GETTING MORE COLORS II

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Abstract. We formulate and prove (in ZFC) a strong coloring theorem which holds at successors of singular cardinals, and use it to answer several questions concerning Shelah's principle $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$ for singular μ.

1. Introduction

In earlier work (see [4]), we obtained the following coloring theorem for successors of singular cardinals:

> Let μ be singular. There is a function $D : [\mu^+]^2 \to [\mu^+]^2 \times \mathrm{cf}(\mu)$ such that for any unbounded $A \subseteq \mu^+$, there is a stationary $S \subseteq \mu^+$ with

$$[S]^2 \times \mathrm{cf}(\mu) \subseteq \mathrm{ran}(D \restriction [A]^2). \tag{1.1}$$

This paper arose out of an attempt at finding similar colorings with even stronger properties. This attempt was successful, with the happy consequence that we can use this new coloring theorem to settle a few questions which have arisen in the author's previous work.

Before proceeding any further, we need to alert the reader to one of our notational conventions.

Definition 1.1. If A and B are sets of ordinals, then we define

$$A \circledast B := \{\langle \alpha, \beta \rangle \in A \times B : \alpha < \beta\}. \tag{1.2}$$

If C is also a set of ordinals, then we define

$$A \circledast B \times C := \{\langle \alpha, \beta, \gamma \rangle : \langle \alpha, \beta \rangle \in A \circledast B \text{ and } \gamma \in C\} \tag{1.3}$$

This notation is a bit *ad hoc* (especially (1.3)), but it makes it much easier to state our main theorem. Note that if κ is a cardinal, then it is quite common to identify $[\kappa]^2$ with those ordered pairs $\langle \alpha, \beta \rangle$ for which $\alpha < \beta$. In our notation, $[\kappa]^2$ therefore corresponds exactly with $\kappa \circledast \kappa$.

Moving on, we can now state the principal result of this paper:

Main Theorem. *Let μ be a singular cardinal. There is a function*

$$D : [\mu^+]^2 \to \mu^+ \times \mu^+ \times \mathrm{cf}(\mu) \tag{1.4}$$

such that whenever $\langle t_\alpha : \alpha < \mu^+ \rangle$ is a family of pairwise disjoint members of $[\mu^+]^{<\mathrm{cf}(\mu)}$, there are stationary subsets S and T of μ^+ such that whenever

$$\langle \alpha^*, \beta^*, \delta \rangle \in S \circledast T \times \mathrm{cf}(\mu), \tag{1.5}$$

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there are $\alpha < \beta < \mu^+$ such that

$$D \restriction t_\alpha \times t_\beta \text{ is constant with value } \langle \alpha^*, \beta^*, \delta \rangle. \tag{1.6}$$

At this point, the reader may well be asking "So what?", as this theorem seems at first glance to be only a much more technical version of the earlier result, which was quite technical in its own right. This first impression is misleading, though. In the first place, we note that this theorem is a ZFC result — a bit of a rarity given the mystery which still clouds the subject of partition relations at successors of singular cardinals. More importantly, though, it turns that this result is powerful enough to have some important consequences for combinatorial set theory. In particular, it puts us in a position to answer several questions concerning a family of combinatorial principles studied extensively by Shelah in his book [11]:

Definition 1.2. Suppose $\kappa+\theta \leq \mu \leq \lambda$ are cardinals with λ infinite. $\mathrm{Pr}_1(\lambda, \mu, \kappa, \theta)$ means that there is a symmetric two-place function c from λ to κ such that whenever $\xi < \theta$ and for each $i < \mu$, $\langle \alpha_{i,\zeta} : \zeta < \xi \rangle$ is a strictly increasing sequence of ordinals less than λ with the $\alpha_{i,\zeta}$ distinct, then for each $\gamma < \kappa$ there are $i < j < \mu$ such that

$$\zeta_1 < \xi \wedge \zeta_2 < \xi \Rightarrow c(\alpha_{i,\zeta_1}, \alpha_{j,\zeta_2}) = \gamma. \tag{1.7}$$

The above definition is lifted almost verbatim from [11], but it is much more general than we need. The following elementary proposition gives an equivalent formulation of Pr_1 in the specific cases of interest to us here:

Proposition 1.3. Suppose μ is a singular cardinal and $\theta \leq \mu^+$. The principle $\mathrm{Pr}_1(\mu^+, \mu^+, \theta, \mathrm{cf}(\mu))$ holds if and only if there is a function $f : [\mu^+]^2 \to \theta$ such that whenever $\langle t_\alpha : \alpha < \mu^+ \rangle$ is a sequence of pairwise disjoint elements of $[\mu^+]^{<\mathrm{cf}(\mu)}$, then for any $\epsilon < \theta$ we can find $\alpha < \beta < \mu^+$ such that $f \restriction t_\alpha \times t_\beta$ is constant with value ϵ.

Note as well that $\mathrm{Pr}_1(\mu^+, \mu^+, \theta, \mathrm{cf}(\mu))$ is a very strong version of more standard negative square-brackets partition relations, as the following implications hold:

$$\mathrm{Pr}_1(\mu^+, \mu^+, \theta, \mathrm{cf}(\mu)) \Longrightarrow \mu^+ \nrightarrow [\mu^+]^2_\theta \Longrightarrow \mu^+ \nrightarrow [\mu^+]^{<\omega}_\theta. \tag{1.8}$$

What sorts of consequences can we deduce from our main theorem? One example of importance to the author's work is the equivalence of the two statements

$$\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu)) \tag{1.9}$$

and

$$\mathrm{Pr}_1(\mu^+, \mu^+, \mu, \mathrm{cf}(\mu)). \tag{1.10}$$

The equivalence of these two statements answers a question that has been around since Shelah's original work on [7] in the mid 1990s. This is important because in many situations it is the first statement which is sought, while the proof only establishes the second. In some situations, an *ad hoc* argument depending on the nature of the function obtained to witness (1.10) has been given to allow one to obtain (1.9). For example, this is what occurs in [7]. In other instances, however, the function constructed to verify (1.10) did not seem to admit such an upgrade, a set of circumstances which occurred in the author's [3] and Chapter IV of Shelah's [11].

The equivalence of (1.9) and (1.10) removes these concerns, and allows us to resolve the associated questions. For example, we can show now that

$$\mathrm{pp}(\mu) = \mu^+ \Longrightarrow \mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu)), \tag{1.11}$$

extending a chain of theorems containing results of Erdős and Hajnal, Shelah, and Todorčević. Another consequence is that the main theorem of [3] can be fully extended to cover singular cardinals of countable cofinality, and this has consequences for stationary reflection. We will discuss these matters in much more detail in the final section of the paper, as well as obtaining several other related results. Readers willing to accept our main theorem as a "black box" can certainly the last section with no problems.

