FORCING AND STABLE ORDERED–UNION ULTRAFILTERS

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Abstract. We investigate the effect of a variant of Matet forcing on ultrafilters in the ground model and give a characterization of those P–points that survive such forcing, answering a question left open by Blass [?]. We investigate the question of when this variant of Matet forcing can be used to diagonalize small filters without destroying P–points in the ground model. We also deal with the question of generic existence of stable ordered–union ultrafilters.

1. Preliminaries

This paper deals with stable ordered–union ultrafilters — ultrafilters that relate to Hindman's partition theorem in much the same way that Ramsey ultrafilters relate to Ramsey's theorem. To define these ultrafilters, we will need a bit of notation.

Definition 1.1.

(1) We let $\mathbb{F}$ be the collection of all finite subsets of ω. For s and t in $\mathbb{F}$, we say $s < t$ if $\max(s) < \min(t)$. A subset X of $\mathbb{F}$ is said to be unmeshed if X can be written as $\{t_n : n < |X|\}$ where $t_n < t_{n+1}$.

(2) If X is a subset of $\mathbb{F}$, then we write $\mathrm{FU}(X)$ for the set of all finite unions of members of X. We let $[X]^n_<$ denote the set of all unmeshed n-tuples $\{s_0 < \cdots < s_{n-1}\}$ of members of X.

(3) The set $(\mathbb{F})^\omega$ is the collection of all infinite unmeshed subsets of $\mathbb{F}$. Given X and Y in $(\mathbb{F})^\omega$, we say X is a condensation of Y, written $X \sqsubseteq Y$, if $X \subseteq \mathrm{FU}(Y)$. If $s \in \mathbb{F}$ and $X \subseteq \mathbb{F}$, we define

$$X \operatorname{past} s = \{t \in X : \min(t) > \max(s)\}. \tag{1.1}$$

If s is a singleton $\{n\}$ then we simply write $X \operatorname{past} n$ in lieu of $X \operatorname{past} \{n\}$. If X and Y are in $(\mathbb{F})^\omega$, we say that X is almost a condensation of Y, written $X \sqsubseteq^* Y$, if there is an n such that $X \operatorname{past} n$ is a condensation of Y.

For those familiar with Hindman's theorem, we will concern ourselves almost exclusively with the finite unions version of it instead of working with addition on the natural numbers.

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Theorem 1 (Hindman [?] Corollary 3.3)**.** *If the set $\mathbb{F}$ is partitioned into finitely many pieces, then there exists a set $D \in (\mathbb{F})^\omega$ such that* $\mathrm{FU}(D)$ *is included in one piece of the partition.*

Theorem 2 (Milliken [?] and Taylor [?])**.** *If $n \in \omega$ and $[\mathbb{F}]^n_<$ is partitioned into finitely many pieces then there exists $D \in (\mathbb{F})^\omega$ with $[\mathrm{FU}(D)]^n_<$ included in one partition class.*

Both of the theorems just cited remain true if, instead of partitioning $\mathbb{F}$ or $[\mathbb{F}]^n_<$, one partitions $\mathrm{FU}(C)$ or $[\mathrm{FU}(C)]^n_<$, where $C = \{c_0 < \cdots < c_i < \dots\}$ is in $(\mathbb{F})^\omega$; the homogeneous set D given by the theorem is then a condensation of C. This follows immediately upon considering the bijection from $\mathbb{F}$ to $\mathrm{FU}(C)$ that sends $s \in \mathbb{F}$ to $\cup\{c_i : i \in s\}$.

Definition 1.2.

(1) A filter $\mathcal{F}$ on $\mathbb{F}$ is said to be an ordered–union filter if it has a basis of the form $\mathrm{FU}(D)$ for $D \in (\mathbb{F})^\omega$.
(2) An ordered–union filter is said to be stable if, whenever it contains $\mathrm{FU}(D_n)$ for each of countably many sets $D_n \in (\mathbb{F})^\omega$, it also contains $\mathrm{FU}(E)$ for some $E \in (\mathbb{F})^\omega$ that is almost a condensation of each D_n.
(3) If $\mathcal{F}$ is an ordered–union filter, we let $\mathcal{F} \restriction (\mathbb{F})^\omega$ be the set of all $A \in (\mathbb{F})^\omega$ such that $\mathrm{FU}(A) \in \mathcal{F}$.

Blass [?, ?] and Hindman [?] have investigated both ordered–union and stable ordered–union ultrafilters; we need only the following result of Blass in the sequel.

Theorem 3 (Blass [?], Theorem 4.2)**.** *For any ordered–union ultrafilter, the following properties are equivalent.*

(1) *Stability*
(2) *Whenever $[\mathbb{F}]^2_<$ is partitioned into two pieces, there is an $X \in U$ with $[X]^2_<$ included in one piece of the partition. (The Ramsey property for pairs.)*
(3) *For $n \in \omega$, whenever $[\mathbb{F}]^n_<$ is partitioned into two pieces, there is an $X \in \mathcal{U}$ with $[X]^n_<$ included in one piece of the partition.*

The following corollary will be used several times in the sequel.

Corollary 1.3. Let $\mathcal{U}$ be a stable ordered–union ultrafilter, and let $\{A_n : n \in \omega\}$ be a $\sqsubseteq$–decreasing sequence in $\mathcal{U} \restriction (\mathbb{F})^\omega$. There is a condensation B of A_0 in $\mathcal{U} \restriction (\mathbb{F})^\omega$ such that for each $b \in \mathrm{FU}(B)$, $B \,\mathrm{past}\, b$ is a condensation of $A_{\max b+1}$. (So in particular, B is almost a condensation of each A_n.)

Proof. Given $\{A_n : n \in \omega\}$ as above, fix a condensation C of A_0 in $\mathcal{U} \restriction (\mathbb{F})^\omega$ such that C is almost a condensation of each A_n. Fix an increasing function $f : \omega \to \omega$ so that for each n, $C \,\mathrm{past}\, f(n)$ is a condensation of A_{n+1}. Define a function g with domain $[\mathrm{FU}(C)]^2_<$ by

$$g(s < t) = \begin{cases} 1 & \text{if } f(\max s) < \min t, \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

By part (2) of Theorem 3, we can find a $B \sqsubseteq C$ in $\mathcal{U} \upharpoonright (\mathbb{F})^\omega$ so that $f \upharpoonright [\mathrm{FU}(B)]^2_<$ is constant. This constant value must be 1, as given $s \in \mathrm{FU}(B)$, we can find $t \in \mathrm{FU}(B)$ with $\min t$ arbitrarily large. Given $s \in \mathrm{FU}(B)$, $f(\max s)$ comes before the next block of B begins, and thus $B \operatorname{past} s$ is a condensation of $A_{\max s+1}$. □

Definition 1.4. If $\mathcal{F}$ is a filter on a set I and f is a function from I to some set J, then $f(\mathcal{F})$ is the filter on J defined by

$$X \in f(\mathcal{F}) \iff f^{-1}(X) \in \mathcal{F}. \tag{1.3}$$

If $\mathcal{F}$ and $\mathcal{G}$ are two filters on ω, we say that $\mathcal{G}$ lies below $\mathcal{F}$ in the Rudin–Blass ordering (written $\mathcal{G} \leq_{\mathrm{RB}} \mathcal{F}$) if there is a finite–to–one function f with

$$f(\mathcal{G}) \subseteq f(\mathcal{F}). \tag{1.4}$$

Blass [**?**] has worked out some of the basic properties of $\leq_{\mathrm{RB}}$. In particular, he shows that this relation is transitive.

