and that the closed discrete set of non-isolated points is not a G_δ-set. Using our terminology, Shelah's ladder system satisfies $\mathcal{P}_0$ and H_2. Burke and Balogh [2] proved it consistent that there is a ladder system defined on a club subset of ω_1 satisfying $\mathcal{M}_{<\omega}$ and H_2.

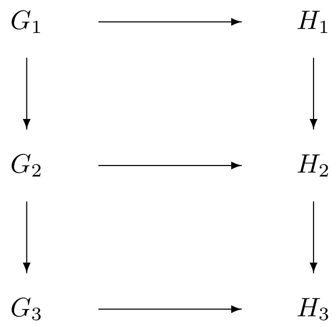

FIGURE 2

In Section 3 we give two proofs establishing the consistency of the existence of a G_1 ladder system. We also consider the G_1 property on ladders on stationary subsets of cardinals greater than ω_1.

In Section 4 we consider the question whether every countably paracompact subspace of ${\omega_1}^2$ is normal. For a ladder L on a stationary set S, we associate a non-normal subspace $Z_L \subseteq {\omega_1}^2$. The main result of that section is that Z_L is countably paracompact if and only if L has properties G_1 and $\mathcal{M}_{<\omega}$. We also consider another closely related construction of a subspace of ω_1.

Section 5 is devoted to the following open problem: Is there (consistently) a countably paracompact, locally compact screenable space which is not paracompact. It is shown the existence of such a space of cardinality ω_1 is equivalent to the existence of a ladder system on some stationary set that is G_1 and $M_{<\omega}$. In addition the existence of an example of larger cardinality is characterized by the existence of a G_1 and $M_{<\omega}$ ladder system with an additional property. Section 6 is devoted to a discussion of the question whether there may exist a G_1 and $M_{<\omega}$ ladder system on ω_1. In the final section we establish some results connecting G_1 to another known ladder system property.

2. CH AND THE UNIFORMIZATION PROPERTIES

In this section we prove that for each $n \in \omega$ it is consistent with CH that every ladder system satisfies $\mathcal{P}_{n+1}$ but fails to satisfy $\mathcal{M}_n$. In the case $n = 1$ this implies the consistence of every ladder system space X_L is countably paracompact in a strong sense but not normal. Our proof is based on a theorem of Shelah (Theorem 9 below). A proof of Shelah's theorem has only been published for the case $n = 0$. The proof of Shelah's theorem for all other $n \in \omega$ is essentially the same.

Let $\bar{C} = \langle C_\delta : \delta \in S\}$ be a ladder system on a stationary set S of countable limit ordinals, and let $f \in {}^{\omega_1}\omega$. We will define a notion of forcing $P = P_{f,\bar{C}}$ that adjoins a function $g : \omega_1 \to [\omega]^{n+2}$ such that for all $\delta \in S$,

- $g \restriction C_\delta$ is eventually constant
- $f(\delta) \in g(\alpha)$ for all but finitely many $\alpha \in C_\delta$.

Definition 2. *A condition $p \in P$ satisfies*

- $p : \alpha + 1 \to [\omega]^{n+2}$ *for some* $\alpha < \omega_1$
- *if* $\delta \leq \alpha$ *is in* S*, then* $p \restriction C_\delta$ *is eventually constant, say with value* F_δ*, and* $p(\delta) \in F_\delta$.

Lemma 3 (Extension Lemma)**.**
Let $p : \alpha + 1 \to [\omega]^{n+2}$ be a condition in P. Given $\beta > \alpha$ and $x \in [\omega]^{n+2}$, there is an extension q of p with dom$(q) = \beta + 1$ and $q(\beta) = x$.

Proof. We prove this by induction on β for all sets $x \in [\omega]^{n+2}$. The case where β is a successor ordinal is trivial. If β is a limit ordinal, we let $\{\beta_n : n < \omega\}$ list $C_\beta \setminus \alpha + 1$ in increasing order. We apply our induction hypothesis repeatedly to obtain a sequence $\{p_n : n < \omega\}$ such that

- $p_0 \leq p$
- $p_{n+1} \leq p_n$
- $\mathrm{dom} p_n = \beta_n + 1$
- $p_n(\beta_n) = \{f(\beta), ..., f(\beta + n + 1)\}$.

Once this is done, we define

$$q = \bigcup_{n\in\omega} p_n \cup \{\langle \beta, x\rangle\}. \tag{1}$$

⊣

Corollary 4. *Given $p : \alpha+1 \to [\omega]^{n+2}$ in P, a finite set A of ordinals in $\omega_1 \setminus (\alpha+1)$, and a set $x \in [\omega]^{n+2}$, there is $q \leq p$ in P such that dom$q = \max(A)+1$ and $q(\beta) = x$ for all $\beta \in A$.*

Proposition 5. *Let $N \prec H(\lambda)$ be countable with $\{f, \bar{C}, P\} \in N$. Let $A \subseteq \delta = N \cap \omega_1$ be any set of order–type ω cofinal in δ, and let $x \in [\omega]^2$. Given $p \in N \cap P$, there is an (N, P)–generic sequence $\{p_n : n \in \omega\}$ below p such that the function defined by $\cup\{p_n : n \in \omega\}$ is eventually constant with value x on A.*

Proof. We can assume that there is a sequence $\langle N_i : i < \omega\rangle$ of countable elementary submodels of $H(\lambda)$ such that

- $N_i \in N_{i+1}$
- $N = \bigcup_{n\in\omega} N_i$
- $\{f, \bar{C}, P, p\} \in N_0$

Let $\{D_n : n \in \omega\}$ list the dense open subsets of P that are elements of N, and let $\delta_i = N_i \cap \omega_1$ for $i < \omega$. By enlarging A, we may assume that $\{\delta_i : i < \omega\} \subseteq A$ — this will not affect the order–type of A as the δ_i's are cofinal in δ. We will define $\{p_n : n \in \omega\}$ by induction on n so that

(1) $p_0 = p$
(2) $p_{n+1} \leq p_n$
(3) $p_{n+1} \in D_n$
(4) if $\alpha \in A \cap \mathrm{dom} p_{n+1} \setminus \mathrm{dom} p_n$, then $p_{n+1}(\alpha) = x$

Given p_n, we first choose i large enough that $\{p_n, D_n\} \in N_i$. Since $A \cap N_i$ is finite, we can apply Corollary 4 inside N_i with p_n and $A \cap N_i$ in place of p and A to obtain a condition which we shall denote q_n. Now inside N_i, we extend q_n to $p_{n+1} \in D_n$. Clearly p_{n+1} has all the properties required of it, as does the sequence $\{p_n : n \in \omega\}$. $\dashv$

Corollary 6. *P is totally proper. More generally, if $N \prec H(\lambda)$ is countable with $\{\bar{C}, P, f\} \in N$, $p \in N \cap P$, and $x \in [\omega]^{n+2}$, then there is a totally generic $q \leq p$ such that dom$q = \delta + 1$ (where $\delta = N \cap \omega_1$) and $q(\delta) = x$.*

Proof. We apply Proposition 5 with C_δ in place of A, and $\{f(\delta), ..., f(\delta) + n + 1\}$ in place of x. The sequence $\{p_n : n < \omega\}$ will have a lower bound q with domain $\delta + 1$, so we can define $q(\delta) = x$ as required. $\dashv$

The following corollary is the place where it is crucial that forcing conditions map into the set of pairs of natural numbers instead of into ω itself.

Corollary 7. *There is a simple $n + 2$–completeness system $\mathbb{D}$ such that P is $\mathbb{D}$–complete.*

Proof. Recall that a completeness system is called simple if there is a first order formula ψ such that

$$\mathbb{D}(N, P, p) = \{A_x : x \text{ a finitary relation on } N\}, \tag{2}$$

where

$$A_x = \{G \in \text{Gen}(N, P) : \langle N \cup \mathcal{P}(N), \in\rangle \models \psi(G, x, p, P, N)\}. \tag{3}$$

In our case, the formula ψ says that "if x is a pair $\langle y, z\rangle$ such that y is an ω–sequence cofinal in $N \cap \omega_1$ and $z \in \omega$, then $\bigcup G\restriction y$ is eventually constant and z is an element of this limit value".