We also point out that recently Assaf Rinot [10] has refined our techniques a bit and discovered a coloring he names "rectangles to squares" that, when combined with our main theorem, establishes that the principle $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$ is equivalent to the much simpler $\mu^+ \nrightarrow [\mu^+]^2_{\mu^+}$ for singular μ.

2. Background material

Finally, the background material we need in this paper is almost identical to that required for the results of its predecessor [4], so we will be a little lazy and assume that our reader is familiar with the notation and definitions laid out in Section 2 of [4] regarding minimal walks on generalized C-sequences.

We will, however, rehash the club-guessing background because it is so crucial for our arguments, which make use of generalized C-sequences that have been carefully selected to interact with certain club-guessing sequences. The sort of club-guessing sequence we use depends on whether or not the cofinality of our singular cardinal μ is uncountable, and we will handle each case separately. In either case, we will be defining a stationary set $S \subseteq \mu^+$, a club-guessing sequence $\bar{C}$, and a generalized C-sequence $\bar{e}$.

If the cofinality of μ is uncountable, then we define

$$S := S^{\mu^+}_{\mathrm{cf}(\mu)} = \{\delta < \mu^+ : \mathrm{cf}(\delta) = \mathrm{cf}(\mu)\}. \tag{2.1}$$

Claim 2.6 on page 127 of [11] (or see Theorem 2 of [8]) gives us a sequence $\langle C_\delta : \delta \in S\rangle$ such that

- C_δ is club in δ,
- $\mathrm{otp}(C_\delta) = \mathrm{cf}(\mu)$,
- $\langle \mathrm{cf}(\alpha) : \alpha \in \mathrm{nacc}(C_\delta)\rangle$ increases to μ, and
- whenever E is club in μ^+, there are stationarily many $\delta \in S$ for which $C_\delta \subseteq E$.

(Recall "$\mathrm{nacc}(C_\delta)$" refers to the non-accumulation points of C_δ, that is, those elements of C_δ that are not limits of points in C_δ.)

We now use the "ladder swallowing" trick (see Lemma 13 of [5]) to build a C-sequence $\langle e_\alpha : \alpha < \mu^+\rangle$ such that for each $\alpha < \mu^+$,

$$|e_\alpha| < \mu, \tag{2.2}$$

and

$$\delta \in S \cap e_\alpha \Longrightarrow C_\delta \subseteq e_\alpha. \tag{2.3}$$

We then construct a (admittedly somewhat trivial) generalized C-sequence $\bar{e} = \langle e^m_\alpha : m < \omega, \alpha < \mu^+\rangle$ by setting $e^m_\alpha = e_\alpha$ for all $m < \omega$.

In the case where μ is of countable cofinality, our definition of S, $\bar{C}$, and $\bar{e}$ is a little more involved because of some open questions concerning club-guessing. A reader interested in these issues can find a more detailed discussion in [8], but we shall rely on technology developed in [3].

Start by setting

$$S := S^{\mu^+}_{\aleph_1} = \{\delta < \mu^+ : \mathrm{cf}(\delta) = \aleph_1\}, \tag{2.4}$$

and assume $\langle \mu_m : m < \omega\rangle$ is an increasing sequence of uncountable cardinals cofinal in μ.

We are going to present a simplified version of the conclusion of Theorem 4 of [3]; the reader can consult that paper for a detailed proof (Proposition 5.8 is particularly relevant). In particular, the work in [3] provides us with a sequence $\langle C_\delta : \delta \in S\rangle$ such that each C_δ is club in δ, and $C_\delta = \bigcup_{m<\omega} C_\delta[m]$ where

$$C_\delta[m] \text{ is closed and unbounded in } \delta, \tag{2.5}$$

$$|C_\delta[m]| \leq \mu_m^+, \tag{2.6}$$

and such that for every club $E \subseteq \mu^+$, there are stationarily many $\delta \in S$ such that for each $m < \omega$, $\mathrm{nacc}(C_\delta[m]) \cap E$ contains unboundedly many ordinals of cofinality greater than μ_m^+. (Note that the use of "nacc" is redundant as the cofinality assumption guarantees such an ordinal cannot be a limit point of $C_\delta[m]$.)

An application of Lemma 5.10 from [3] provides us with a generalized C-sequence $\bar{e} = \langle e^m_\alpha : \alpha < \mu^+, m < \omega\rangle$ satisfying

$$|e^m_\alpha| \leq \mathrm{cf}(\alpha) + \mu_m^+ \tag{2.7}$$

and

$$\delta \in S \cap e^m_\alpha \Longrightarrow C_\delta[m] \subseteq e^m_\alpha. \tag{2.8}$$

No matter what the cofinality of μ, the reader should give the phrase "choose $\delta \in S$ such that C_δ guesses E" the obvious interpretation in light of the above discussion.

Our notation regarding scales $(\vec{\mu}, \vec{f})$ at successors of singular cardinals is fairly standard, the reader can consult [4] for a refresher if needed. The arguments in this paper require a little more sophistication in scale combinatorics, however. In particular, the following result from [5] will be needed:

Lemma 2.1. Let μ be singular, and suppose $(\vec{\mu}, \vec{f})$ is a scale for μ. Then there is a closed unbounded $C \subseteq \mu^+$ such that the following holds for every $\beta \in C$:

$$(\forall^* i < \mathrm{cf}(\mu))(\forall \eta < \mu_i)(\forall \nu < \mu_{i+1})(\exists^* \alpha < \beta)\,[f_\alpha(i) > \eta \wedge f_\alpha(i+1) > \nu]\,. \tag{2.9}$$

Proof. The first step is to establish the following statement:

$$(\forall^* i < \mathrm{cf}(\mu))(\forall \eta < \mu_i)(\forall \nu < \mu_{i+1})(\exists^* \alpha < \mu^+)[f_\alpha(i) > \eta \wedge f_\alpha(i+1) > \nu]. \tag{2.10}$$

Assume by way of contradiction that the above statement fails. It follows that there is an unbounded $I \subseteq \kappa$ (without loss of generality satisfying $i \in I \rightarrow i+1 \notin I$) such that a "bad pair" (η_i, ν_i) exists for every $i \in I$. We now define a function $f \in \prod_{i<\mathrm{cf}(\mu)} \mu_i$ by

$$f(i) = \begin{cases} \eta_i & i \in I \\ \nu_{i-1} & i-1 \in I \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

Upon consideration of the negation of 2.10, we see that for each $i \in I$, there is an $\alpha_i < \mu^+$ such that for each α satisfying $\alpha_i \leq \alpha < \mu^+$, either $f_\alpha(i) \leq \eta_i$ or $f_\alpha(i+1) \leq \nu_i$. But we can choose $\alpha^* < \mu^+$ greater than each such α_i and such that $f <^* f_{\alpha^*}$, and a contradiction falls out immediately.