At some point, we will assume the reader has a (very) basic knowledge of the cardinal invariant $\mathrm{cov}(B)$ — the covering number for the ideal of meager sets. We use only the well–known characterization of $\mathrm{cov}(B)$ as the least cardinal κ such that there is a model N of ZFC* (some large unspecified finite fragment of ZFC) of cardinality κ with no $c \in {}^\omega\omega$ Cohen over N.

2. Matet Forcing

The following notion of forcing is due to Matet [**?**]. His notation differs from ours slightly; in particular, our stable ordered–union ultrafilters are called by him Milliken–Taylor ultrafilters.

Definition 2.1. Given a stable ordered–union ultrafilter $\mathcal{U}$, the notion of forcing $P(\mathcal{U})$ consists of all pairs $\langle s, A\rangle$ such that $s \in \mathcal{F}$, $A \in \mathcal{U} \upharpoonright (\mathbb{F})^\omega$, and $s < a$ for every $a \in A$. A condition $\langle s, A\rangle$ extends a condition $\langle t, B\rangle$, written $\langle s, A\rangle \leq \langle t, B\rangle$, if A is a condensation of B and $s \setminus t \in \mathrm{FU}(B)$. If we have that $s = t$, we call $\langle s, A\rangle$ a pure extension of $\langle t, B\rangle = \langle s, B\rangle$.

We note that any two conditions with the same first coordinate are compatible, so the forcing $P(\mathcal{U})$ is σ–centered.

Let $\mathcal{U}$ be a stable ordered–union ultrafilter, fixed for the rest of this section. If G is any generic subset of $P(\mathcal{U})$, then the union of the first components of conditions in G is a subset of ω that we call a Matet real. In the sequel, $\dot{M}_{\mathcal{U}}$ is assumed to be a name for the Matet real. A condition $\langle s, A\rangle$ in $P(\mathcal{U})$ forces s to be an initial segment of $\dot{M}_{\mathcal{U}}$, and furthermore promises that $\dot{M}_{\mathcal{U}} \setminus \max s$ will be a union of members of A.

The next definition is the key one that allows us to formulate the main result of this section.

Definition 2.2. The core of $\mathcal{U}$, denoted by $\Phi(\mathcal{U})$ is defined by

$$X \in \Phi(\mathcal{U}) \iff \exists Y \in \mathcal{U} \text{ with } \cup Y \subseteq X. \tag{2.1}$$

The following proposition summarizes some simple facts about $\Phi(\mathcal{U})$. Part (4) shows that Matet forcing has some relevance to the problem of zapping small filters (see Laflamme's work in [**?**]).

Proposition 2.3.

(1) $\Phi(\mathcal{U}) = \{X \subseteq \omega : [X]^{<\omega} \in \mathcal{U}\}$
(2) $\Phi(\mathcal{U})$ is a P–filter.
(3) $\Phi(\mathcal{U})$ is not diagonalized, i.e, there is no infinite $Z \subseteq \omega$ such that Z is almost included in each member of $\Phi(\mathcal{U})$.
(4) Forcing with $P(\mathcal{U})$ adjoins a set that diagonalizes $\Phi(\mathcal{U})$.

Proof. For (1), we need only note that if $D \in \mathcal{U} \restriction (\mathbb{F})^\omega$, then $[\cup D]^{<\omega} \supseteq \mathrm{FU}(D)$, and hence is in $\mathcal{U}$ as well. Statement (2) follows immediately from (1) and the stability of $\mathcal{U}$.

For (3), fix a subset X of ω, and partition $[\mathbb{F}]^2_<$ by

$$f(s<t) = \begin{cases} 1 & \text{if the interval } (\max s, \min t) \text{ meets } X, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Because $\mathcal{U}$ satisfies the Ramsey property for pairs, there is a $D \in \mathcal{U} \restriction (\mathbb{F})^\omega$ so that $f \restriction [\mathrm{FU}(D)]^2_<$ is constant. Since D contains elements with arbitrarily high minimums, the constant value of f must be 1. Since $\cup D \in \Phi(\mathcal{U})$ and $X \setminus \cup D$ is infinite, X does not diagonalize $\Phi(\mathcal{U})$.

For (4), we note that if $\langle s, A\rangle \in P(\mathcal{U})$ and $X \in \Phi(\mathcal{U})$, there is a condensation B of A in $\mathcal{U} \restriction (\mathbb{F})^\omega$ with $\cup B \subseteq X$. The condition $\langle s, B\rangle$ extends $\langle s, A\rangle$ and forces $\dot{M}_{\mathcal{U}} \setminus \max s$ to be a subset of X. □

Both Blass and Laflamme proved (independently and unpublished) that "traditional" Matet forcing (where a condition consists of a pair $\langle s, A\rangle$ where $A \in (\mathbb{F})^\omega$ — no ultrafilters involved) preserves P–point ultrafilters in the ground model, i.e., if $\mathcal{V}$ is a P-point, then in the generic extension ever subset of ω either contains or is disjoint from a set in $\mathcal{V}$. Their results can be recovered from ours by viewing Matet forcing as a two–step iteration, where one first adjoins a generic stable–ordered ultrafilter $\mathcal{U}$ and then forces with $P(\mathcal{U})$.

Proposition 2.4. Let $\mathcal{V}$ be an ultrafilter on ω, and suppose there is a finite–to–one function f for which $f(\Phi(\mathcal{U})) \subseteq \mathcal{V}$. Then forcing with $P(\mathcal{U})$ destroys $\mathcal{V}$.

Proof. Let $\dot{X}$ be a $P(\mathcal{U})$–name for $f[\dot{M}_{\mathcal{U}}]$. It suffices to show that for any condition $\langle s, A\rangle$ and any set $Y \in \mathcal{V}$, there is an extension of $\langle s, A\rangle$ that forces both $Y \cap \dot{X}$ and $Y \setminus \dot{X}$ to be non-empty. Since $f[\cup A] \in f[\Phi(\mathcal{U})] \subseteq \mathcal{V}$, we know that $f[\cup A] \cap Y$ is infinite and so we can find a $t \in \mathrm{FU}(A)$ such that $f[t]$ meets Y. The condition $\langle s \cup t, A \operatorname{past} t\rangle$ is an extension of $\langle s, A\rangle$ that forces $\dot{X}$ to meet Y.

We next partition $[\mathrm{FU}(A\,\mathrm{past}\,t)]^2_<$ by defining
(2.3)
$$g(r<u)=\begin{cases}1 & \text{if } f[r]<f[u] \text{ and the interval } (\max f[r],\min f[u]) \text{ meets } Y,\\ 0 & \text{otherwise}\end{cases}$$

Since $\mathcal{U}$ is stable ordered–union, there is a condensation B of $A\,\mathrm{past}\,t$ such that $g \restriction [\mathrm{FU}(B)]^2_<$ is constant. Since f is finite–to–one, this constant value must be 1. Notice that

$$\langle s\cup t, B\rangle \Vdash Y\setminus \dot{X} \text{ is infinite} \tag{2.4}$$

and since $\langle s\cup t, B\rangle$ is an extension of $\langle s\cup t, A\,\mathrm{past}\,t\rangle$, we are done. □

Blass [?] has shown that the ultrafilters $\min(\mathcal{U})$ and $\max(U)$ on ω, obtained by considering min and max as functions from $\mathbb{F}$ to ω, are selective ultrafilters. Both of these ultrafilters contain $\Phi(\mathcal{U})$, and hence they fail to generate ultrafilters in the generic extension.

We can squeeze a little more information out of Proposition 2.4.