Note that P is $\mathbb{D}$–complete, as if $x = \langle C_\delta, f(\delta)\rangle$, then any member of A_x has a lower bound. Thus we need only show that $\mathbb{D}$ is an $n+2$–completeness system. To see this, let $\{x_i : i < n + 2\}$ be a set of $n + 2$ finitary relations on N; we must show that $\bigcap_{i<n+2} A_{x_i}$ is non–empty.

The non–trivial case is where all x_i and x_1 satisfy the hypothesis of the implication in the formula ψ. Let $x_i = \langle y_i, z_i\rangle$. Let $A = \bigcup_{i<n+2} y_i$. We apply Proposition 5 to this A with $\{z_0, ..., z_n + 1\}$ in place of x. The sequence $\{q_n : n \in \omega\}$ that the conclusion gives us then generates a member of A_{x_i} for every $i < n + 2$. $\dashv$

Proposition 8. *P is $< \omega_1$–proper.*

Proof. That P is α-proper for every α follows by induction on α using Corollary 6.

Ensuring the failure of $\mathcal{M}_n$: We need the following result of Shelah (see [10] Chapter VIII Claim 4.10 for the the proof in the case that $n = 0$; the proof of the general case is similar).

Theorem 9. *Let $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha : \alpha < \alpha_0\rangle$ be an iteration with countable support such that each $\dot{Q}_\alpha$ is $< \omega_1$–proper and $\mathbb{D}_\alpha$–complete for some simple $n+2$–completeness system $\mathbb{D}_\alpha$. Suppose that $\langle N_i : i \leq \beta\rangle$ is a countable increasing continuous sequence of countable models such that*

- $\langle N_j : j \leq i\rangle \in N_{i+1}$
- $\xi \leq \zeta \in N_0 \cap \alpha_0 + 1$

- $\mathbb{P} \in N_0$.

Suppose further that $\langle N_i \cap \alpha_0 : i \leq \beta \rangle$ is long for (ξ, ζ), $(G_i : i < n+2)$ are directed subsets of $P_\xi \cap N_\beta$, $r_i \in P_\xi$ is a lower bound for G_i for each $i < n+2$, $G_i \cap N_0 = G_j \cap N_0$ for all $i, j < n+2$, and for all $\eta \leq \beta$, $G_i \cap N_\eta \in Gen(N_\eta, P_\xi)$. Suppose also that $p \in P_\zeta \cap N_0$ and $p \restriction \xi \in G_0$.

Then there is a directed $G^* \subseteq P_\zeta \cap N_0$ such that $G_0 \cap N_0 \subseteq G^*$, $G^* \in Gen(N_0, P_\zeta, p)$, and

$$r_i \Vdash_{P_\xi} \text{“}\{q \restriction [\xi, \zeta) : q \in G^*\} \text{ has a lower bound in } P_\zeta / P_\xi\text{”} \tag{4}$$

for all $i < n+2$.

Our iteration $\mathbb{P}$ is a countable support iteration where at even stages we force with $({}^{<\omega_1}\omega, \supseteq)$ and at odd stages we force with $P_{f,\bar{C}}$ for some f and $\bar{C}$. Clearly a bookkeeping argument will take care that all ladders satisfy $\mathcal{P}_{n+1}$ in the extension so the crux of the matter is to prove that none of the ladder system spaces satisfy $\mathcal{M}_n$.

Suppose that $p \in P_{\omega_2}$ forces that the ladder system space built from $(\dot{S}, \dot{\bar{C}})$ is $\mathcal{M}_n$. We may assume that $(S, \bar{C})$ are in the ground model as we can find some limit stage $\alpha < \omega_2$ with $(S, \bar{C}) \in V[G_\alpha]$. The first thing we do is force with $({}^{<\omega_1}\omega, \supseteq)$. This adjoins a function $f : \omega_1 \to \omega$. Let $\dot{g}$ be a P_{ω_2} name such that

$$p \Vdash \dot{g} : \omega_1 \to [\omega]^{n+1} \text{ uniformizes } \check{f} \text{ in the sense of } \mathcal{M}_n. \tag{5}$$

Let $\langle N_i : i \leq \beta \rangle$ be a sequence of models as in the assumptions of Theorem 9 with $\xi = 1$ and $\zeta = \omega_2$, and with $\dot{g}$, f all in N_0.

Let $\delta = N_0 \cap \omega_1$, and let G_i be chosen as in the assumptions of Theorem 9 so that $\cup G_i(\delta) = i$ for each $i < n+2$. This is easy as $({}^{<\omega_1}\omega, \supseteq)$ is countably complete.

Let G^* be as in the conclusion of the theorem. From G^*, we can decode the values of $\dot{g}(\gamma)$ for all $\gamma < \delta$. Since $p \in G^*$, we know that $\dot{g}$ must uniformize f. Consider the decided sequence of values $(g(\gamma) : \gamma \in C_\delta)$. These sets are of size $n+1$, so there are at most $n+1$ values k such that $k \in g(\gamma)$ for all but finitely many $\gamma \in C_\delta$. This means that from G^* we can decode the value of $f(\delta)$ up to a set of $n+1$ possible values. Take $i < n+2$ such that i is not one of these values. This is a contradiction, as $r_i \Vdash f(\delta) = i$ and r_i can be extended to a lower bound for G^*.

3. Consistency of G_1.

In this section we give two proofs of the consistency of the existence of G_1 ladder systems. The first proof is from $V = L$, more specifically from Devlin's $\diamondsuit^\#$, (see [5]). For our purposes we will say that a sequence $\{\mathcal{N}_\alpha : \alpha \in \omega_1\}$ is a $\diamondsuit^\#$-sequence, if

1. Each $\mathcal{N}_\alpha$ is a countable transitive model of a suitable portion of ZFC,
2. $\{\mathcal{N}_\alpha \cap \mathcal{P}(\alpha) : \alpha \in \omega_1\}$ forms a $\diamondsuit^+$ sequence, and
3. $\{\alpha : \alpha = (\omega_1)^{\mathcal{N}_\alpha}\}$ is stationary.

For a more precise formulation of $\diamondsuit^\#$ see [5].

Theorem 10. *$\diamondsuit^\#$ implies the existence of a thin ladder system on a stationary subset of ω_1.*

REMARK: Kunen has shown that $V = L$ implies the existence of a G_1 ladder system defined on all of ω_1.

Proof. Fix $\{\mathcal{N}_\alpha : \alpha \in \omega_1\}$ a $\diamondsuit^\#$-sequence and let $S = \{\alpha : \alpha = (\omega_1)^{\mathcal{N}_\alpha}\}$. So S is stationary. We define the ladder system on S. Fix $\alpha \in S$ and let $\{f_k : k \in \omega\}$ be an enumeration of $\mathcal{N}_\alpha \cap {}^\alpha\omega$. Define L_α recursively: Fix n_0 minimal such that

$$\mathcal{N}_\alpha \models \text{“}f_0^{-1}(n_0) \text{ is uncountable.”}$$

We can do this since $\mathcal{N}_\alpha$ “thinks” that α is ω_1 and hence that f_0 is a function from ω_1 to ω. Let $I_0 = f_0^{-1}(n_0)$. Having defined I_k so that $\mathcal{N}_\alpha \models$ “I_k is uncountable”, fix n_{k+1} so that

$$\mathcal{N}_\alpha \models \text{“}I_k \cap f_{k+1}^{-1}(n_{k+1}) \text{ is uncountable.”}$$

So $I_0 \supseteq ... \supseteq I_k \supseteq ...$ and each I_k is unbounded in α. Choose L_α an increasing cofinal ω-sequence in α such that $L_\alpha \subseteq^* I_k$ for each $k \in \omega$. Clearly $f_k \restriction L_\alpha$ is eventually constant for each $k \in \omega$.

To see that $\mathcal{L} = \{L_\alpha : \alpha \in S\}$ satisfies G_1, fix $f : \omega_1 \to \omega$ arbitrary. By the property of being a $\diamondsuit^\#$ sequence, $C = \{\alpha : f \restriction \alpha \in \mathcal{N}_\alpha\}$ is club. Clearly, by construction, $f \restriction L_\alpha$ is eventually constant for any $\alpha \in S \cap C$. $\dashv$

Our next proof is a forcing construction of a ladder system satisfying G_1. We obtain the model by collapsing a Mahlo cardinal to ω_1. First we define a single forcing that collapses ω_1. Let Q be the set of triples (p, F, r) such that

(1) p is a function from some $n \in \omega$ to ω_1 such that $n < m < dom(p)$ implies $p(n) < p(m)$.
(2) F is a finite subset of ${}^{\omega_1}\omega$.
(3) $r : F \to \omega$
(4) $\{\alpha \in \omega_1 : \forall f \in F(f(\alpha) = r(f))\}$ is uncountable.