Now that we have (2.10), fix $i^* < \mathrm{cf}(\mu)$ large enough so that for all i in the interval $[i^*, \mathrm{cf}(\mu))$ we have

$$(\forall \eta < \mu_i)(\forall \nu < \mu_{i+1})(\exists^* \alpha < \mu^+)[f_\alpha(i) > \eta \wedge f_\alpha(i+1) > \nu]. \tag{2.12}$$

For each i satisfying $i^* \leq i < \mathrm{cf}(\mu)$ and each pair $(\eta, \nu) \in \mu_i \times \mu_{i+1}$, we let $A^i(\eta, \nu)$ be the set of $\alpha < \mu^+$ satisfying

$$\eta < f_\alpha(i) \text{ and } \nu < f_\alpha(i+1). \tag{2.13}$$

Each such $A^i(\eta, \mu)$ is unbounded in μ^+, and since there are only μ sets of this form, the collection of $\beta < \mu^+$ that are limit points of all $A^i(\eta, \nu)$ simultaneously is closed and unbounded in μ^+ as required. □

3. The minimal walk lemma

This section begins with a technical *ad hoc* definition. The definition actually depends on a generalized C-sequence $\bar{e}$, a scale $(\vec{\mu}, \vec{f})$, and a sequence $\langle t_\alpha : \alpha < \mu^+ \rangle$ of pairwise disjoint elements of $[\mu^+]^{<\mathrm{cf}(\mu)}$, but we suppress this dependence in the notation. We also remind the reader that our notation concerning minimal walks is exactly as laid out in Section 2 of [4].

Definition 3.1. Suppose $k < \mathrm{cf}(\mu)$ and $m < \omega$. The formula $\psi_{k,m}(\eta, \eta^+, \beta^*, \gamma, \beta)$ asserts

$$\eta \leq \eta^+ < \beta^* < \gamma \leq \min(t_\beta), \tag{3.1}$$

$$\eta = \min\{\eta^m(\beta^*, \epsilon) : \epsilon \in t_\beta\}, \tag{3.2}$$

$$\eta^+ = \sup\{\eta^m(\beta^*, \epsilon) : \epsilon \in t_\beta\}, \tag{3.3}$$

$$\gamma = \min\{\gamma^m(\beta^*, \epsilon) : \epsilon \in t_\beta\}, \tag{3.4}$$

and

$$(\forall \epsilon \in t_\beta)\left[f_\eta \restriction [k, \mathrm{cf}(\mu)) \leq f_{\eta^m(\beta^*, \epsilon)} \restriction [k, \mathrm{cf}(\mu)) \leq f_{\eta^+} \restriction [k, \mathrm{cf}(\mu))\right]. \tag{3.5}$$

The above definition probably seems quite mysterious, but it isolates a certain configuration whose importance will become clear as the proof progresses. The next result is one of two technical lemmas crucial for our arguments.

Lemma 3.2. Suppose μ is a singular cardinal, and S, $\bar{C}$, and $\bar{e}$ are as in the preceding section. Further suppose that $(\vec{\mu}, \vec{f})$ is a scale for μ and $\langle t_\beta : \beta < \mu^+ \rangle$ is a sequence of pairwise disjoint elements of $[\mu^+]^{<\mathrm{cf}(\mu)}$. Then there are an $m < \omega$ and $k < \mathrm{cf}(\mu)$ such that

$$(\exists^* \eta < \mu^+)(\exists \eta^+ < \mu^+)(\exists^{\mathrm{stat}} \beta^* < \mu^+) (\exists^* \gamma < \mu^+)(\exists \beta < \mu^+)[\psi_{k,m}(\eta, \eta^+, \beta^*, \gamma, \beta)]. \tag{3.6}$$

Proof. The proof of this lemma comes in two stages: first, we will find specific objects satisfying the appropriate instance of ψ, and then we use standard elementary submodel arguments to prove that there are "many" objects which do the job.

We start by defining $x := \{\mu, S, (\vec{\mu}, \vec{f})), \bar{C}, \bar{e}, \langle t_\beta : \beta < \mu^+\rangle\}$, that is, x consists of the parameters needed to comprehend ψ. We then let $\langle M_\alpha : \alpha < \mu^+\rangle$ be a μ^+-approximating sequence over x. It is clear that the set

$$E := \{\delta < \mu^+ : M_\delta \cap \mu^+ = \delta\} \tag{3.7}$$

is closed and unbounded in μ^+, so can fix $\delta \in S$ for which C_δ guesses E.

Choose β such that

$$\delta < \min(t_\beta), \tag{3.8}$$

and define

$$\gamma^\otimes = \sup\{\gamma^{m,-}(\delta, \epsilon) : \epsilon \in t_\beta, m < \omega\}. \tag{3.9}$$

We observe

$$\max\{\gamma^\otimes, M_0 \cap \mu^+\} < \delta, \tag{3.10}$$

as $\delta \in E$ and $|t_\beta| + \aleph_0 < \mathrm{cf}(\delta)$. (Recall here that $\mathrm{cf}(\delta)$ is uncountable and $\mathrm{cf}(\mu) \leq \mathrm{cf}(\delta)$ by our choice of S.)

Proposition 3.3. We can find $\beta^* < \delta$ and $m < \omega$ such that

$$\beta^* \in \mathrm{nacc}(C_\delta[m]) \cap E, \tag{3.11}$$

$$\mathrm{cf}(\beta^*) > \mathrm{cf}(\mu) + \sup\{|e^m_{\gamma^m(\delta,\epsilon)}| : \epsilon \in t_\beta\}, \tag{3.12}$$

and

$$\max\{\gamma^\otimes, M_0 \cap \mu^+\} < \max(C_\delta[m] \cap \beta^*). \tag{3.13}$$

Proof. Our first observation is that for any $m < \omega$, we have

$$\mathrm{cf}(\mu) + \sup\{|e^m_{\gamma^m(\delta,\epsilon)}| : \epsilon \in t_\beta\} < \mu \tag{3.14}$$

as $|t_\beta| < \mathrm{cf}(\mu)$.