Corollary 2.5. *If $\mathcal{V}$ is an ultrafilter above $\Phi(\mathcal{U})$ in the Rudin–Blass ordering, then forcing with $P(\mathcal{U})$ destroys $\mathcal{V}$.*

Proof. Let f be a finite–to–one function with $f(\Phi(\mathcal{U}))\subseteq f(\mathcal{V})$. We apply the previous proposition to the ultrafilter $f(\mathcal{V})$, and note that if $f(\mathcal{V})$ fails to generate an ultrafilter in the generic extension, then so does $\mathcal{V}$. □

Our goal is to show that if $\mathcal{V}$ is a P–point in the ground model, then forcing with $P(\mathcal{U})$ destroys $\mathcal{V}$ if and only if $\mathcal{V}$ is not above $\Phi(\mathcal{U})$ in the Rudin–Blass ordering. The necessity is given by the previous corollary; before we can prove this is sufficient, we need some lemmas about $P(\mathcal{U})$. All of the next few lemmas use standard arguments, and they are essentially the same lemmas used by Blass in his investigation of Matet forcing.

Lemma 2.6. *If $\langle s,A\rangle\in P(\mathcal{U})$ and θ is a sentence of the forcing language, then $\langle s,A\rangle$ has a pure extension deciding θ, i.e., there is a condensation B of A in $\mathcal{U}\restriction(\mathbb{F})^\omega$ such that either*

$$\langle s,B\rangle\Vdash\theta \text{ or } \langle s,B\rangle\Vdash\neg\theta \tag{2.5}$$

Proof. Let A be listed in increasing order as $\{a_n : n\in\omega\}$. If $t\in\mathrm{FU}(A)$, let the rank of t be the least n for which $t\in\mathrm{FU}(\{a_i : i\le n\})$. Our first goal is to find a condensation B of A in $\mathcal{U}\restriction(\mathbb{F})^\omega$ with the following property:

$$\forall t\in\mathrm{FU}(B),\ \langle s\cup t, B\,\mathrm{past}\,t\rangle \text{ has a pure extension deciding } \theta \iff \langle s\cup t, B\,\mathrm{past}\,t\rangle \text{ decides } \theta. \tag{2.6}$$

To achieve this, we define a sequence $\{A_n : n\in\omega\}$ in $\mathcal{U}\restriction(\mathbb{F})^\omega$ such that

1. $A_0=A$, $A_{n+1}\sqsubseteq A_n$
2. Given $t\in\mathrm{FU}(A)$ with rank n, there is a pure extension of $\langle s\cup t, A_{n+1}\,\mathrm{past}\,t\rangle$ deciding θ if and only if $\langle s\cup t, A_{n+1}\,\mathrm{past}\,t\rangle$ decides θ.

Given A_n, we construct A_{n+1} by first listing $\mathrm{FU}(\{a_i : i \in \omega\})$ as t_0, $t_1, \dots$, t_m for some natural number m. Our A_{n+1} will be built in $m+1$ stages —- for $i < m+1$, we define $B_i \in \mathcal{U} \restriction (\mathbb{F})^\omega$ with $B_0 = A_n$ and $B_{i+1} \sqsubseteq B_i$, and A_{n+1} will be B_m. Given B_i, we ask if $\langle s \cup t_i, B_i \rangle$ has a pure extension $\langle s \cup t_i, B \rangle$ that decides θ. If so we let $B_{i+1} = B$, if not we let $B_{i+1} = B_i$. This done, the set $A_{n+1} = B_m$ is as required.

By Corollary 1.3, we can fix a condensation C of A in $\mathcal{U} \restriction (\mathbb{F})^\omega$ so that for each $t \in \mathrm{FU}(C)$, we have $C \operatorname{past} t \sqsubseteq A_{\max t+1}$. Notice that this is true for any condensation of C as well.

We next partition $\mathrm{FU}(C)$ into two pieces — a good piece and a bad piece — by putting t into the good piece if $\langle s \cup t, C \operatorname{past} t \rangle$ decides θ. By Theorem 3, we can find a $D \sqsubseteq C$ in $\mathcal{U} \restriction (\mathbb{F})^\omega$ so that $\mathrm{FU}(D)$ is contained in one piece of the partition. We claim that every member of $\mathrm{FU}(D)$ is good; by the homogeneity of D it suffices to verify that $\mathrm{FU}(D)$ contains at last one good member.

To see this, extend the condition $\langle s, D \rangle$ to a condition $\langle s \cup t, E \rangle$ that decides θ. Since $D \operatorname{past} t$ is a condensation of $A_{\max(t)+1}$, we have that $\langle s \cup t, E \rangle$ is a pure extension of $\langle s \cup t, A_{\max t+1} \rangle$ that decides θ. Since t has rank at most $\max t$, our construction of $A_{\max t+1}$ guarantees that $\langle s \cup t, A_{\max t+1} \rangle$ decides θ. Now $C \operatorname{past} t$ is a condensation of $A_{\max t+1}$ and so t satisfies the requirement to be good.

We are almost done — partition $\mathrm{FU}(D)$ into two pieces, called the pros and the cons, by putting t into the pros if and only if $\langle s \cup t, D \operatorname{past} t \rangle \Vdash \theta$. Since every element of $\mathrm{FU}(D)$ is good, we have that t is put into the cons if and only if $\langle s \cup t, D \operatorname{past} t \rangle \Vdash \neg\theta$. Since $\mathcal{U}$ is an ordered–union ultrafilter, we can find a condensation E of D in $\mathcal{U} \restriction (\mathbb{F})^\omega$ so that $\mathrm{FU}(E)$ consists entirely of pros, or entirely of cons. The condition $\langle s, E \rangle$ is a pure extension of $\langle s, A \rangle$ that decides θ. □

Lemma 2.7. Let $\dot{X}$ be a $P(\mathcal{U})$–name for a subset of ω, and let $\langle s, A \rangle$ be a condition in $P(\mathcal{U})$. There is a condensation B of A in $\mathcal{U} \restriction (\mathbb{F})^\omega$ such that for all $t \in \mathrm{FU}(B)$, if $i \leq \max t$ then $\langle s \cup t, B \operatorname{past} t \rangle$ decides "$i \in \dot{X}$", i.e. either $\langle s \cup t, B \operatorname{past} t \rangle \Vdash i \in \dot{X}$ or $\langle s \cup t, B \operatorname{past} t \rangle \Vdash i \notin \dot{X}$.

Proof. The proof is similar to the argument given in Lemma 2.6. We define a sequence $\{A_n : n \in \omega\}$ in $\mathcal{U} \restriction (\mathbb{F})^\omega$ satisfying

(1) $A_0 = A$, $A_{n+1} \sqsubseteq A_n$
(2) If $t \in \mathrm{FU}(\{a_j : j \leq n\})$ (where $\{a_n : n \in \omega\}$ is an enumeration of A in increasing order), then for all $i \leq \max(t)$,

$$\langle s \cup t, A_{n+1} \rangle \text{ decides "} i \in \dot{X} \text{"}. \tag{2.7}$$

To get A_{n+1} from A_n, list $\mathrm{FU}(\{a_j : j \leq n\})$ in some order and starting with A_n, successively apply Lemma 2.6 to achieve (2).

Next we appy Corollary 1.3 to the sequence of A_n's produced above. This gives us a condensation B of A in $\mathcal{U} \restriction (\mathbb{F})^\omega$ so that for all $t \in \mathrm{FU}(B)$, $B \operatorname{past} t \sqsubseteq A_{\max t+1}$. If n is the least natural number for which $t \in \mathrm{FU}(\{a_j :$

$j \leq n\}$), then clearly $n \leq \max t$ and this implies that $B \operatorname{past} t$ is a condensation of A_{n+1}. Thus if $i \leq \max t$, $\langle s \cup t, B \operatorname{past} t\rangle$ decides if i is in $\dot{X}$. □

Lemma 2.8. If $\dot{X}$ and $\langle s, B\rangle$ are as in the conclusion of the previous lemma, we can find a condensation C of B in $\mathcal{U} \restriction (\mathbb{F})^\omega$ such that for each $t \in \mathrm{FU}(C)$ and $i \in \omega$, the decisions of $\langle s \cup t \cup r, C \operatorname{past} r\rangle$ about "$i \in \dot{X}$" are the same for all $r \in \mathrm{FU}(C \operatorname{past} t)$ that begin late enough.