We order Q by $(p, F, r) \leq (q, G, s)$ if

(1) p extends q, $F \supseteq G$, r extends s, and
(2) $p(n) \in g^{-1}(s(g))$ for each $n \in dom(p) \setminus dom(q)$ and each $g \in G$.

Since $|Q| = 2^{\aleph_1}$ we have that

Lemma 11. *Q is $(2^{\aleph_1})^+$-cc*

This cannot be improved: Given $\{f_\alpha : \alpha < 2_1^{\aleph}\} \subseteq {}^{\omega_1}\omega$ such that $\{f_\alpha^{-1}(0) : \alpha < 2^{\aleph_1}\}$ forms an almost disjoint family, the conditions $(\emptyset, \{f_\alpha\}, \langle f_\alpha, 0\rangle)$ are pairwise incompatible (any common extension of two such conditions would violate (4)).

Let g be a Q-name for the generic function $\bigcup\{p : \exists F, r\ (p, F, r) \in \Gamma\}$ and let L be a Q-name for the range of g. Then L is an ω-sequence cofinal in $(\omega_1)^V$, hence ω_1 is collapsed by Q.

It is easy to verify that for each $f \in {}^{\omega_1}\omega$, the set $D_f = \{(p, F, r) \in Q : f \in F\}$ is dense in Q. Thus, the following lemma holds:

Lemma 12. *For any $f \in {}^{\omega_1}\omega$, $\Vdash_Q$ “$\check{f}$ is eventually constant on L.”*

Let κ be Mahlo. So, the set of inacessible ordinals less than κ is stationary in κ. Let $C = \{\kappa_i : i \in \kappa\}$ be an increasing enumeration of all cardinals in κ. So C is club. And let S be the set of inaccesibles. We fix an iteration $\langle P_i, Q_i : i \in \kappa\rangle$ as follows. For each $i \in \kappa$ let Q_i be defined recursively as follows.

(1) If i is a successor and $\Vdash_{P_i}$ “κ_i is uncountable,” let Q_i be $\mathrm{Fn}(\omega, \kappa_i)$. So Q_i collapses κ_i to ω.
(2) If κ_i is inaccesible, let Q_i be a P_i name for Q.
(3) Otherwise let Q be the trivial poset.

Let P_κ be the finite support iteration of the Q_i's (countable also works). It easily follows that P_κ has the κ-cc. It is also follows that for each inaccessible κ_i, P_i is κ_i-cc and collapses all κ_j for $j < i$ to countable ordinals. Hence $\Vdash_{P_i} \kappa_i = \omega_1$. So Q_i adds an ω-sequence cofinal in κ_i. For each $\delta \in S$ let L_δ be the $P_{\delta+1}$-name for the ladder added by Q_δ. We work with the ladder system $L = \{L_\delta : \delta \in S\}$.

Given any P_κ-name f for a function $f : \kappa \to \omega$, by κ-cc we have that the set of δ for which there is a P_δ name f_δ such that $\Vdash_{P_\kappa} f_\delta = f \restriction \delta$ is club. Thus, by the lemma above, for any P_κ-name f for a function $\kappa \to \omega$, the set of δ for which $\Vdash_{P_\kappa}$ "$\tau \restriction L_\delta$ is eventually constant" is club on S.

Finally, by κ-cc it follows that S remains stationary in V^{P_κ}. Thus G_1 holds in V^{P_κ}.

Thus, the existence of G_1 ladders is consistent. However, such ladder systems are very unstable. If P is a finite support iteration of length ω_1, then P adds a function $g : \omega_1 \to \omega$ that is $Fn(\omega_1, \omega, < \omega)$-generic over the universe. For a ladder system L in the ground model, this g has the property that $g \restriction L_\alpha$ has infinite range for every $\alpha \in \omega_1$. Thus, the ladder L fails to have property H_1 in the extension. On the other hand, if P is a countable support iteration of length at least ω_1, the P adds a function g that is $Fn(\omega_1, \omega, < \omega_1)$-generic over the universe. This g has the property that $g \restriction L_\alpha$ is eventually one-to-one for stationary many α. Thus, the ladder L fails to have property G_3 in the extension.

Thus we have the following

Theorem 13. *Suppose that L is a ladder system and that $\mathbb{P}$ is a finite or countable support iteration of length at least ω_1. Then $\Vdash_{\mathbb{P}}$ L* does not satisfy G_1.

Next we consider ladder systems on stationary subsets of cardinals $\kappa > \omega_1$. For these cardinals, relatively weak assumptions imply no such ladder is G_1.

Theorem 14. *Let κ be a regular cardinal. Suppose there is a cardinal λ such that $\lambda^\omega < \kappa \le 2^\lambda$. Then there is no thin ladder system on any stationary subset S of $\{\alpha < \kappa : cf(\alpha) = \omega\}$.*

Proof. Let $\mathcal{L} = \{L_\alpha : \alpha \in S\}$ be a ladder system. Identify κ with a subset of the space 2^λ, and let $\mathcal{B}$ be any base for 2^λ of cardinality λ. Given any countable subset $A = \{a_n\}_{n\in\omega}$ of κ, find $B_n \in \mathcal{B}$ with $a_n \in B_n$ and $B_n \cap \{a_i : i < n\} = \emptyset$. Then

$$\mathcal{P} = \{\kappa \cap B_n \setminus \cup_{i<n} B_i\} : n \in \omega\} \cup \{\kappa \setminus \bigcup_{i\in\omega} B_i\}$$

is a partition of κ each element of which contains at most one member of A. Since the hypothesis implies $\kappa > \mathfrak{c}$, we see that there is a set $\mathcal{F}$ of $\lambda \cdot \mathfrak{c}$-many, in particular less than κ-many, functions from κ into ω such that every countable function from κ into ω is extended by some function from $\mathcal{F}$.

Now, for each $\alpha \in S$, there is $f_\alpha \in S$ such that the range of $f_\alpha \restriction L_\alpha$ is unbounded in ω. Since $|\mathcal{F}| < \kappa$ and κ is regular, there is an $f \in \mathcal{F}$ and a stationary $S' \subset S$ such that $f_\alpha = f$ for each $\alpha \in S'$. I.e, the coloring f is unbounded on a stationary set of ladders, so $\mathcal{L}$ does not satisfy G_1. ⊣

Corollary 15. (1) *Assume the Continuum Hypothesis. Then there is no thin ladder system on any stationary subset of $\{\alpha < \omega_2 : cf(\alpha) = \omega\}$.*

(2) *Assume the Singular Cardinal Hypothesis. If $\kappa > \mathfrak{c}$ is regular, and not strongly inaccessible or the successor of a singular strong limit of countable cofinality, then there is no G_1 ladder system on any stationary subset of $\{\alpha < \kappa : cf(\alpha) = \omega\}$.*

Proof. For (1), CH implies that the hypothesis of the theorem is satisfied with $\kappa = \omega_2$ and $\lambda = \omega_1$. For (2), it is not difficult to show that under the Singular Cardinal Hypothesis, the hypothesis on κ in (2) is equivalent to the hypothesis on κ in the theorem. ⊣

4. Subspaces of ${\omega_1}^2$

Recently N. Kemoto and others have been systematically studying separation properties of products of ordinals and their subspaces. One of the more interesting questions left open by this investigation is the following:

Question 3. *Is every countably paracompact subspace of ${\omega_1}^2$ normal?*

In [8] the following characterization theorem was proven.

Theorem 16. *For each $X \subseteq {\omega_1}^2$, X is normal if and only if X is countably paracompact and strongly collectionwise Hausdorff.*

Hence every normal subspace of ${\omega_1}^2$ is countably paracompact. In addition it was shown in the same paper that ${\omega_1}^2$ is hereditarily collectionwise Hausdorff. Therefore any countably paracompact non-normal subspace of ${\omega_1}^2$ would be an example of a countably paracompact first countable collectionwise Hausdorff not strongly collectionwise Hausdorff space. In any model of ZFC where first countable countably paracompact spaces are strongly collectionwise Hausdorff, we have that a positive answer to Question 3. Although it is an open question whether V=L implies that countably paracompact first countable spaces are strongly collectionwise Hausdorff (see [9]) it was shown in [8] that Question 3 still has a positive answer assuming V=L.