If μ is of uncountable cofinality, then we set $m = 1$ (recall that $\bar{e}$ is somewhat trivial in this case). Our choice of $\bar{C}$ and δ tells us that all sufficiently large elements of $\mathrm{nacc}(C_\delta)$ satisfy (3.11) and (3.12), and so we can easily find β^* satisfying (3.13) as well.

If the cofinality of μ is countable, then we note that t_β is finite. Thus, we can fix a single $m_0 < \omega$ such that

$$\langle \beta^m_i(\delta, \epsilon) : i < \rho^m_2(\delta, \epsilon)\rangle = \langle \beta^{m_0}_i(\delta, \epsilon) : i < \rho^{m_0}_2(\delta, \epsilon)\rangle \tag{3.15}$$

whenever $\epsilon \in t_\beta$ and $m \geq m_0$.

Now choose $m \geq m_0$ such that

$$\mu^+_m \geq \max\{\mathrm{cf}(\gamma^{m_0}(\delta, \epsilon)) : \epsilon \in t_\beta\}. \tag{3.16}$$

In light of (3.15), we see

$$\mu^+_m \geq \max\{\mathrm{cf}(\gamma^m(\delta, \epsilon)) : \epsilon \in t_\beta\}. \tag{3.17}$$

Now $\mathrm{nacc}(C_\delta[m]) \cap E$ contains unboundedly many elements of cofinality greater than μ^+_m, so we can find a β^* with the required properties. □

Proposition 3.4. For any $\epsilon \in t_\beta$, we have

$$(\forall i < \rho_2^m(\delta,\epsilon))\,[\beta_i^m(\beta^*,\epsilon) = \beta_i^m(\delta,\epsilon)]\,, \tag{3.18}$$

$$\rho_m^2(\beta^*,\epsilon) = \rho_m^2(\delta,\epsilon), \tag{3.19}$$

$$\gamma^m(\beta^*,\epsilon) = \gamma^m(\delta,\epsilon), \tag{3.20}$$

and

$$\beta^* \in \operatorname{nacc}(e^m_{\gamma^m(\delta,\epsilon)}). \tag{3.21}$$

Proof. The first statement holds as $\gamma^\otimes < \beta^* < \delta$. Given $\epsilon \in t_\beta$, we note that $\delta \in e^m_{\gamma^m(\delta,\epsilon)}$ by definition, and so $C_\delta[m] \subseteq e^m_{\gamma^m(\delta,\epsilon)}$ by our choice of $\bar{e}$. It follows that $\beta^* \in e^m_{\gamma^m(\delta,\epsilon)}$ and both (3.19) and (3.20) are immediate. Given that $\beta^* \in e^m_{\gamma^m(\delta,\epsilon)}$ was established earlier in the proof, the statement (3.21) follows from (3.12). □

Now that we have isolated β^* and m, we define

$$\eta := \min\{\eta^m(\beta^*,\epsilon) : \epsilon \in t_\beta\}, \tag{3.22}$$

$$\eta^+ := \sup\{\eta^m(\beta^*,\epsilon) : \epsilon \in t_\beta\}, \tag{3.23}$$

and

$$\gamma := \min\{\gamma^m(\beta^*,\epsilon) : \epsilon \in t_\beta\}. \tag{3.24}$$

Proposition 3.5. We have

$$M_0 \cap \mu^+ < \eta \le \eta^+ < \beta^* = M_{\beta^*} \cap \mu^+ < \delta = M_\delta \cap \mu^+ < \gamma. \tag{3.25}$$

Proof. It is clear that $\eta \le \eta^+ \le \beta^* < \delta$. By our choice of β^* we know

$$M_0 \cap \mu^+ < \max(C_\delta[m] \cap \beta^*).$$

Since $C_\delta[m] \subseteq e^m_{\gamma^m(\delta,\epsilon)} = e^m_{\gamma^m(\beta^*,\epsilon)}$ for each $\epsilon \in t_\beta$, it follows that

$$M_0 \cap \mu^+ < \max(C_\delta[m] \cap \beta^*) \le \eta.$$

By (3.20) and (3.21), we know that $\eta^m(\beta^*,\epsilon) < \beta^*$ for every $\epsilon \in t_\beta$. Since $\operatorname{cf}(\beta^*) > |t_\beta|$, it follows that $\eta^+ < \beta^*$.

Finally, (3.20) and the definition of γ imply that $\delta < \gamma$, and the proof is complete. □

Now let $A := \{\eta^m(\beta^*,\epsilon) : \epsilon \in t_\beta\}$. It is clear that $|A| < \operatorname{cf}(\mu)$ so there must exist a single $k < \operatorname{cf}(\mu)$ such that

$$\xi_0 < \xi_1 \text{ in } A \Longrightarrow f_{\xi_0} \restriction [k, \operatorname{cf}(\mu)) < f_{\xi_1} \restriction [k, \operatorname{cf}(\mu)).$$

Given this choice of k together with the preceding work, we see that

$$\psi_{k,m}(\eta,\eta^+,\beta^*,\gamma,\beta) \tag{3.26}$$

holds.

We now finish the proof of the lemma by a standard elementary submodel argument. Since $x \cup \{\eta,\eta^+,\beta^*,k,m\} \in M_\delta$ and $\gamma > M_\delta \cap \mu^+$, we know

$$(\exists^*\gamma < \mu^+)(\exists\beta < \mu^+)[\psi_{k,m}(\eta,\eta^+,\beta^*,\gamma,\beta)].$$

Since $x \cup \{\eta,\eta^+,k,m\} \in M_{\beta^*}$ and $\beta^* = M_{\beta^*} \cap \mu^+$, we know

$$(\exists^{\text{stat}}\beta^* < \mu^+)(\exists^*\gamma < \mu^+)(\exists\beta < \mu^+)[\psi_{k,m}(\eta,\eta^+,\beta^*,\gamma,\beta)].$$

Finally, since $M_0 \cap \mu^+ < \eta$ and $x \cup \{k, m\} \in M_0$, we conclude

$$(\exists^* \eta < \mu^+)(\exists \eta^+ < \mu^+)(\exists^{\text{stat}} \beta^* < \mu^+)(\exists^* \gamma < \mu^+)(\exists \beta < \mu^+)[\psi_{k,m}(\eta, \eta^+, \beta^*, \gamma, \beta)],$$

as required.