Proof. Partition the unmeshed 4-tuples of $\mathrm{FU}(B)$ into two classes by (2.8)

$$F(t < u < v < w) = \begin{cases} 1 & \text{if } \forall i \leq \max u, \langle s \cup t \cup v, B \operatorname{past} v\rangle \Vdash i \in \dot{X} \\ & \iff \langle s \cup t \cup w, B \operatorname{past} w\rangle \Vdash i \in \dot{X}, \\ 0 & \text{otherwise} \end{cases}$$

There is a $C \sqsubseteq B$ with $\mathrm{FU}(C) \in \mathcal{U}$ such that $F \restriction [\mathrm{FU}(C)]^4_<$ is constant. This constant value must be 1 — it it were not, then letting $t < u$ be the first two elements of C, we have that as v ranges over the rest of C that no two v's give the same tuple of decisions for $\dot{X} \cap (\max u + 1)$. This is clearly impossible, as there are only finitely many such tuples available. □

Definition 2.9. Let $\dot{X}$ be a $P(\mathcal{U})$–name for a subset of ω. We say a condition $\langle s, A\rangle \in P(\mathcal{U})$ is neat for $\dot{X}$ if for each natural number i and $t \in \mathrm{FU}(A)$, the decisions of $\langle s \cup t \cup r, A \operatorname{past} r\rangle$ about "$i \in \dot{X}$" are the same for all $r \in \mathrm{FU}(A)$ that begin late enough.

The preceding sequence of lemmas establishes that for a fixed name $\dot{X}$, the set of conditions that are neat for $\dot{X}$ is dense — for fixed i, we can take "late enough" to mean "after the next element of $A \operatorname{past} t$ that ends after i".

Theorem 4. *If $\mathcal{V}$ is a P–point that is not above $\Phi(\mathcal{U})$ in the Rudin–Blass ordering, then $\mathcal{V}$ continues to generate an ultrafilter after we force with $P(\mathcal{U})$.*

Proof. Let $\dot{X}$ be a $P(\mathcal{U})$–name for a subset of ω, and let $\langle s, A\rangle$ be a condition in $P(\mathcal{U})$. Without loss of generality, assume the condition $\langle s, A\rangle$ is neat for $\dot{X}$. We must produce a set $Y \in \mathcal{V}$ and an extension of $\langle s, A\rangle$ that forces $\dot{X}$ to include Y or forces $\dot{X}$ to be disjoint from Y. For each t in $\mathrm{FU}(A)$, let

$$(2.9) \quad X_t = \{i \in \omega : \text{ For all } r \in \mathrm{FU}(A) \text{ that start late enough, } \langle s \cup t \cup r, A \operatorname{past} r\rangle \Vdash i \in \dot{X}\}.$$

Notice that our assumptions on A imply that

$$(2.10) \quad \omega \setminus X_t = \{i \in \omega : \text{ For all } r \in \mathrm{FU}(A) \text{ that start late enough, } \langle s \cup t \cup r, A \operatorname{past} r\rangle \Vdash i \notin \dot{X}\}.$$

Since $\mathcal{U}$ is an ordered-union ultrafilter, we can use Hindman's Theorem to refine A if necessary to ensure that all or none of the X_t's are in $\mathcal{V}$, so without loss of generality assume each X_t is in $\mathcal{V}$. (Otherwise replace $\dot{X}$

with its complement.) Since $\mathcal{V}$ is a P–point, there is a $Z \in \mathcal{V}$ that is almost included in each X_t.

Fix a partition $\Pi = \{I_n : n \in \omega\}$ of ω into finite intervals $[m, n)$ in such a way that

(1) If $i \leq \min I_n$ and $\max t \leq \min I_n$ (for $t \in \mathrm{FU}(A)$), then those $r \in \mathrm{FU}(A)$ that begin beyond $\max I_n$ are "late enough" in the sense of 2.9,
(2) If $\max t \leq \min I_n$, then $Z \setminus \max I_n$ is a subset of X_t.

Notice that these properties are preserved if we replace A by any condensation in $\mathcal{U} \restriction (\mathbb{F})^\omega$ or if we replace Z by any subset.

Define a partition of $[\mathrm{FU}(A)]^2_<$ into two pieces by setting

(2.11)
$$F(r < u) = \begin{cases} 1 & \text{If the interval } (\max r, \min u) \text{ contains at least two } \Pi\text{–intervals,} \\ 0 & \text{otherwise} \end{cases}$$

Since $\mathcal{U}$ is a stable ordered–union ultrafilter, we can find a condensation B of A in $\mathcal{U} \restriction (\mathbb{F})^\omega$ so that F is constant when restricted to $[\mathrm{FU}(B)]^2_<$; clearly this constant value must be 1.

Let $\Pi' = \{J_n : n \in \omega\}$ be a partition of ω obtained by merging blocks of consecutive Π–intervals so that

(3) Each $b \in B$ is contained in an interval of Π'
(4) If $J \in \Pi'$ contains a $b \in B$, then J also contains a Π–interval that begins beyond $\max b$ and one that ends before $\min b$.

We can achieve this because of the way B was obtained.

Let $\pi : \omega \to \omega$ be the function that takes the value n on the interval J_n. Since $\Phi(\mathcal{U}) \not\leq_{\mathrm{R}B} \mathcal{V}$, we know that $\pi(\Phi(\mathcal{U})) \not\subseteq \pi(\mathcal{V})$, so choose $W \in \pi(\Phi(\mathcal{U}))$ with $W \notin \pi(\mathcal{V})$. As $\pi(\mathcal{V})$ is an ultrafilter, this means $\omega \setminus W \in \pi(\mathcal{V})$.

Define sets W_0 and W_1 by

$$W_0 = \bigcup_{n \in W} J_n \in \Phi(\mathcal{U}) \tag{2.12}$$

and

$$W_1 = \bigcup_{n \notin W} J_n \in \mathcal{V}. \tag{2.13}$$

Let $Y = Z \cap W_1$. Since W_0 is in the filter $\Phi(\mathcal{U})$, there is a condensation D of B in $\mathcal{U} \restriction (\mathbb{F})^\omega$ such that $\cup D \subseteq W_0$. Since $\mathcal{U}$ is closed under supersets, we may assume that $D \subseteq B$ — breaking up the blocks of D into the pieces of B from which they were built does not take us out of $\mathcal{U} \restriction (\mathbb{F})^\omega$. Thus each $d \in D$ is contained in a unique interval $J \in \Pi'$ that does not meet Y. This taken together with our construction of Π' guarantees that between Π–intervals met by a member of D and Π–intervals met by Y, there is at least one "buffer" Π–interval.

The condition $\langle s, D\rangle$ is an extension of $\langle s, B\rangle$ in $P(\mathcal{U})$; we claim that for any $t \in \mathrm{FU}(D)$,

$$\langle s \cup t, D \operatorname{past} t\rangle \Vdash Y \setminus \max t \subseteq \dot{X}. \tag{2.14}$$

This clearly suffices. To prove it, assume that it fails. Then there is an $i \in Y \setminus \max t$ and an extension of $\langle s \cup t, D \operatorname{past} t\rangle$ that forces "$i \notin \dot{X}$". Write this extension as $\langle s \cup t \cup u \cup v, E\rangle$, where our notation is chosen so that $\max(t \cup u) < i < \min v$. (Remember that Y and $\cup D$ are disjoint). Note that although u may be empty, we can assume that v is not by making a further extension if necessary.