In this section we consider a natural non-normal subspace of ${\omega_1}^2$ constructed from a ladder system on a stationary subset of ω_1 and prove that this space is countably paracompact if and only if the ladder system has properties G_1 and $\mathcal{M}_{<\omega}$. We are left with the question whether it is consistent to assume the existence of a ladder system which has properties $\mathcal{M}_{<\omega}$ and G_1. See Section 6 for further discussion of this question.

Fix a ladder system L on a stationary $S \subseteq \omega_1$ with the property that each L_α consists of successor ordinals. We denote the following subspace of ${\omega_1}^2$ by Z_L:

$$\{(\alpha, \alpha+1) : \alpha \in S\} \cup \{(\beta, \gamma) : \exists \alpha \in S, \beta \in L_\alpha \text{ and } \gamma = \alpha + 1 \text{ or } \gamma > \alpha \text{ is a limit}\}.$$

Stationarity of S easily implies that Z_L is not strongly collectionwise Hausdorff. Notice that by assumption on the ladder system, for each $(\beta, \gamma) \in Z_L$, if $\beta \notin S$ then β is a successor.

Theorem 17. *Z_L is countably paracompact if and only if L has properties G_1 and $\mathcal{M}_{<\omega}$.*

Proof. Assume that L has properties G_1 and $\mathcal{M}_{<\omega}$. Fix a decreasing sequence $(D_n)_{n\in\omega}$ of closed subsets of Z_L such that $\bigcap D_n = \emptyset$. Let $E_n = \{\alpha \in S : (\alpha, \alpha+1) \in D_n\}$ and define $f_0 : S \to \omega$ by $f_0(\alpha) = \max\{n : \alpha \in E_n\}$. Fix a corresponding $g_0 : \omega_1 \to [\omega]^{<\aleph_0}$ uniformizing f_0 in the sense of property $\mathcal{M}_{<\omega}$. Let

$$W'_n = \{(\beta, \alpha+1) : \beta \in L_\alpha, f_0(\alpha) = n \text{ and } n \in g_0(\beta)\}.$$

And let

$$W_n = (\{(\alpha, \alpha+1) : \alpha \in S\} \cap D_n) \cup \bigcup_{i\geq n} W'_i.$$

Notice that W_n is an open neighborhood of the sets $D_n \cap \{(\alpha, \alpha+1) : \alpha \in S\}$.

To define neighborhoods at the rest of the points of D_n first note that since ω_1 is countably compact, for each β a successor, there is an n_β such that $\{\beta\}\times\omega_1\cap D_n = \emptyset$ for each $n \geq n_\beta$. Letting $f_1(\beta) = n_\beta$ and by applying G_1, we may fix a a club C consisting of limit ordinals such that $f_1''L_\alpha$ is finite for each $\alpha \in C \cap S$.

For each $\gamma \in C$ let γ^+ be the minimal element of C above γ. For each $\gamma \in C$ the space $Z_L \cap (\gamma, \gamma^+]^2$ is a clopen metrizable subspace of Z_L. Thus, there are open sets $O^\gamma_n \supseteq D_n \cap (\gamma, \gamma^+]^2$ such that

$$\bigcap_{n\in\omega} \overline{O^\gamma_n} = \emptyset.$$

For each $(\beta, \eta) \in Z_L \cap D_n$ for which $\beta \notin S$ fix an open set

$$U_n(\beta, \eta) \subseteq \{\beta\} \times (0, \eta] \cap Z_L$$

so that
(a) if $(\beta, \eta) \in (\gamma, \gamma^+]^2$ for some $\gamma \in C$ then $U_n(\beta, \eta) \subseteq O^\gamma_n \cap (\gamma, \gamma^+]^2$.
(b) if $\beta \in (\gamma, \gamma^+)$ for some $\gamma \in C$ and $\eta > \gamma^+$ then $U_n(\beta, \eta) \subseteq \{\beta\} \times (\gamma^+, \eta]$.

Now let $O_n = \bigcup\{U_n(\beta, \eta) : (\beta, \eta) \in D_n,\ \beta \notin S\}$ and let

$$G_n = W_n \cup O_n.$$

Claim 18. $\bigcap_{n\in\omega} \overline{G_n} = \emptyset$.

Proof. First fix $\beta \notin S$. Fix n large enough so that for every $m \geq n$ both $\{\beta\} \times \omega_1 \cap D_m = \emptyset$ and $m \notin g_0(\beta)$. Then $\{\beta\} \times \omega_1 \cap G_m = \emptyset$ for each $m \geq n$.

Next, fix $\alpha \in S$ and consider the point $(\alpha, \alpha+1)$. For each $m \neq f_0(\alpha)$ it is clear that $\alpha + 1 \times \{\alpha+1\} \cap W'_m = \emptyset$. So

$$(L_\alpha \cup \{\alpha\}) \times \{\alpha+1\} \cap W_m = \emptyset$$

for each $m > f_0(\alpha)$. Thus for $m > f_0(\alpha)$, if $(\alpha, \alpha+1) \in \overline{G}_m$ then $(\alpha, \alpha+1) \in \overline{O}_m$. Now consider two cases:

Case 1: $\alpha \in C$. In this case, $f_1''L_\alpha$ is finite, so we can fix $n_1 \in \omega$ such that $\{\beta\} \times \omega_1 \cap D_m = \emptyset$ for all $\beta \in L_\alpha$ and all $m \geq n_1$. Thus

$$(L_\alpha \times \omega_1) \cap O_m = \emptyset \text{ for each } m \geq n_1.$$

Case 2: $\alpha \notin C$. In this case, fix $\gamma \in C$ such that $\gamma < \alpha < \gamma^+$. Fix n_1 such that $(\alpha, \alpha+1) \notin \overline{O^\gamma_m}$ for each $m \geq n_1$. Thus, by choice of the open sets $U_m(\beta, \eta)$ for those $\beta < \gamma$ and by choice of the open sets $U_m(\beta, \eta)$ for those $\eta > \gamma^+$ we have $(\alpha, \alpha+1) \notin \overline{O_m}$ for all $m \geq n_1$.

In either case, $(\alpha, \alpha+1) \notin \overline{O}_m$ for all $m \geq n_1$ so Z_L is countably paracompact.

For the converse, first suppose that that L does not satisfy the weak uniformizability property $\mathcal{M}_{<\omega}$ and fix a partition $f : S \to \omega$ such that for any $g : \omega_1 \to \mathcal{P}(\omega)$

if for each $\alpha \in S$ $f(\alpha) \in g(\beta)$ for all but finitely many $\beta \in L_\alpha$, then there is a β such that $g(\beta)$ is infinite. Consider the closed sets $E_n = \{(\alpha, \alpha+1) : f(\alpha) = n\}$. Fix open sets $W_n \supseteq E_n$. For each $\beta \notin S$ let $g(\beta) = \{n : (\beta, \alpha+1) \in W_n$ for some $\alpha \in f^{-1}(n)\}$. Each W_n is open so $f(\alpha) \in g(\beta)$ for all but finitely many $\beta \in L_\alpha$. Thus we may fix β such that $g(\beta)$ is infinite. Fix $\alpha_n \in f^{-1}(n)$ for each $n \in g(\beta)$. Let γ be a limit of the set $\{\alpha_n : n \in g(\beta)\}$. Then the family $\{W_n : n \in \omega\}$ is not locally finite at (β, γ). So if L does not satisfy $\mathcal{M}_{<\omega}$, then Z_L is not countably paracompact.

On the other hand, assume that L doesn't satisfy G_1 and fix a function $f : \omega_1 \to \omega$ such that $T = \{\alpha : |f''L_\alpha| = \aleph_0\}$ is stationary. Let

$$D_n = \{(\beta, \gamma) \in Z_L : f(\beta) \geq n \ \ \beta < \gamma \text{ and } \gamma \text{ is a limit}\}.$$

Notice that each D_n is closed and that $\bigcap_{n\in\omega} D_n = \emptyset$. Suppose that for each n, U_n is an open set containing D_n such that $U_0 \supseteq U_1 \supseteq \ldots$. Fix a countable elementary submodel $\mathcal{M}$ such that everything relevant, e.g., T, $\{D_n : n \in \omega\}$, f, $\{U_n : n \in \omega\}$, lies in $\mathcal{M}$. Then $\mathcal{M} \cap \omega_1 \in T$. Clearly, the following claim will complete the proof.