□

4. The Γ lemma

This section makes heavy use of scales and the associated combinatorics, so let us begin by assuming $(\vec{\mu}, \vec{f})$ is a scale for some singular cardinal μ. There is a natural (and well-known) function $\Gamma : [\mu^+]^2 \to \text{cf}(\mu)$ associated with our scale, namely

$$\Gamma(\alpha, \beta) := \sup\{i < \text{cf}(\mu) : f_\beta(i) \leq f_\alpha(i)\}. \tag{4.1}$$

Notice that $\Gamma(\alpha, \beta)$ is defined whenever $\alpha < \beta < \mu^+$ because $(\vec{\mu}, \vec{f})$ is a scale. We make the convention that $\Gamma(\alpha, \alpha)$ is defined and equal to some symbol ∞ as well.

We also define a partial function $\Gamma^+ : [\mu^+]^2 \to \text{cf}(\mu)$ by setting

$$\Gamma^+(\alpha, \beta) := \max\{i < \text{cf}(\mu) : f_\beta(i) \leq f_\alpha(i)\}, \tag{4.2}$$

should this maximum exist, and leaving Γ^+ undefined in all other situations. Clearly Γ and Γ^+ agree whenever the latter function is defined.

Lemma 4.1. Assume μ is a singular cardinal and $(\vec{\mu}, \vec{f})$ is a scale for μ. Further assume:

- $M_0 \in M_1 \in M_2$ are elementary submodels of $\mathfrak{A}$ of cardinality μ such that $M_i \cap \mu^+$ is an initial segment of μ^+,
- $(\vec{\mu}, \vec{f}) \in M_0$,
- $\beta^* = M_0 \cap \mu^+$,
- $\bar{s} = \langle s_\alpha : \alpha < \mu^+ \rangle$ is a sequence of pairwise disjoint elements of $[\mu^+]^{<\text{cf}(\mu)}$ with $\bar{s} \in M_0$, and
- $t \in [\mu^+]^{<\text{cf}(\mu)}$ with $M_2 \cap \mu^+ \leq \min(t)$.

Then for all sufficiently large $i < \text{cf}(\mu)$, there are unboundedly many $\alpha < \beta^*$ such that for all $\epsilon_a \in s_\alpha$ and $\epsilon_b \in t$, we have

$$\Gamma^+(\epsilon_a, \epsilon_b) = i, \tag{4.3}$$

but

$$f_{\beta^*}(i+1) < f_{\epsilon_a}(i+1). \tag{4.4}$$

Proof. Our first observation is that it suffices to prove the lemma under the following assumptions about $\bar{s}$:

- $\alpha < \beta \implies \sup(t_\alpha) < \min(t_\beta)$,
- $\alpha \leq \min(t_\alpha)$, and
- there is an $i_0 < \text{cf}(\mu)$ such that for all $\alpha < \mu^+$,

$$f_{\min(t_\alpha)} \restriction [i_0, \text{cf}(\mu)) \leq f_\epsilon \restriction [i_0, \text{cf}(\mu)) \text{ for all } \epsilon \in t_\alpha. \tag{4.5}$$

The point is that given a sequence $\bar{s}$ as in the assumptions of the lemma, we can find an unbounded $J \subseteq \mu^+$ such that $\langle s_\alpha : \alpha \in J \rangle$ enjoys the three additional properties we want. (For the last property, we need to use that $|s_\alpha| < \text{cf}(\mu)$ for each α.) Thus, there will be such a J in M_0, and obtaining the conclusion for $\langle s_\alpha : \alpha \in J \rangle \in M_0$ is enough.

Next, we observe that if we define $\vec{g} = \langle g_\alpha : \alpha < \mu^+\rangle$ where

$$g_\alpha := f_{\min(t_\alpha)}, \tag{4.6}$$

then $(\vec{\mu}, \vec{g})$ forms a scale for μ. This new scale is definable from parameters in M_0, and hence $(\vec{\mu}, \vec{g})$ is an element of M_0 as well. In particular, we can apply Lemma 2.1 to $(\vec{\mu}, \vec{g})$ in the model M_0. Taken in conjunction with the fact that $\beta^* = M_0 \cap \mu^+$, we see that there is an $i_1 < \mathrm{cf}(\mu)$ such that whenever $i_1 \leq i < \mathrm{cf}(\mu)$, $\xi_0 < \mu_i$, and $\xi_1 < \mu_{i+1}$,

$$(\exists^* \alpha < \beta^*)[f_{\min(t_\alpha)}(i) > \xi_0 \wedge f_{\min(t_\alpha)}(i+1) > \xi_1]. \tag{4.7}$$

Now let C be a closed unbounded subset of β^* of order-type $\mathrm{cf}(\beta^*)$, and define

$$x = \{\mu, (\vec{\mu}, \vec{f}), \bar{s}, \beta^*\} \cup \mathrm{cf}(\mu) \cup C. \tag{4.8}$$

We define M to be the Skolem hull of x in the structure $\mathfrak{A}$.

Note that M is of cardinality $\max\{\mathrm{cf}(\beta^*), \mathrm{cf}(\mu)\}$, and $M \in M_2$ as M can be computed in M_2 by taking the hull of x using the restrictions of the Skolem functions to M_1. Let g denote the characteristic function of M in $\vec{\mu}$, that is,

$$g = \mathrm{Ch}_M^{\vec{\mu}}. \tag{4.9}$$

It is clear that

$$g \in M_2 \cap \prod_{i<\mathrm{cf}(\mu)} \mu_i. \tag{4.10}$$

Let $i_2 < \mathrm{cf}(\mu)$ be least with $|M| < \mu_{i_2}$, so whenever $i_2 \leq i < \mathrm{cf}(\mu)$ we have

$$g(i) = \sup(M \cap \mu_i). \tag{4.11}$$

Note as well that $C \subseteq M$ and so $M \cap \beta^*$ is unbounded in β^*.

Since $g \in M_2 \cap \prod_{i<\mathrm{cf}(\mu)} \mu_i$, it follows that $g <^* f_\gamma$ whenever $M_2 \cap \mu^+ \leq \gamma < \mu^+$. Since $|t| < \mathrm{cf}(\mu)$, we can fix $i_3 < \mathrm{cf}(\mu)$ such that

$$g \restriction [i_3, \mathrm{cf}(\mu)) < f_\epsilon \restriction [i_3, \mathrm{cf}(\mu)) \text{ for all } \epsilon \in t. \tag{4.12}$$

Now define

$$i^* = \max\{i_0, i_1, i_2, i_3\}. \tag{4.13}$$

We claim that whenever $i^* \leq i < \mathrm{cf}(\mu)$, there are unboundedly many $\alpha < \beta^*$ for which the conclusion of the lemma holds.