Since there is a Π–interval between $\max t \cup u$ and i, we get from $i \in Y \subseteq Z$ that $i \in X_{t\cup u}$. Since there is a Π–interval between i and $\min v$, it follows that

$$\langle s \cup t \cup u \cup v, A \operatorname{past} v\rangle \Vdash i \in \dot{X}. \tag{2.15}$$

This is absurd, as E is a condensation of $A \operatorname{past} v$. □

3. Matet–adequate Families

In this section, we will be interested in subsets $\mathbb{F}$ partially ordered by $\sqsubseteq^*$ viewed as notions of forcing. We begin with the bare minimum needed to prove that such a forcing adjoins a stable–ordered union ultrafilter.

Definition 3.1. We say a set $\mathcal{H} \subseteq (\mathbb{F})^\omega$ is Matet–adequate if

1. $\mathcal{H}$ is closed under finite changes, i.e. if $A \in \mathcal{H}$ and $A =^* B$, then $B \in \mathcal{H}$. Note that this is different than $\cup A =^* \cup B$.
2. $\mathcal{H}$ is closed upwards: $A \in \mathcal{H}$ and $A \sqsubseteq^* B$ implies $B \in \mathcal{H}$
3. $(\mathcal{H}, \sqsubseteq^*)$ is countably closed, i.e. if $\{A_n : n \in \omega\} \subseteq \mathcal{H}$ and $A_{n+1} \sqsubseteq^* A_n$ then there is a $B \in H$ such that $B \sqsubseteq^* A_n$ for each n.
4. If $A \in H$ and $\mathrm{FU}(A)$ is partitioned into 2 pieces, then there is a $B \sqsubseteq A$ in $\mathcal{H}$ so that $\mathrm{FU}(B)$ is included in a single piece of the partition.

We refer to condition (4) as the Hindman property.

Note that the first condition in the definition of Matet–adequate means that in most cases, we can replace $\sqsubseteq^*$ by $\sqsubseteq$. The second condition isn't necessary — we include it solely for convenience.

Proposition 3.2. Let $\mathcal{H}$ be Matet–adequate. If G is any generic subset of $\mathcal{H}$, then in the generic extension $V[G]$, the set

$$\mathcal{U}_G = \{X \subseteq \mathbb{F} : X \supseteq \mathrm{FU}(A) \text{ for some } A \in G\} \tag{3.1}$$

is a stable ordered–union ultrafilter.

Proof. It follows immediately that $\mathcal{U}_G$ is an ordered–union filter. The stability of $\mathcal{U}_G$ follows because of the countable closure of $\mathcal{H}$ under $\sqsubseteq$ — given $\{A_n : n \in \omega\} \subseteq \mathcal{H}$, the set of B that are incompatible with some A_n or almost a condensation each A_n is dense in $\mathcal{H}$. The Hindman property implies that for any partition of $\mathbb{F}$ in the ground model, there is an $A \in G$ with $\mathrm{FU}(A)$ included in one piece of the partition; since $\mathcal{H}$ is countably closed we know that there are no new partitions of $\mathbb{F}$ in the extension and clearly this implies that $\mathcal{U}_G$ is an ultrafilter. □

Corollary 3.3. If $\mathcal{H}$ is Matet–adequate and $[\mathbb{F}]^n_<$ is partitioned into two pieces, then every D in $\mathcal{H}$ has a condensation $E \in \mathcal{H}$ with $[\mathrm{FU}(E)]^2_<$ included in one piece of the partition.

Proof. We use Theorem 3 and an absoluteness argument. Given $D \in \mathcal{H}$, force with all condensations of D in $\mathcal{H}$. This adjoins a stable ordered–union ultrafilter $\mathcal{U}$ with $\mathrm{FU}(D) \in \mathcal{U}$. We then apply part (3) of Theorem 3 to get a condensation E of D with $\mathrm{FU}(E) \in \mathcal{U}$ and $[\mathrm{FU}(E)]^2_<$ included in one piece of the partitition. Since no new reals are added, we know E is in the ground model. Since $\mathcal{H}$ is closed upwards, we know that $E \in \mathcal{H}$. □

With the previous corollary in mind, we make the following definition.

Definition 3.4. A family $\mathcal{H} \subseteq (\mathbb{F})^\omega$ has the Milliken–Taylor property if for every partition of $[\mathbb{F}]^2_<$ into finitely many pieces, each $D \in \mathcal{H}$ has a condensation $E \in \mathcal{H}$ with $[\mathrm{FU}(E)]^2_<$ included in one piece of the partition.

The most obvious example of a Matet–adequate family is $(\mathbb{F})^\omega$ itself. If $\mathcal{U}$ is a stable ordered–union ultrafilter, then

$$\mathcal{U} \restriction (\mathbb{F})^\omega = \{D \in (\mathbb{F})^\omega : \mathrm{FU}(D) \in \mathcal{U}\} \tag{3.2}$$

is another example of a Matet–adequate family — the Hindman property follows because $\mathcal{U}$ is an ordered–union ultrafilter, while the countable closure follows because $\mathcal{U}$ is stable. Both of these examples have been investigated in the context of forcing, so it is natural to ask what happens in the more general situation.

Definition 3.5. Let $\mathcal{H}$ be a Matet–adequate family. We define a notion of forcing $P(\mathcal{H})$, Matet forcing with respect to $\mathcal{H}$ as follows: A condition p is a pair $\langle s, D\rangle$ where $s \in \mathbb{F}$, $D \in \mathcal{H}$, and $\max s < \min d$ for $d \in D$. A condition $\langle s, D\rangle$ extends $\langle t, E\rangle$ if $s \supseteq t$, $D \sqsubseteq E$, and $s \setminus t \in \mathrm{FU}(E)$.

If G is any generic subset of $P(\mathcal{H})$, then

$$\bigcup \{s : \text{ for some } D \in \mathcal{H},\ \langle s, H\rangle \in G\ \} \tag{3.3}$$

is a subset of ω that we call a Matet real. A condition $\langle s, D\rangle \in P(\mathcal{H})$ should be thought of as a promise that the Matet real will have s as an initial segment and will consist of unions of elements of D.

A standard argument establishes that $P(\mathcal{H})$ can be viewed as a two step iteration. If we let $\dot{\mathcal{U}}$ be the canonical name for the stable ordered–union ultrafilter adjoined by forcing with $(\mathcal{H}, \sqsubseteq^*)$, then the map that sends $\langle s, A\rangle$ to $A * \langle s, A\rangle$ takes $P(\mathcal{H})$ to a dense subalgebra of the iteration $\mathcal{H} * P(\dot{\mathcal{U}})$. This point of view will prove useful when we investigate the effect of $P(\mathcal{H})$ on P–points in the ground model.

In the case where $\mathcal{H}$ is $(\mathbb{F})^\omega$, the associated notion of forcing is the standard Matet forcing. For more general $\mathcal{H}$, $P(\mathcal{H})$ also shares some of the nice combinatorial properties enjoyed by standard Matet forcing and Matet forcing with respect to a stable ordered–union ultrafilter. For example, if θ is a sentence of the forcing language and $\langle s, A\rangle \in P(\mathcal{H})$, then there is a condensation B of A in $\mathcal{H}$ such that $\langle s, B\rangle$ decides whether θ is true or false. The reader interested in a proof of this fact should check that the proof of the corresponding property for Matet forcing with respect to a stable–ordered union ultrafilter $\mathcal{U}$ given in Section 2 uses only the Matet–adequacy of $\mathcal{U} \upharpoonright (\mathbb{F})^\omega$.