Claim 19. *: Let $\alpha_0 = \mathcal{M} \cap \omega_1$. Then $(\alpha_0, \alpha_0+1) \in \bigcap\{\overline{U_n} : n \in \omega\}$.*

Proof. For each $\beta \in L_{\alpha_0}$, let $Z_\beta = \{(\beta, \gamma) : \gamma > \beta \text{ is a limit}\} \cap Z_L$. Notice that $Z_\beta \subseteq D_{f(\beta)} \subseteq U_{f(\beta)}$. By the pressing down lemma, there is a γ_β such that $(\{\beta\} \times \omega_1 \setminus \gamma_\beta) \cap Z_L \subseteq U_n$ for each $n \geq f(\beta)$. Since all objects under consideration are in $\mathcal{M}$ we may assume that $\gamma_\beta \in \mathcal{M}$ for each $\beta \in L_{\alpha_0}$. Now fix $n \in \omega$ arbitrary and fix $\eta < \alpha_0$. Fix $\beta \in L_{\alpha_0}$ such that $\beta > \eta$ and such that $f(\beta) \geq n$. This can be done since $\alpha_0 \in T$. Thus, $(\beta, \alpha+1) \in U_{f(\beta)} \cap (\eta, \alpha_0] \times \{\alpha+1\}$. Thus, $(\alpha_0, \alpha_0+1) \in \overline{U_n}$ for every $n \in \omega$. Thus Z_L is not countably paracompact. ⊣

We now present an alternate construction of a subspace of ${\omega_1}^2$ that is a natural candidate for an anti-dowker subspace. Let LIM and SUCC denote respectively the set of limit ordinals in ω_1 and the set of successor ordinals in ω_1. Let $S \subseteq$ LIM be stationary and let $B \subseteq S$ be such that B is discrete in itself (hence B is nonstationary). Let L be a ladder system on S. Without loss of generality, suppose that for $\alpha < \beta$ in B, all ordinals of l_α are below all ordinals of l_β. Also, assume that $\bigcup L \subseteq$ SUCC H_L denotes the following subspace of ${\omega_1}^2$:

$$\{(\beta, \alpha+1) : \alpha \in S \setminus B \text{ and } \beta \in l_\alpha \cup \{\alpha\}\} \ \cup \ \{(\beta, \gamma) : \beta \in B \text{ and } \gamma \in S \setminus \beta+1\}$$

Let X_L be the associated ladder system topology on $\omega_1 \times \{0\} \cup S \times \{1\}$ described above. While it is possible to characterize when H_L is anti-dowker completely in terms of combinatorial properties of L, it is simpler to consider the corresponding properties of X_L:

Theorem 20. *H_L is antidowker if and only if X_L satisfies all of the following conditions:*
(1) *Sets $S \setminus B$ and B have no disjoint open neighborhoods in X_L.*
(2) *Every countable partition of B can be extended to a countable family of open sets of X_L which is locally finite at every point of $S \setminus B$.*
(3) *Every countable partition of $S \setminus B$ can be extended to a countable family of open sets of X_L which is locally finite at every point of B.*

Proof. To see that H_L is not normal, consider the closed subsets $H_1 = \{(\alpha, \alpha+1) : \alpha \in S \setminus B\}$ and $H_2 = \{(\beta, \gamma) : \beta \in B \text{ and } \gamma \in S \setminus \beta+1\}$. Suppose that U is an open set containing H_1. It suffices to show that $\overline{U} \cap H_2 \not\!/$ For each $\alpha \in S \setminus B$, fix $l'_\alpha \subseteq^* l_\alpha$, such that $l'_\alpha \times \{\alpha+1\} \subseteq U$. Let U_X be the corresponding open set in X_L determined by the l'_α's. There is a $\beta \in B$ such that the set of $\alpha \in S \setminus B$ such that $\beta \in \overline{U_X}$ is stationary–call this set T. If there is no such β, then B and $S \setminus B$ would be separated in X_L. Choose $\gamma \in S \setminus B$ a limit of T. Then $(\beta, \gamma) \in \overline{U}$ as required. This is also a direct proof that H_L is not strongly collectionwise Hausdorff.

To see that H_L is countably paracompact, suppose that $(D_n : n \in \omega)$ is a decreasing sequence of closed subsets of H_L with empty intersection. For each n let $S_n = \{\alpha \in S \setminus B : (\alpha, \alpha+1) \in D_n\}$. Let $B_n = \{\beta \in B : \{\gamma \in S : (\beta, \gamma) \in D_n\}$ is uncountable$\}$. Then both (S_n) and (B_n) are decreasing sequences of closed subsets of X_L with empty intersection. By (3) we may fix an open expansion (U_n) of (S_n) in X_L such that $B \cap \bigcap \overline{U_n} = \emptyset$. Similarly, by (3) we may fix an open expansion (V_n) of (B_n) and $(S \setminus B) \cap \bigcap \overline{V_n} = \emptyset$. For each $\alpha \in S_n$ let $\alpha_n < \alpha$ such that $l_\alpha \setminus \alpha_n \subseteq U_n$, similarly define $\beta_n < \beta$ for each $\beta \in B_n$. Let C be club in ω_1 such that for each $\beta \in B$ each $n \in \omega$ and each $\delta > \beta$ with $\delta \in C$, if $\beta \notin B_n$, then the countable set $\{\gamma : (\beta, \gamma) \in D_n\} \subseteq \delta$. As in the proof that Z_L is countably paracompact, let O_n^γ be an open expansion of $D_n \cap (\gamma, \gamma^+]$ for each $\gamma \in C$ such that the corresponding sequence of open sets $O_n = \bigcup\{O_n^\gamma : \gamma \in C\}$ is locally finite. For $\beta \in B_n$, fix $\gamma \in C$ such that $\gamma < \beta < \gamma^+$ and let $W_n(\beta) = (\beta_n, \beta] \times (\gamma^+, \omega_1) \cap H_L$. Let S'_n be the set of $\alpha \in S_n$ such that $(\alpha, \alpha+1) \notin O_n$, let $W_n(\alpha) = (\alpha_n, \alpha] \times \{\alpha+1\} \cap H_L$. Finally let

$$W_n = O_n \cup \bigcup\{W_n(\alpha) : \alpha \in S'_n\} \cup \bigcup\{W_n(\beta) : \beta \in B_n\}.$$

By the choice of the sets O_n, U_n and V_n it follows that $\bigcap \overline{W_n} = \emptyset$. ⊣

Note that if X_L is countably paracompact, then it satisfies items (2) and (3) from Theorem 20. Clearly, items (2) and (3) can be similarly characterized by appropriately uniformizing functions $f : B \to \omega$ and $g : S \setminus B \to \omega$.

We now have two sets of sufficient conditions on a ladder system for the existence of anti-dowker subspaces of ${\omega_1}^2$. We conjecture that one set of these properties or something closely related will characterize the existence of an anti-dowker subspaces of a finite power of ω_1. We do have the following theorem relating our two sufficient conditions:

Theorem 21. *There is a ladder system $\tilde{L}$ satisfying $\mathcal{M}_{<\omega}$ and G_1 if there is a ladder system $L = \{l_\alpha : \alpha \in S\}$ with both of the following properties.*
(4) *X_L is countably paracompact (e.g., this happens if L satisfies $\mathcal{P}_{<\omega}$).*
(5) *There is an uncountable set $B \subseteq S$ which is discrete in itself if considered with a usual topology of ω_1, such that for every open neighborhood U of B in X_L, $\overline{Succ \setminus U}^{X_L}$ is not stationary in ω_1.*

Moreover, if X_L satisfies (4) and (5), then $X_{\tilde{L}}$ satisfies (1) - (3) of Theorem 20.