Let such an i be given, and define

$$N := \mathrm{Sk}_{\mathfrak{A}}(M \cup \mu_i), \tag{4.14}$$

and

$$h := \mathrm{Ch}_N^{\vec{\mu}}. \tag{4.15}$$

By a standard result (see Corollary 2.7 of [4], for example) we know

$$g \restriction [i+1, \mathrm{cf}(\mu)) = h \restriction [i+1, \mathrm{cf}(\mu)). \tag{4.16}$$

Now let us define

$$\xi_0 := \sup\{f_\epsilon(i) : \epsilon \in t\}, \tag{4.17}$$

and

$$\xi_1 := f_{\beta^*}(i+1). \tag{4.18}$$

Both ξ_0 and ξ_1 are in N — ξ_0 gets in as $\mu_i \subseteq N$, while ξ_1 is in N because β^* and the other parameters needed to define it are in N.

The next two claims constitute the heart of the matter:

Claim 1. Suppose $\alpha \in N \cap \beta^*$ satisfies

- $f_{\min(s_\alpha)}(i) > \xi_0$, and
- $f_{\min(s_\alpha)}(i+1) > \xi_1$.

Then for all $\epsilon_a \in s_\alpha$, we have

- $\Gamma^+(\epsilon_a, \epsilon_b) = i$ for all $\epsilon_b \in t$, and
- $f_{\beta^*}(i+1) < f_{\epsilon_a}(i+1)$.

Proof. Let $\alpha \in N \cap \beta^*$ satisfy the assumptions of the claim, and fix $\epsilon_a \in s_\alpha$. Since $i_0 \leq i$, we know

$$f_{\beta^*}(i+1) = \xi_1 < f_{\min(s_\alpha)}(i+1) \leq f_{\epsilon_a}(i+1), \tag{4.19}$$

and therefore $\Gamma(\epsilon_a, \beta^*) \neq i$.

Now fix $\epsilon_b \in t$. Since $i_0 \leq i$, we know

$$f_{\epsilon_b}(i) \leq \xi_0 < f_{\min(s_\alpha)}(i) \leq f_{\epsilon_a}(i). \tag{4.20}$$

On the other hand, since $s_\alpha \in N$ and $|s_\alpha| < \mathrm{cf}(\mu) \subseteq N$, we know that $\epsilon_a \in N$. Since $i \geq i_2$, it follows that

$$f_{\epsilon_a} \restriction [i+1, \mathrm{cf}(\mu)) \leq h \restriction [i+1, \mathrm{cf}(\mu)). \tag{4.21}$$

Since $i \geq i_3$, a glance at (4.12) and (4.16) shows us that

$$f_{\epsilon_a} \restriction [i+1, \mathrm{cf}(\mu)) \leq g \restriction [i+1, \mathrm{cf}(\mu)) < f_{\epsilon_b} \restriction [i+1, \mathrm{cf}(\mu)). \tag{4.22}$$

The conjunction of (4.20) and (4.22) establishes

$$\Gamma^+(\epsilon_a, \epsilon_b) = i, \tag{4.23}$$

finishing the proof. □

Claim 2. The set of $\alpha \in N \cap \beta^*$ satisfying the hypotheses of the preceding claim is unbounded in β^*.

Proof. The scale $(\vec{\mu}, \vec{g})$ is in N, and so (4.7) holds in N by elementarity. We have explicitly ensured that $N \cap \beta^*$ is unbounded in β^* (as we demanded $C \subseteq M \subseteq N$), and the result follows immediately after another application of elementarity. □

□

5. Main Theorem

We come now to the proof of the main theorem stated in the introduction. We remind the reader that our notation concerning various functions arising in minimal walks along generalized C-sequences is exactly as in [4].

Theorem 1 (Main Theorem). *Let μ be a singular cardinal. There is a function*

$$D : [\mu^+]^2 \to \mu^+ \times \mu^+ \times \mathrm{cf}(\mu) \tag{5.1}$$

such that whenever $\langle t_\alpha : \alpha < \mu^+ \rangle$ is a family of pairwise disjoint members of $[\mu^+]^{<\mathrm{cf}(\mu)}$, there are stationary subsets S and T of μ^+ such that whenever

$$\langle \alpha^*, \beta^*, \delta \rangle \in S \circledast T \times \mathrm{cf}(\mu), \tag{5.2}$$

there are $\alpha < \beta < \mu^+$ *such that*

$$D \restriction t_\alpha \times t_\beta \text{ is constant with value } \langle \alpha^*, \beta^*, \delta\rangle. \tag{5.3}$$

Proof. Let $(\vec{\mu}, \vec{f})$ be a scale for μ, and fix a generalized C-sequence $\bar{e}$ as in Lemma 3.2. Let $\iota : \mathrm{cf}(\mu) \to \omega \times \mathrm{cf}(\mu)$ be a function such that for any $m < \omega$ and $\delta < \mathrm{cf}(\mu)$, there are unboundedly many $i < \mathrm{cf}(\mu)$ with $\iota(i) = \langle m, \delta\rangle$.

Given $\alpha < \beta$, we let m and δ denote functions defined by the recipe

$$\iota(\Gamma(\alpha,\beta)) = \langle m(\alpha,\beta), \delta(\alpha,\beta)\rangle. \tag{5.4}$$

Following [5] and [8], given an natural number m, we let $c_m(\alpha,\beta)$ be defined as $\beta^m_i(\alpha,\beta)$, where i is the least number for which

$$\Gamma(\alpha,\beta) \neq \Gamma(\alpha, \beta^m_i(\alpha,\beta)). \tag{5.5}$$

Note that $c_m(\alpha,\beta)$ is always defined when $\alpha < \beta$ because of our convention that $\Gamma(\alpha,\alpha)$ is equal to ∞.