4. An example

Our work in this section will require a bit more background material — Hindman's survey paper [**?**] is an excellent guide. We will state the definitions and results that we will need to quote.

Definition 4.1. If $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on $\mathbb{F}$, then $\mathcal{U}\dot\cup\mathcal{V}$ is the ultrafilter defined by

$$X \in \mathcal{U}\dot\cup\mathcal{V} \Leftrightarrow \{s : \{t : s \cup t \in X\} \in \mathcal{V}\} \in \mathcal{U}. \tag{4.1}$$

That $\mathcal{U}\dot\cup\mathcal{V}$ is an ultrafilter is not immediately obvious, but the details are worked out in [**?**]. We will adopt Blass' convention of viewing ultrafilters as a type of generalized quantifier. Couched in this (hopefully self–explanatory) notation, (4.1) becomes

$$X \in \mathcal{U}\dot\cup\mathcal{V} \Leftrightarrow \text{ for } \mathcal{U}\text{-most } s, \text{ for } \mathcal{V}\text{-most } t,\ s \cup t \in X \tag{4.2}$$

The operation $\dot\cup$ is associative, and hence $\beta\mathbb{F}$ (the space of all ultrafilters on $\mathbb{F}$) forms a semigroup under the operation $\dot\cup$. We note that it is well–known that the natural topology on $\beta\mathbb{F}$ is a compact, zero–dimensional, and Hausdorff — basic clopen sets are those of the form $\{\mathcal{U} : X \in \mathcal{U}\}$, where X is a subset of $\mathbb{F}$. The following proposition is a special case of a result on topological semigroups and is a standard tool for proving partition theorems. For a proof, again see the survey paper [**?**].

Proposition 4.2. *If $\mathcal{X}$ is a closed (hence compact) subsemigroup of $(\beta\mathbb{F}, \dot\cup)$, then there is an ultrafilter $\mathcal{U} \in \mathcal{X}$ such that $\mathcal{U}\dot\cup\mathcal{U} = \mathcal{U}$.*

Such an ultrafilter is called idempotent. Writing out the definition of idempotent, we see that if $\mathcal{U}$ is idempotent,

$$X \in \mathcal{U} \Leftrightarrow \text{ for } \mathcal{U}\text{-most } s, \text{ for } \mathcal{U}\text{-most } t,\ s \cup t \in X. \tag{4.3}$$

We shall almost exclusively deal with ultrafilters on $\mathbb{F}$ with the property that for each X in $\mathcal{U}$, the set $\min'' X \subseteq \omega$ is infinite. We refer to such ultrafilters as min-unbounded; it is not hard to show that the min-unbounded ultrafilters are a closed subsemigroup of $\beta\mathbb{F}$.

Proposition 4.3. *Let $\mathcal{F}$ be an ordered–union filter. Then there is an idempotent, min-unbounded ultrafilter $\mathcal{U} \in \beta\mathbb{F}$ that extends $\mathcal{F}$.*

Proof. We let

$$K = \{\mathcal{V} \in \beta\mathbb{F} : \mathcal{V} \text{ is min-unbounded and } \mathcal{F} \subseteq \mathcal{V}\} \tag{4.4}$$

Clearly $K \neq \emptyset$ as $\mathcal{F}$ is min-unbounded, and K is closed in $\beta\mathbb{F}$, hence compact. We show that K is in a fact a subsemigroup of $(\beta\mathbb{F}, \dot\cup)$.

Fix $\mathcal{V}_0$ and $\mathcal{V}_1$ in K. We first show that $\mathcal{F} \subseteq \mathcal{V}_0 \dot\cup \mathcal{V}_1$. Recall that $X \in \mathcal{V}_0 \dot\cup V_1$ if and only if

$$\text{for } \mathcal{V}_0\text{–most } s, \text{ for } \mathcal{V}_1\text{–most } t,\ s \cup t \in X. \tag{4.5}$$

Now $\mathcal{F}$ is an ordered-union filter, so there is some $D \in (\mathbb{F})^\omega$ with $\mathrm{FU}(D) \subseteq X$ and $\mathrm{FU}(D) \in \mathcal{F}$. Since $\mathrm{FU}(D) \in \mathcal{V}_0 \cap \mathcal{V}_1$ and $\mathrm{FU}(D)$ is closed under finite unions, we have (4.5), and hence $X \in \mathcal{V}_0 \dot\cup \mathcal{V}_1$. A similar argument tells us that $\mathcal{V}_0 \dot\cup \mathcal{V}_1$ is min-unbounded if both $\mathcal{V}_0$ and $\mathcal{V}_1$ are.

We now apply Proposition 4.2 to get an ultrafilter $\mathcal{U}$ as required. □

Theorem 5. *Let $\mathcal{F}$ be an ordered–union filter generated by $< \mathrm{cov}(B)$ sets. Suppose $\mathbb{F}$ is partitioned into finitely many pieces. Then there is a $D \in (\mathbb{F})^\omega$ such that $\mathrm{FU}(D)$ is contained in one piece of the partition and $D \cap X$ is infinite for each $X \in \mathcal{F}$.*

Proof. Fix a family $\{D_\alpha : \alpha < \kappa < \mathrm{cov}(B)\} \subseteq (\mathbb{F})^\omega$ such that $\mathcal{F}$ is generated by $\{\mathrm{FU}(D_\alpha) : \alpha < \kappa < \mathrm{cov}(B)\}$. We produce a $D \in (\mathbb{F})^\omega$ such that $\mathrm{FU}(D)$ is included in one piece of the partition and $D \cap \mathrm{FU}(D_\alpha)$ is infinite for each $\alpha < \kappa$. Clearly this implies that $D \cap X$ is infinite for each $X \in \mathcal{F}$.

By Proposition 4.3, we can fix an idempotent, min-unbounded ultrafilter $\mathcal{U}$ that extends $\mathcal{F}$. Also fix an enumeration of $\mathbb{F}$ in order-type ω.

Let N be a model of ZFC^* (an unspecified large finite fragment of ZFC) that has cardinality κ, and such that $\mathcal{F}$, $\mathcal{U}$, $\{D_\alpha : \alpha < \kappa\}$, our partition, and our enumeration of $\mathbb{F}$ are all members of N. Since $\kappa < \mathrm{cov}(B)$, there is a $c \in \omega^\omega$ that is Cohen generic over N.

We follow the Galvin–Glazer proof of Hindman's Theorem (see [**?**]), using the Cohen real to make sure the D we build has infinite intersection with each $\mathrm{FU}(D_\alpha)$. For purposes of this proof, if $X \subseteq \mathbb{F}$ and $t \in \mathbb{F}$, we set

$$X \ominus t = \{s : s \cup t \in X\} \tag{4.6}$$

Notice that if $\mathcal{U}$ is idempotent, then for each $X \in \mathcal{U}$, the set $\{t : X \ominus t \in \mathcal{U}\}$ is in $\mathcal{U}$.

Define, for $n \in \omega$ sets $X_n \in N \cap \mathcal{U}$ and $d_n \in \mathbb{F}$ as follows:

(1) X_0 is the piece of the partition that is in $\mathcal{U}$
(2) d_n is the $c(n) + 1$st element of

$$\{d \in X_n : X_n \ominus d \in \mathcal{U} \text{ and } \min(d) > \max(d_i) \text{ for } i < n\} \tag{4.7}$$

(3) $X_{n+1} = X_n \cap (X_n \ominus d_n)$

Since $\mathcal{U} = \mathcal{U} \dot\cup \mathcal{U}$, the set defined by (4.7) will be in $\mathcal{U}$, and thus we can successfully carry out the construction.

Let $D = \{d_n : n \in \omega\}$. That $\mathrm{FU}(D) \subseteq X_0$ follows as in the Galvin–Glazer proof; we give a proof for the sake of completeness.