Proof. Let $L = \{l_\alpha : \alpha \in S\}$ be such that X_L satisfy (4) and (5). Without loss of generality, assume that $Succ = \cup L$. Because B is not stationary, we can choose a cofinite subset $l'_\alpha \subset l_\alpha$ for every $\alpha \in B$ so that $\{l'_\alpha : \alpha \in B\}$ is a disjoint family. Denote the family $\{l'_\alpha \cup \{\alpha\} : \alpha \in B\}$ by $\tilde{Succ}$ (for the reason described below). We can assume that $\beta < \gamma$ whenever $\alpha_1, \alpha_2 \in B$ with $\alpha_1 < \alpha_2$ and $\beta \in l_{\alpha_1}$ and $\gamma \in l_{\alpha_2}$. Finally, we assume that $\cup\{l'_\alpha : \alpha \in B\} = Succ$ (if not, remove all the points of $Succ \setminus (\cup\{l'_\alpha : \alpha \in B\})$ from ω_1). Denote a natural quotient map from $Succ \cup S$ onto $\tilde{Succ} \cup (S \setminus B)$ by q. (So that (i) for every $\alpha \in B$, q collapses $l'_\alpha \cup \{\alpha\}$ into a singleton, and (ii) for every $\beta \in S \setminus B$, $q(\beta) = \beta$.) Then q preserves order in an obvious sense so that $\tilde{S}ucc \cup (S \setminus B)$ is orderly isomorphic to a subset of ω_1 with $\tilde{S}ucc$ being the set of successor ordinals and $S \setminus B$ being a stationary set. Define a ladder system $\tilde{L}$ on the stationary set $S \setminus B$ by the following rule. For every $\alpha \in S \setminus B$ and every $\beta \in \tilde{S}ucc$, $\beta \in \tilde{l}_\alpha$ if and only if $l_\alpha \cap l'_\beta \neq \emptyset$. It then follows that $X_{\tilde{L}}$ satisfies (1).

Now we show that $\tilde{L}$ satisfies G_1. Let $\{\tilde{S}ucc^n : n \in \omega\}$ be a countable partition of $\tilde{S}ucc$. Then $\mathcal{B} = \{B^n : n \in \omega\}$ is a partition of B if we set $B^n = B \cap q^{-1}(\tilde{S}ucc^n)$. Because X_L is countably paracompact, $\mathcal{B}$ can be extended to a countable family $\mathcal{U} = \{U^n \subset X_L : n \in \omega\}$ of open subsets of X_L which is locally finite in X_L. Because B is collectionwise Hausdorff in X_L, we can assume that $U^k \cap U^n = \emptyset$ whenever $k \neq n$. Denote the set $\overline{Succ \setminus \cup\mathcal{U}}^{X_L} \cap S$ by F, then F is not stationary by (5). Pick up $\beta \in S \setminus (B \cup F)$. Consider the ladder $\tilde{L}_\beta$ from $\tilde{L}$.

Because $\beta \notin F$, there are only finitely many $k \in \omega$ such that $L_\beta \cap (\cup\{L'_\gamma \setminus \cup\mathcal{U} : \gamma \in B_k\}) \neq \emptyset$.

Because $\mathcal{U}$ is locally finite in X_L, there are only finitely many $k \in \omega$ such that $L_\beta \cap U^k \neq \emptyset$. (Remember that $\mathcal{U}$ is a disjoint family.)

Hence if $\beta \in S \setminus (B \cup F)$, then $\{k \in \omega : L_\beta \cap (\cup\{L'_\gamma : \gamma \in B^k\}) \neq \emptyset\}$ is a finite set. Because $\{q(L'_\gamma) : \gamma \in B^k\} = \tilde{S}ucc^k$, we have that $\{k \in \omega : \tilde{L}_\beta \cap \tilde{S}ucc^k \neq \emptyset\}$ is a finite set as required.

For the rest of the proof, It is easy to check that (4) implies (2) and (3), and that (5) implies (1). ⊣

Theorem 22. *If there is a model of ♣ in which every ladder system space is countably paracompact, then there is a ladder system space in this model which satisfies (1) - (3) of Theorem20. In particular, there is an antidowker subset of* $\omega_1{}^2$ *in this model.*

Proof. Denote $\{\alpha + \omega : \alpha \in \omega_1\}$ by B. Then B is discrete in itself in the usual topology of ω_1. Let $\{L_\alpha : \alpha \in Lim\}$ be a ♣-sequence in this model. We prove that X_L satisfies (1) - (3). Indeed, (2) and (3) hold since X_L is countably paracompact. To prove (3), assume towards contradiction that U and V are disjoint open neighborhoods of B and $Lim \setminus B$ respectively. Since U is an uncountable subset of ω_1, there is $\beta \in Lim$ such that $L_\beta \subset U$. It follows that $\beta \subset Lim \setminus B$, hence L_β is contained in V modulo a finite subset. A contradiction with the assumption that U and V are disjoint. ⊣

It is easy to see that ♣ can be replaced in Theorem 22 with a much weaker principle. For example, it is enough to assume an existence of a ladder system $\{l_\alpha : \alpha \in S\}$ such that for every uncountable $A \subset Succ$ there is $\alpha \in S$ with the set $A \cap L_\alpha$ being infinite.

5. Screenable countably paracompact spaces

A space is *screenable* if every open cover has a σ-disjoint open refinement. Z. Balogh [1] showed that normal locally compact screenable spaces are paracompact (in ZFC). But the question whether or not the same is true with "normal" replaced by "countably paracompact" remains open. P. Daniels showed that it holds under $V = L$ [4] or under MA_{ω_1}+Axiom R [3] . In the result below, we obtain an equivalence of the problem in terms of ladder systems. Note that Daniels' results follow from this equivalence, together with Theorem 24 and Corollary 27 in the next section.

Theorem 23. *The following are equivalent:*

(1) *There is a countably paracompact, locally compact screenable space which is not paracompact;*
(2) *There is an uncountably regular cardinal κ and a thin, countably metacompact ladder system L on a stationary subset S of $\{\alpha < \kappa : cf(\alpha) = \omega\}$, such that the ladder space restricted to any $\alpha < \kappa$ is CWH.*

Proof. (2) $\Rightarrow$ (1). Let $\{L_\alpha : \alpha \in S\}$ be a ladder system satisfying the conditions of (2). The set for our space X is $(\kappa \times 2) \cup E$, where $E = \{\{\alpha, \beta\} : \beta \in L_\alpha$ or $\alpha \in L_\beta\}$. The set E is a set of isolated points. Let F be a finite set. A neighborhood of a point of the form $(\alpha, 0)$, is

$$N(\alpha, 0, F) = \{(\alpha, 0)\} \cup \{\{\beta, \alpha\} \in E : \alpha \in L_\beta\} \backslash F,$$

and a neighborhood of $(\alpha, 1)$ is

$$N(\alpha, 1, F) = \{(\alpha, 1)\} \cup \{\{\beta, \alpha\} \in E : \beta \in L_\alpha\} \backslash F.$$

Note the following:

(1) $\kappa \times 2$ is a closed discrete set in X;
(2) $N(\alpha, e, F)$ is the one-point compactification of a subset of E;
(3) For fixed $e < 2$, the sets $N(\alpha, e, \emptyset)$, $\alpha < \omega_2$, are pairwise disjoint;
(4) $N(\alpha, 0, \emptyset) \cap N(\beta, 1, \emptyset)$ equals $\{\alpha, \beta\}$ if $\alpha \in L_\beta$, and is empty otherwise.

It follows that X is locally compact, screenable, and 2-boundedly metacompact. It is not collectionwise Hausdorff, so not paracompact, by the pressing down lemma.

Note that we have not yet used any of the special properties of the ladder system. We will use them in proving that X is countably paracompact. Let $c : \kappa \times 2 \to \omega$ code a countable partition of $\kappa \times 2$. It suffices to show that there is a locally finite expansion.

Let C be a club witnessing thinness of the ladder system with respect to the coloring $c_0(\alpha) = c(\alpha, 0)$. Let $F_\alpha \in [S_\alpha]^{<\omega}$ witness countable metacompactness of the ladder system for the partition $c_1(\alpha) = c(\alpha, 1)$; i.e., if $\beta \in \kappa$, then the set $\{c_1(\alpha) : \beta \in L_\alpha \backslash F_\alpha\}$ is finite. By the CWH property, we may assume that for $\alpha, \alpha' \notin C$, $L_\alpha \backslash F_\alpha \cap L_{\alpha'} \backslash F_{\alpha'} = \emptyset$.

Now let $F_{\alpha 0} = \{\{\beta, \alpha\} : \beta \in L_\alpha \backslash F_\alpha, \alpha \notin C\}$, and let $F_{\alpha 1} = \{\{\beta, \alpha\} : \beta \in F_\alpha\}$. Note that these are finite sets. Let $U_n = \bigcup_{c(\alpha,1)=n} N(\alpha, 1, F_{\alpha 1})$ and $V_n = \bigcup_{c(\beta,0)=n} N(\beta, 0, F_{\beta 0})$.