We now define

$$\beta^*(\alpha,\beta) = c_{m(\alpha,\beta)}(\alpha,\beta) \tag{5.6}$$

and

$$\eta^*(\alpha,\beta) = \eta^{m(\alpha,\beta)}(\beta^*(\alpha,\beta),\beta). \tag{5.7}$$

If $\eta^*(\alpha,\beta) < \alpha$, we define

$$\alpha^*(\alpha,\beta) = c_{m(\alpha,\beta)}(\eta^*(\alpha,\beta),\alpha), \tag{5.8}$$

and then set

$$D(\alpha,\beta) := \langle \alpha^*(\alpha,\beta), \beta^*(\alpha,\beta), \delta(\alpha,\beta)\rangle. \tag{5.9}$$

If $\eta^*(\alpha,\beta) \geq \alpha$ we define $D(\alpha,\beta)$ arbitrarily.

The informal description associated with the preceding definition is much clearer than the notation required to write it down precisely. Given $\alpha < \beta$, we compute $D(\alpha,\beta)$ in the following manner. First, we use Γ and ι to obtain a natural number m and an ordinal $\delta < \mathrm{cf}(\mu)$. The number m tells us which piece of the generalized C-sequence $\bar{e}$ we will be using for our minimal walks, while δ appears in the final output of D. The next step is to take the m-walk from β down to α until we reach a spot β^* where $\Gamma(\alpha,\beta^*)$ is different from $\Gamma(\alpha,\beta)$ — this is the ordinal $\beta^*(\alpha,\beta)$. Given β^*, we proceed as in the preceding section and isolate the ordinal $\eta^m(\beta^*,\beta)$, which we name $\eta^* = \eta^*(\alpha,\beta) \leq \beta^*$. If it happens that $\eta^* < \alpha$, then we m-walk from α down to η^* until we reach a point α^* where $\Gamma(\eta^*,\alpha^*)$ is different from $\Gamma(\eta^*,\alpha)$. The function D now returns the value $\langle\alpha^*,\beta^*,\delta\rangle$.

The rest of the proof consists in showing that the function D has the required properties, so let us assume $\langle t_\alpha : \alpha < \mu^+\rangle$ is a pairwise disjoint collection of members of $[\mu^+]^{<\mathrm{cf}(\mu)}$. After a bit of culling and re-indexing, we may assume that

$$\alpha \leq \min(t_\alpha), \tag{5.10}$$

and

$$\alpha < \beta \Longrightarrow \sup(t_\alpha) < \min(t_\beta). \tag{5.11}$$

We now apply Lemma 3.2 to obtain $m < \omega$ and $k < \mathrm{cf}(\mu)$ such that

$$(5.12)\quad (\exists^* \eta < \mu^+)(\exists \eta^+ < \mu^+)(\exists^{\mathrm{stat}} \beta^* < \mu^+)(\exists^* \gamma < \mu^+)(\exists \beta < \mu^+)[\psi_{k,m}(\eta, \eta^+, \beta^*, \gamma, \beta)],$$

and then fix ordinals $\eta_a \leq \eta_a^+$ such that

$$(5.13)\quad (\exists^{\mathrm{stat}} \alpha^* < \mu^+)(\exists^* \gamma < \mu^+)(\exists \alpha < \mu^+)[\psi_{k,m}(\eta_a, \eta_a^+, \alpha^*, \gamma, \alpha)].$$

The next definition and claim are quite technical, but they are critical for our argument.

Definition 5.1. We say that $\{\eta_b, \eta_b^+\}$ is an η-*candidate* if

- $\eta_a^+ < \eta_b \leq \eta_b^+ < \mu^+$, and
- $(\exists^{\mathrm{stat}} \beta^* < \mu^+)(\exists^* \gamma < \mu^+)(\exists \beta < \mu^+)[\psi_{k,m}(\eta_b, \eta_b^+, \beta^*, \gamma, \beta)]$.

We say that $\{\eta_b, \eta_b^+\}$ *works for* $\{\alpha^*, \alpha\}$ *at* i_a if

- $\eta_b \leq \eta_b^+ < \alpha^* < \min(t_\alpha)$,
- $i_a < \mathrm{cf}(\mu)$,
- $\Gamma^+(\eta, \beta_i^m(\alpha^*, \epsilon)) = i_a$ for all $\epsilon \in t_\alpha$, $i < \rho_2^m(\alpha^*, \epsilon)$, and $\eta \in \{\eta_b, \eta_b^+\}$, and
- $f_{\eta_b}(i_a + 1) > f_{\alpha^*}(i_a + 1)$.

The next lemma says that there are many η-candidates that will be suitable for our construction.

Lemma 5.2. For all sufficiently large $i_a < \mathrm{cf}(\mu)$, we can find an η-candidate $\{\eta_b, \eta_b^+\}$ such that

$$(5.14)\quad (\exists^{\mathrm{stat}} \alpha^* < \mu^+)(\exists^* \gamma < \mu^+)(\exists \alpha < \mu^+)[\psi_{k,m}(\eta_a, \eta_a^+, \alpha^*, \gamma, \alpha) \text{ and } \{\eta_b, \eta_b^+\} \text{ works for } \{\alpha^*, \alpha\} \text{ at } i_a].$$

Proof. Our choice of k and m tells us that there are unboundedly many η for which we can find an η^+ such that $\{\eta, \eta^+\}$ is an η-candidate, so we can construct a sequence $\langle s_\alpha : \alpha < \mu^+ \rangle$ of pairwise disjoint subsets of μ^+ such that each s_α is an η-candidate.

Let $M_0 \in M_1 \in M_2$ be elementary submodels of $\mathfrak{A}$ as in Lemma 4.1, chosen so that

$$(5.15)\quad \{\eta_a, \eta_a^+, \langle s_\alpha : \alpha < \mu^+ \rangle\} \in M_0,$$

and, for $\alpha^* = M_0 \cap \mu^+$,

$$(5.16)\quad (\exists^* \gamma < \mu^+)(\exists \alpha < \mu^+)[\psi_{k,m}(\eta_a, \eta_a^+, \alpha^*, \gamma, \alpha)].$$

This can be done because, by way of (5.13), there are stationarily many α^* satisfying (5.16).