We show by induction on $|F|$, where $F \in \mathbb{F}$, that

$$\bigcup_{i \in F} d_i \in X_{\min F}. \tag{4.8}$$

Since the sequence of X_n's is decreasing, this will establish $\mathrm{FU}(D) \subseteq X_0$. For F a singleton, (4.8) reduces to the statement $d_n \in X_n$, which holds by construction.

If $|F| > 1$, let $m = \min F$. Our induction hypothesis implies that

$$\bigcup_{i \in F \setminus \{m\}} d_i \in X_{\min(F \setminus \{m\})} \subseteq X_{m+1}. \tag{4.9}$$

Since X_{m+1} is a subset of $X_m \ominus d_m$, this means

$$\bigcup_{i \in F} d_i = d_m \cup \bigcup_{i \in F \setminus \{m\}} d_i \in X_m \tag{4.10}$$

as required.

To finish, we show that $D \cap \mathrm{FU}(D_\alpha)$ is infinite for each $\alpha < \kappa$. For purposes of contradiction, suppose that for some $\alpha < \kappa$ and $r \in \omega$ that

$$D \operatorname{past} r \cap \mathrm{FU}(D_\alpha) = \emptyset. \tag{4.11}$$

Since our construction can be done in the generic extension $N[c]$, there is a $p \in^{<\omega} \omega$ such that

$$p \Vdash D \operatorname{past} r \cap \mathrm{FU}(D_\alpha) = \emptyset. \tag{4.12}$$

We produce an extension q of p that forces $\mathrm{FU}(D_\alpha)$ to meet $D \operatorname{past} r$, yielding a contradiction.

First, let $m = \max\{\mathrm{dom}(p), r\} + 1$, and extend p arbitrarily to a condition $p' : m + 1 \to \omega$. Note that p' decides particular values for $d_0, \ldots, d_m$ and $X_0, \ldots, X_m$. We know that

$$\{d \in X_m : X_m \ominus d \in \mathcal{U} \text{ and } \min d > \max d_i \text{ for } i < m\} \in \mathcal{U} \tag{4.13}$$

and hence meets $\mathrm{FU}(D_\alpha)$. Fix some $t \in \mathbb{F}$ common to both sets, and let k be such that t is the $k + 1$st set in our enumeration of $\mathbb{F}$. Extend our condition p' to q by setting $q(m + 1) = k$. Then

$$q \Vdash D \operatorname{past} r \cap \mathrm{FU}(D_\alpha) \neq \emptyset, \tag{4.14}$$

and this contradicts (4.12). □

Definition 4.4. Let $\mathcal{F}$ be an ordered–union filter. A set $D \in (\mathbb{F})^\omega$ is said to be $\mathcal{F}$–fat if $D \cap X$ is infinite for every $X \in \mathcal{F}$.

Theorem 6. *Let $\mathcal{F}$ be an ordered–union filter generated by $< \mathrm{cov}(B)$ sets. The set of $D \in (\mathbb{F})^\omega$ that are $\mathcal{F}$–fat is Matet-adequate.*

Proof. Fix a sequence $\{D_\alpha : \alpha < \kappa < \mathrm{cov}(B)\}$ in $(\mathbb{F})^\omega$ so that $\mathcal{F}$ is generated by $\{\mathrm{FU}(D_\alpha) : \alpha < \kappa\}$, and let $D \in (\mathbb{F})^\omega$ be $\mathcal{F}$–fat. Notice that $\mathcal{F} \cup \{\mathrm{FU}(D)\}$ is included in the ordered–union filter generated by $\{\mathrm{FU}(D \cap \mathrm{FU}(D_\alpha)) : \alpha < \kappa\}$. Given a partition of $\mathbb{F}$ into finitely many pieces, we can apply the previous theorem to this filter and get an $A \in (\mathbb{F})^\omega$ with $\mathrm{FU}(A)$ included in one piece of the partition so that

$$A \cap \mathrm{FU}(D \cap \mathrm{FU}(D_\alpha)) \text{ is infinite for each } \alpha < \kappa. \tag{4.15}$$

Let $B = A \cap \mathrm{FU}(D)$. Then B is a condensation of D with $\mathrm{FU}(B)$ included in one piece of the partition, and by (4.15) we have that B is $\mathcal{F}$–fat.

The argument that the $\mathcal{F}$–fat sets are countably closed under $\sqsubseteq^*$ is very similar to the proof of Theorem 5, and so we only give a sketch. Given a $\sqsubseteq$–decreasing sequence of $\mathcal{F}$–fat sets, we fix a model M of ZFC^* of cardinality κ that contains everything relevant. Since $\kappa < \mathrm{cov}(B)$, there is a c Cohen generic over M. We define a sequence $\{d_n : n < \omega\}$ recursively by letting d_{n+1} be the $c(n) + 1$st element of D_{n+1} past d_n. Clearly $\{d_n : n \in \omega\}$ is almost a condensation of each D_n, and the fact that c is a Cohen real, just as in the proof of Theorem 5, ensures that $\{d_n : n \in \omega\}$ will be $\mathcal{F}$–fat. □

The following is an immediate consequence of Corollary 3.3.

Corollary 4.5. If $\mathcal{F}$ is an ordered–union filter generated by $< \mathrm{cov}(B)$ sets and $[\mathbb{F}]^n_<$ is partitioned into two pieces, then every $\mathcal{F}$–fat A has an $\mathcal{F}$–fat condensation B with $[\mathrm{FU}(B)]^n_<$ included in one piece of the partition.

5. Some Applications

In this final section, we give two applications. The first of these is in the spirit of Laflamme's paper [**?**] — we give sufficient conditions for a filter to be diagonalizable in a forcing extension that preserves all P–points in the ground model. Our work in the last section shows that this condition is satisfied by filters generated by $< \mathrm{cov}(B)$ sets.

Theorem 7. *Let $\mathcal{F}$ be a filter, and let $\bar{\mathcal{F}}$ be the ordered–union filter generated by $\{[X]^{<\omega} : X \in \mathcal{F}\}$. If $\mathcal{H} = \{A \in (\mathbb{F})^\omega : A \text{ is } \bar{\mathcal{F}}\text{–fat}\}$ is Matet adequate, then forcing with $P(\mathcal{H})$ diagonalizes $\mathcal{F}$ while destroying no P–points from the ground model.*

Proof. We will begin with an easy claim whose proof is left to the reader.

Claim 1.
1. $\{a_n : n \in \omega\} \in (\mathbb{F})^\omega$ is $\bar{\mathcal{F}}$–fat if and only if for every $X \in \mathcal{F}$ there are infinitely many n with $a_n \subseteq X$.
2. If $A \in \mathcal{H}$ and $X \in \mathcal{F}$, then $\{a \in A : a \subseteq X\} \in \mathcal{H}$.

Now recall that the notion of forcing $P(\mathcal{H})$ can be viewed as a composition $\mathcal{H} * P(\dot{\mathcal{U}})$, where $\dot{\mathcal{U}}$ is the canonical name for the stable ordered–union ultrafilter adjoined by forcing with $(\mathcal{H}, \sqsubseteq^*)$. By the second part of the preceding claim, if $X \in \mathcal{F}$ then $\{A \in \mathcal{H} : \cup A \subseteq X\}$ is dense in $\mathcal{H}$, and so

$$V[\dot{\mathcal{U}}] \models \mathcal{F} \subseteq \Phi(\dot{\mathcal{U}}). \tag{5.1}$$

Thus the composition $\mathcal{H} * P(\dot{\mathcal{U}})$ diagonalizes $\mathcal{F}$ by part 4 of Proposition 2.3.