It remains to prove that $\{U_n : n \in \omega\}$ and $\{V_n : n \in \omega\}$ are locally finite. Any limit point of $\{V_n : n \in \omega\}$ would have the form $(\alpha, 1)$. Suppose $N(\alpha, 1, F_{\alpha 1}) \cap V_n \neq \emptyset$. then there is β with $c(\beta, 0) = n$ and $N(\alpha, 1, F_{\alpha 1}) \cap N(\beta, 0, F_{\beta 0}) \neq \emptyset$. Thus $\beta \in L_\alpha \backslash F_\alpha$ and $\alpha \in C$. But for $\alpha \in C$, the set $\{c(\beta, 0) : \beta \in L_\alpha\}$ is finite. It follows that $N(\alpha, 1, F_{\alpha 1})$ meets only finitely many V_n's.

Finally let us see that $\{U_n : n \in \omega\}$ is locally finite. If not, there would be a limit point of the form $(\beta, 0)$. If $N(\beta, 0, \emptyset) \cap U_n \neq \emptyset$, then there is $\alpha \in S$ with $N(\beta, 0, \emptyset) \cap N(\alpha, 1, F_{\alpha 1}) \neq \emptyset$ and $c(\alpha, 1) = n$. Then $\beta \in L_\alpha \backslash F_\alpha$ and since the F_α's witness countable metacompactness of the ladder space, the set of possible "colors" $n = c(\alpha, 1)$ is finite. So, $N(\beta, 0, \emptyset)$ meets only finitely many U_n's.

(1) $\Rightarrow$ (2): Assume that there is a countably paracompact, locally compact, screenable space which is not paracompact. We will be using the following results of P. Daniels [3]. Let $Z(\kappa)$ denote the space $\kappa \cup [\kappa]^2$, where $[\kappa]^2$ is the set of isolated points, and a neighborhood of $\alpha \in \kappa$ has the form $N(\alpha, F) = \{\alpha\} \cup \{\alpha, \beta\} \in [\kappa]^2 : \beta \in \kappa \setminus F\}$, where F is finite. Note that κ is a closed discrete set of points in $Z(\kappa)$.

(i) If there is a space as in (1), then there is a countably paracompact subspace X of $Z(\kappa)$ for some uncountable regular cardinal κ such that X contains κ and and κ is unseparated in X, but every initial segment of κ is separated in X;

(ii) Let $A_\alpha = \{\beta : \{\alpha, \beta\} \in X\}$, and let $\Gamma = \{\alpha < \kappa : \exists \beta \geq \alpha(|A_\beta \cap \alpha| \geq \omega)\}$. Then Γ is stationary.

(iii) For each $\alpha \in \Gamma$, choose $\theta(\alpha) \geq \alpha$ such that $A_{\theta(\alpha)} \cap \alpha$ is infinite. Then there is a stationary $\Delta \subset \Gamma$ such that $\theta \restriction \Delta$ is one-to-one. Further, the set $\Omega = \{\alpha \in \Delta : cf(\alpha) = \omega \text{ and } sup(A_{\theta(\alpha)} \cap \alpha) = \alpha\}$ is also stationary.

Let Ω be as in (iii) above. By passing to the intersection of Ω with a club if necessary, we may assume that $\alpha < \beta \in \Omega$ implies $\theta(\alpha) < \beta$. For each $\alpha \in \Omega$, let L_α be a subset of $A_{\theta(\alpha)} \cap \alpha$ which is cofinal in α and has order type ω. We claim that $L = \{L_\alpha : \alpha \in \omega\}$ is a ladder system having the desired properties.

That the ladder space of L restricted to any $\alpha < \kappa$ is CWH follows easily from the fact (see (i) above) that X is $< \kappa - CWH$.

Claim 1 L is thin.

To see this, suppose $g : \kappa \to \omega$. Since X is countably paracompact, for each α we can find $F_\alpha \in [\kappa]^{<\omega}$ such that, if $U_n = \bigcup_{g(\alpha)=n} N(\alpha, F_\alpha)$, then $\{U_n\}_{n\in\omega}$ is locally finite, and further, local finiteness at α is witnessed by $N(\alpha, F_\alpha)$. Suppose g has infinite range on L_α for all α in a stationary set T. Find $\alpha \in T$ such that $F_\beta \subset \alpha$ for each $\beta < \alpha$. Then for all sufficiently large $\beta \in L_\alpha$, the point $\{\beta, \theta(\alpha)\}$ is in $N(\beta, F_\beta) \cap N(\theta(\alpha), F_{\theta(\alpha)})$. It follows that $N(\theta(\alpha), F_{\theta(\alpha)})$ meets U_n for infinitely many n, contradiction.

It remains to prove:

Claim 2 L is countably metacompact.

Let $h : \Omega \to \omega$. We need to find finite sets H_α, $\alpha \in \Omega$, such that for each β, the set $\{h(\alpha) : \beta \in L_\alpha \setminus H_\alpha\}$ is finite.

To this end, first extend h to $g : \kappa \to \omega$ so that $g(\theta(\alpha)) = h(\alpha)$. Apply countable paracompactness of X to obtain the finite sets F_α, $\alpha \in \kappa$, with the same properties as in Claim 1. We claim that $H_\alpha = F_\alpha \cup F_\theta(\alpha)$ works. Suppose on the contrary that β is such that $\{h(\alpha) : \beta \in L_\alpha \setminus H_\alpha\}$ is infinite. Choose $k_0 \in \omega$ such that $g(F_\beta) \subset k_0$. Then find $k_0 < k_1 < \dots$ such that $\beta \in L_{\alpha_n} \setminus H_{\alpha_n}$ for some $\alpha_n \in \Omega$ with $g(\theta(\alpha_n)) = h(\alpha_n) = k_n$. Then the point $\{\beta, \theta(\alpha_n)\}$ is in $N(\beta, F_\beta) \cap N(\theta(\alpha_n), F_{\theta(\alpha_n)})$ for all n, whence $N(\beta, F_\beta)$ meets U_{k_n} for all n, contradiction. $\dashv$

6. Consistency of $G_1 + \mathcal{M}_{<\omega}$

In the last two sections we have shown two different topological statements to be equivalent to $G_1 + \mathcal{M}_{<\omega}$. In this section we discuss the consistency of this conjunction.

We do not know if this conjunction is in fact consistent. We do know various models in which it is inconsistent. For example, we have already seen that uncountable finite-support or countable-support iterations, or uniformizabilty of ladder systems (and hence MA_{ω_1}), kills G_1 by itself. We have also seen that $\diamondsuit^{\#}$ implies G_1. However, a weaker diamond principle, $\diamondsuit(S)$, kills the conjunction $G_1 + \mathcal{M}_{<\omega}$ on S.

Theorem 24. *Let S be a stationary subset of the ordinals of countable cofinality in the regular uncountable cardinal κ. Then $\diamondsuit(S)$ implies there is no thin countably metacompact ladder system on S.*

Proof. $\diamondsuit(S)$ implies that there are $F_\alpha : \alpha \to [\omega]^{<\omega}$ for $\alpha \in S$ such that, given any $F : \kappa \to [\omega]^{<\omega}$, there are stationarily many $\alpha \in S$ with $F \restriction \alpha = F_\alpha$.

Suppose $\mathcal{L}$ is a thin, countably metacompact ladder system on S. For each $\alpha \in S$, choose, if possible, $f(\alpha) \in \omega$ such that $f(\alpha) \notin F_\alpha(\beta)$ for cofinally many $\beta \in L_\alpha$; otherwise, let $f(\alpha) = 0$. By countable metacompactness, there is $F : \kappa \to [\omega]^{<\omega}$ such that for each $\alpha \in S$, $f(\alpha) \in F(\beta)$ for all but finitely many $\beta \in L_\alpha$. By thinness, F is eventually constant on a club C of ladders. Considering $\alpha \in C \cap S$ such that $F \restriction \alpha = F_\alpha$ yields a contradiction. ⊣

Remark. It is a corollary to the above proof that a ladder system satisfying $G_1 + \mathcal{M}_{<\omega}$ also satisfies $\mathcal{P}_{<\omega}$ and that $\diamondsuit(S)$ implies no ladder on S satisfies $\mathcal{P}_{<\omega}$.