Now let $\mathcal{V}$ be a P–point in the ground model. By Theorem 4, it suffices to prove that

$$V[\dot{\mathcal{U}}] \models \Phi(\dot{\mathcal{U}}) \not\leq_{\mathrm{RB}} \mathcal{V}. \tag{5.2}$$

Since forcing with $\mathcal{H}$ adjoins no new reals, it suffices to prove that for every finite–to–one $f \in {}^\omega\omega$ in the ground model

$$\{A \in \mathcal{H} : f[\cup A] \notin f(\mathcal{V})\} \text{ is dense in } \mathcal{H}. \tag{5.3}$$

So let $A \in \mathcal{H}$ and finite–to–one f be given. Define a partition $F_0 : [A]^2_< \to 2$ by setting $F_0(s < t) = 1$ if and only if $f[s] \cap f[t] = \emptyset$. There is an extension B of A in $\mathcal{H}$ with $F_0 \restriction [B]^2_<$ constant; since f is finite–to–one the constant value must be 1.

Next define $F_1 : \mathrm{FU}(B) \to 2$ by setting $F(s) = 1$ if and only if s is the union of at least two members of B. There is an extension C of B in $\mathcal{H}$ such that $F_1 \restriction \mathrm{FU}(C)$ is constant. Again, this constant value must be 1.

If $C = \{c_n : n \in \omega\}$, then for each n we can find (non–empty) d_n and e_n in $\mathrm{FU}(B)$ such that $c_n = d_n \cup e_n$. This is because F_1 is constant with value 1 on $\mathrm{FU}(C)$. Let $D = \{d_n : n \in \omega\}$ and $E = \{e_n : n \in \omega\}$. By part 1 of the above claim, both D and E are $\bar{\mathcal{F}}$–fat because C is. By construction, both D and E are therefore extensions of B (not of C!) in $\mathcal{H}$.

By our choice of B, we have $f[\cup D] \cap f[\cup E] = \emptyset$. This means that one of $f[\cup D]$ and $f[\cup E]$ is not in $f(\mathcal{V})$. Without loss of generality assume that $f[\cup D] \notin f(\mathcal{V})$. Since D is an extension of A in $\mathcal{H}$, we have established (5.3). □

Corollary 5.1. If $\mathcal{F}$ is a filter on ω generated by $< \mathrm{cov}(B)$ sets, then we can force the existence of a set $X \subseteq \omega$ almost included in each member of $\mathcal{F}$ and ensure that every P–point in the ground model continues to generate an ultrafilter in the generic extension.

Proof. Assume that $\mathcal{F}$ is generated by sets $\{X_\alpha : \alpha < \kappa < \mathrm{cov}(B)\}$. Let $\bar{\mathcal{F}}$ be the ordered–union filter generated by sets of the form $[X_\alpha]^{<\omega}$ for $\alpha < \kappa$. By Theorem 6, the collection $\mathcal{H}$ of $\bar{\mathcal{F}}$–fat sets is Matet–adequate. Now we apply the previous theorem. □

The above arguments suggest a line of attack at the old problem of whether $\mathfrak{u} < \mathfrak{a}$ is consistent. If one can somehow arrange (in ZFC or after a preliminary forcing that preserves P–points) that the filter dual to a maximal almost disjoint family is contained in a filter $\mathcal{F}$ with the property that the family of $\bar{\mathcal{F}}$–fat sets is Matet–adequate, then we can destroy the MAD family without destroying P–points from the ground model. This was the original motivation behind the research of this paper, although the author wasn't able to get very far along this avenue.

Our last application is a characterization of the statement "$\mathrm{cov}(B) = \mathfrak{c}$" in the same spirit as that obtained indepenently Canjar [**?**] and Bartoszynski–Judah (see Theorem 4.5.6 of [**?**]). We will also need at some point the following result of Blass and Hindman.

Proposition 5.2 (Corollary of Theorem 2 in [**?**])**.** If $\mathcal{U}$ is an ordered–union ultrafilter, then $\min(\mathcal{U})$ and $\max(\mathcal{U})$ are selective ultrafilters.

Recall that we think of min and max as functions from $\mathbb{F}$ to ω, so that $\min(\mathcal{U})$ and $\max(\mathcal{U})$ are ultrafilters on ω.

Theorem 8. *The following statements are equivalent:*

(1) $\operatorname{cov}(B) = \mathfrak{c}$
(2) *Every ordered–union filter generated by* $< \mathfrak{c}$ *sets can be extended to a stable ordered–union ultrafilter.*
(3) *Every ordered–union filter generated by* $< \mathfrak{c}$ *sets can be extended to an ordered–union ultrafilter.*
(4) *Every filter on* ω *generated by* $< \mathfrak{c}$ *sets can be extended to a selective ultrafilter.*

Proof. The implication (2) implies (3) is trivial, and the implication (4) implies (1) is Theorem 4.5.6 of [**?**].

We take care of (3) implies (4) first. Let $\mathcal{F}$ be a filter on ω generated by $\{X_\alpha : \alpha < \kappa < \mathfrak{c}\}$. We note that $\{[X_\alpha]^{<\omega} : \alpha < \kappa\}$ generates an ordered–union filter $\mathcal{G}$ on $\mathbb{F}$ as $[X_\alpha]^{<\omega} = \mathrm{FU}(\{\{n\} : n \in X\}$, and therefore by our assumptions we can find an ordered–union ultrafilter $\mathcal{U}$ that extends $\mathcal{G}$. By Proposition 5.2, the ultrafilter $\min(\mathcal{U})$ on ω is a selective ultrafilter, and since $\mathcal{U}$ contains $[X_\alpha]^{<\omega}$ for each $\alpha < \kappa$, $\min(\mathcal{U})$ extends $\mathcal{F}$.

For (1) implies (2), let $\mathcal{F}$ be an ordered–union filter generated by $< \mathfrak{c}$ sets. Fix an enumeration $\{f_\alpha : \alpha < \mathfrak{c}\}$ of all functions $f : [\mathbb{F}]^2_< \to 2$. For each $\alpha < \mathfrak{c}$, we will define an ordered–union filter $\mathcal{F}_\alpha$ so that

(1) $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{F}_{\alpha+1} \supseteq \mathcal{F}_\alpha$
(2) $\mathcal{F}_\alpha$ is generated by $< \mathfrak{c}$ sets
(3) for limit λ, $\mathcal{F}_\lambda$ is the union of all previous $\mathcal{F}_\alpha$
(4) $\mathcal{F}_{\alpha+1}$ will contain a set X with $f_\alpha \restriction [X]^2_<$ constant.

First we note that limit stages of the construction cause us no problem, so we concentrate instead on the successor stages. Given $\mathcal{F}_\alpha$ and f_α, we can apply Corollary 4.5 to get $D \in (\mathbb{F})^\omega$ that is $\mathcal{F}_\alpha$–fat and such that $f_\alpha \restriction [\mathrm{FU}(D)]^2_<$ is constant. The $\mathcal{F}_\alpha$–fatness of D ensures that $\mathcal{F}_\alpha \cup \{\mathrm{FU}(D)\}$ is contained in an ordered–union filter (generated by $< \mathfrak{c}$ sets) just as in the proof of Theorem 6, and hence the construction continues.

In the end, let $\mathcal{U}$ be the union of all the $\mathcal{F}_\alpha$'s. $\mathcal{U}$ is certainly an ordered–union filter that contains sets homogeneous for each partition of $[\mathbb{F}]^2_<$, and this is easily seen to imply that $\mathcal{U}$ is an ultrafilter. By part 2 of Theorem 3, $\mathcal{U}$ is a stable ordered–union ultrafilter. Since $\mathcal{U}$ extends $\mathcal{F}$, we are done. □

References

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