As far as higher cardinal versions are concerned, we do not know if it consistent for there to be any thin ladder system on any stationary subset of a regular cardinal greater than ω_1. Under Fleissner's Axiom R [7], there is no need to consider cardinals higher than ω_1 for the topological application of Section 5 (see Theorem 23).

Theorem 25. [7] *(Axiom R) If X is ω_1-cwH, has local density $\leq \omega_1$, and has countable tightness, then X is cwH.*

Corollary 26. *(Axiom R) Let $\kappa > \omega_1$ be regular. Then there is no ladder system on a stationary subset of κ such that the ladder space restricted to any $\alpha < \kappa$ is cwH.*

Proof. A ladder space with the stated properties easily satisfies the hypotheses of Fleissner's theorem but not the conclusion. ⊣

Corollary 27. *(MA_{ω_1}+Axiom R) There is no ladder system satisfying the conditions of Theorem 23 (2).*

Finally, we mention that we also don't know if $G_1 + \mathcal{M}_0$ is consistent, i.e., if there could be a thin, normal ladder system.

7. $\clubsuit^+_{NS}$ AND G_1.

In [12] the following $\clubsuit$- principles are introduced.

Definition 28. *$\clubsuit_{NS}$ is the statement: there is a ladder system $L = \{L_\alpha : \alpha \in \omega_1\}$ such that*

1. *for each club $C \subseteq \omega_1$, the set of α such that $L_\alpha \subseteq^* C$ contains a club, and*
2. *for each $A \subseteq \omega_1$ there is a club C such that for each $\alpha \in C$ either $L_\alpha \subseteq^* A$ or $L_\alpha \cap A$ is finite.*

Lemma 29. *If L satisfies $\clubsuit_{NS}$, then for every stationary $S \subseteq \omega_1$ and every $\{A_n : n \in \omega\}$ a partition of ω_1,*

$$\{\alpha \in S : (\exists n(L_\alpha \subseteq^* A_n)) \vee (\forall n(|L_\alpha \cap A_n| < \aleph_0)\}$$

is stationary

Proof. If not, there is n_0 such that the set of $\alpha \in S$ such that

$$|L_\alpha \cap A_{n_0}| = |L_\alpha \cap (\bigcup_{n>n_0} A_n)| = \aleph_0$$

is stationary. This contradicts $\clubsuit_{NS}$.

Corollary 30. *If L is a $\clubsuit_{NS}$ sequence, and L does not satisfy G_1 on any stationary set, then for every stationary set there is $\{A_n : n \in \omega\}$ a partition of ω_1 such that*

$$\{\alpha \in S : \forall n(|A_n \cap L_\alpha| < \aleph_0\}$$

is stationary.

For the next definition we need some notation. Let T be stationary and $h : [\omega_1]^{<\omega} \to P(\omega_1)$,

$$\mathcal{F}_{T,L} = \{A \subseteq \omega_1 : \{\alpha : L_\alpha \subseteq^* A\} \text{ is club on } T\}$$
$$Z_{h,L} = \{\alpha : \exists \beta \in L_\alpha \text{ such that } \forall \eta \in L_\alpha \setminus \beta \ (\eta \in h(L_\alpha \cap \eta))\}$$

Definition 31. *$\clubsuit^+_{NS}$ is the statement: there is a ladder system $L = \{L_\alpha : \alpha \in \omega_1\}$ satisfying $\clubsuit_{NS}$ and such that, for all $X \in [P(X)]^{\omega_1}$, for all S stationary, there is a $T \subseteq S$ stationary and an ultrafilter u on ω_1 such that*

(1) *$\mathcal{F}_{T,L} \cap X = u \cap X$*
(2) *for all $h : [\omega_1]^{<\omega} \to X \cap u$, $T \setminus Z_{h,L}$ is nonstationary.*

Theorem 32. *If L satisfies $\clubsuit^+_{NS}$ and does not satisfy G_1, then $\mathfrak{d} = \omega_1$.*

Proof. Fix $\{A_n : n \in \omega\}$ a partition of ω_1 such that

$$S = \{\alpha : \forall n|L_\alpha \cap A_n| < \aleph_0\}$$

is stationary.

Let $B_n = \bigcup m > nA_n$ and let $X \subseteq P(\omega_1)$ be any family of sets such that $B_n \in X$ for all n. Let $T \subseteq S$ be stationary and u be an ultrafilter be given by the definition of $\clubsuit^+_{NS}$. Note that since $T \subseteq S$, $B_n \in \mathcal{F}_{T,L} \cap X$ for each n. So $B_n \in u \cap X$ for each $n \in \omega$.

For each $\alpha \in \omega_1$, let $\{\alpha_n : n \in \omega\}$ be the increasing enumeration of L_α and define $g_\alpha : \omega \to \omega$ by

$$g_\alpha(n) = \min\{m : B_m \cap \{\alpha_i : i \leq n\} = \emptyset\}.$$

We claim that the family of g_α is dominating. Fix $f : \omega \to \omega$ arbitrary. Define $h : [\omega]^{<\omega} \to u \cap X$ by $h(a) = B_f(|a|)$. By definition of $\clubsuit^+_{NS}$, we may fix $\alpha \in Z_{h,L} \cap T$. By definition of $Z_{h,L}$ we may fix $\beta \in L_\alpha$ such that $\eta \in h(L_\alpha \cap \eta)$ for all $\eta \in L_\alpha \setminus \beta$. Fix n_0 such that $\alpha(n_0) = \beta$ and fix $n > n_0$. Then

$$\alpha(n) \in h(L_\alpha \cap \alpha(n)) = B_{f(|L_\alpha \cap \alpha(n)|)} = B_{f(n)}.$$

Since $\alpha(n) \notin B_{g_\alpha(n)}$ it follows that $g_\alpha(n) > f(n)$. It follows that $f <^* g_\alpha$ and hence that $\{g_\alpha : \alpha < \omega_1\}$ is dominating. ⊣

In [12] chapter 8, the consistency of $\clubsuit^+_{NS}$ with saturation of the nonstationary ideal in a variant of a P_{max} extension. The continuum is $\aleph_2$ in this model. Unfortunately, $\mathfrak{d} = \omega_1$ also holds in this model [13]. Nonetheless, we conjuecture that consistency of a G_1-ladder system satisfying weak uniformization properties should be obtainable using some P_{max} variation. Indeed, the statement that there is a stationary set carrying a ladder system satisfying G_1 is the negation of a Π_2 sentence, and the statement that every ladder system satisfies $M_{<\omega}$ is also Π_2.

References

[1] Z. Balogh, *Paracompactness in locally Lindelf spaces*, Canad. J. Math. **38** (1986), 719-727.

[2] D.K. Burke and Z. Balogh *A total ladder system space by ccc forcing* Top. Appl. 44 (1992), no. 1-3, 37–44.

[3] P. Daniels, *When countably paracompact, locally compact, screenable spaces are paracompact*, Topology Appl.**26** (1987), 271-279.

[4] P. Daniels, *On collectionwise Hausdorffness in countably paracompact, locally compact spaces*, Special issue on set-theoretic topology, Topology Appl.**28** (1988), no. 2, 113-125.

[5] K. Devlin *The combinatorial principle* $\diamondsuit^\sharp$ J.Symb.Log. 47 (1982), no. 4, 888–899 (1983).

[6] Devlin and Shelah *A weak version of* $\diamondsuit$ *which follows from* $2^{\aleph_0} < 2^{\aleph_1}$ Israel J. Math. 29 (1978), no. 2-3, 239–247.

[7] W.G. Fleissner *Left separated spaces with point-countable bases* Trans. Amer. Math. Soc. 294 (1986), no. 2, 665–677.

[8] N. Kemoto, K.D. Smith, P.J. Szeptycki *Countable paracompactness versus normality in* ω_1^2 Top. Appl. 104 (2000), no. 1-3, 141–154

[9] P. Nyikos

[10] S. Shelah *Proper and Improper Forcing* Springer Verlag (1998)

[11] S. Shelah *A consistent counterexample in the theory of collectionwise Hausdorff spaces* Israel J. Math. 65 (1989), no. 2, 219–224

[12] W.H. Woodin *The Axiom of Determinacy, Forcing Axioms, and the Nonstationary Ideal* de Gruyter Series in Logic and its Applications (1999).

[13] W.H. Woodin *Private communication*