Choose $\gamma \geq M_2 \cap \mu^+$ and $\alpha < \mu^+$ such that $\psi_{k,m}(\eta_a, \eta_a^+, \alpha^*, \gamma, \alpha)$ holds, and define

$$(5.17)\quad t := \{\beta_i^m(\alpha^*, \epsilon_a) : \epsilon_a \in t_\alpha \text{ and } i < \rho_2^m(\alpha^*, \epsilon_a)\}.$$

We can now apply Lemma 4.1 to conclude that for all sufficiently large $i_a < \mathrm{cf}(\mu)$, there is an $\alpha < \alpha^*$ such that

$$(5.18)\quad \Gamma^+(\eta, \epsilon) = i_a \text{ for all } \eta \in s_\alpha \text{ and } \epsilon \in t,$$

and

$$f_{\alpha^*}(i_a+1) < f_\eta(i_a+1) \text{ for all } \eta \in s_\alpha. \tag{5.19}$$

It should be clear that $s_\alpha = \{\eta_b, \eta_b^+\}$ works for $\{\alpha^*, \alpha\}$ at i_a. Since $\{\eta_b, \eta_b^+\} \in M_0$, the conclusion of the lemma now follows by a standard elementary submodel argument like that used to finish the proof of Lemma 3.2. □

In light of the preceding claim, we can fix $i_a > k$ and an η-candidate $\{\eta_b, \eta_b^+\}$ such that

$$(\exists^{\text{stat}}\alpha^* < \mu^+)(\exists^*\gamma < \mu^+)(\exists\alpha < \mu^+) \\ [\psi_{k,m}(\eta_a, \eta_a^+, \alpha^*, \gamma, \alpha) \text{ and } \{\eta_b, \eta_b^+\} \text{ works for } \{\alpha^*, \alpha\} \text{ at } i_a]. \tag{5.20}$$

Let us now define

$$S^{:} = \{\alpha^* < \mu^+ : (\exists^*\gamma < \mu^+)(\exists\alpha < \mu^+) \\ [\psi_{k,m}(\eta_a, \eta_a^+, \alpha^*, \gamma, \alpha) \text{ and } \{\eta_b, \eta_b^+\} \text{ works for } \{\alpha^*, \alpha\} \text{ at } i_a]\}$$

and

$$T^* := \{\beta^* < \mu^+ : (\exists^*\gamma < \mu^+)(\exists\beta < \mu^+)[\psi_{k,m}(\eta_b, \eta_b^+, \beta^*, \gamma, \beta)].$$

Our choices make it clear that S^* and T^* are both stationary (see Definition 5.1). We will thin out these sets a bit to obtain the promised stationary sets S and T. To do this, let

$$x := \{\mu, S, (\vec{\mu}, \vec{f}), \bar{C}, \bar{e}, \langle t_\beta : \beta < \mu^+\rangle\} \tag{5.21}$$

(that is, x consist of those parameters needed to comprehend $\psi_{k,m}$), and let

$$y := x \cup \{S^*, T^*, \eta_a, \eta_a^+, \eta_b, \eta_b^+\}.$$

Let $\langle M_\delta : \delta < \mu^+\rangle$ be a μ^+-approximating sequence over y, and let

$$E := \{\delta < \mu^+ : M_\delta \cap \mu^+ = \delta\}.$$

We define

$$S := S^* \cap E,$$

and

$$T := T^* \cap E.$$

Now suppose $\langle \alpha^*, \beta^*, \delta\rangle \in S \circledast T \times \text{cf}(\mu)$. We must produce $\alpha < \beta$ such that $D \restriction t_\alpha \times t_\beta$ is constant with value $\langle \alpha^*, \beta^*, \delta\rangle$. To this point, we know

$$\eta_a \leq \eta_a^+ < \eta_b \leq \eta_b^+ < \alpha^* < \beta^*. \tag{5.22}$$

We choose now γ_b and β such that

- $\psi_{k,m}(\eta_b, \eta_b^+, \beta^*, \gamma_b, \beta)$, and
- $M_{\beta^*+2} \cap \mu^+ \leq \gamma_b$.

This can be done by the definition of T. Next, we define

$$t := \{\beta_i^m(\beta^*, \epsilon_b) : \epsilon_b \in t_\beta \text{ and } i < \rho_2^m(\beta^*, \epsilon_b)\}. \tag{5.23}$$

By definition, $\gamma_b = \min(t)$ and so $M_{\beta^*+2} \cap \mu^+ \leq \min(t)$.

Finally, define J to be the set of all $\alpha < \mu^+$ such that for some $\gamma < \mu^+$, we have

- $\psi_{k,m}(\eta_a, \eta_a^+, \alpha^*, \gamma, \alpha)$, and
- $\{\eta_b, \eta_b^+\}$ works for $\{\alpha^*, \alpha\}$ at i_a.

It is clear that J is unbounded in μ^+ since $\alpha^* \in S$. Furthermore, the set J is definable in the model M_{α^*+1}, hence $J \in M_{\beta^*}$.

Thus, the objects $M_{\beta^*} \in M_{\beta^*+1} \in M_{\beta^*+2}$, β^*, $\langle t_\alpha : \alpha \in J\rangle$, and t satisfy the hypotheses of Lemma 4.1.

We conclude that for all sufficiently large $i_b < \mathrm{cf}(\mu)$, there are unboundedly many $\alpha \in J \cap \beta^*$ such that for all $\epsilon_a \in t_\alpha$ and all $\epsilon \in t$,

$$\Gamma^+(\epsilon_a, \epsilon) = i_b, \tag{5.24}$$

and

$$f_{\beta^*}(i_b + 1) < f_{\epsilon_a}(i_b + 1). \tag{5.25}$$

In particular, we can choose α and i_b in such a way that the above conditions are satisfied, and in addition such that

$$\iota(i_b) = \langle m, \delta\rangle \tag{5.26}$$

Notice that for any $\epsilon_a \in t_\alpha$ and $\epsilon_b \in t_\beta$, we have

$$\eta_a \le \eta_a^+ < \eta_b \le \eta_b^+ < \alpha^* < \epsilon_a < \beta^* < \epsilon_b. \tag{5.27}$$

The rest of the proof consists of showing that $D \restriction t_\alpha \times t_\beta$ is constant with value $\langle \alpha^*, \beta^*, \delta\rangle$, so assume now that $\epsilon_a \in t_\alpha$ and $\epsilon_b \in t_\beta$.

Right away, we see that (5.24) implies $\Gamma(\epsilon_a, \epsilon_b) = i_b$ because $\epsilon_a \in t_\alpha$ and $\epsilon_b \in t$. As an immediate corollary, it follows that

$$m(\epsilon_a, \epsilon_b) = m \tag{5.28}$$

and

$$\delta(\epsilon_a, \epsilon_b) = \delta. \tag{5.29}$$

Claim 3. $\beta^*(\epsilon_a, \epsilon_b) = \beta^*$.

Proof. Since $\psi_{k,m}(\epsilon_b, \epsilon_b^+, \beta^*, \gamma_b, \beta)$ holds, we know

$$\eta_b \le \eta^m(\beta^*, \epsilon_b) \le \eta_b^+. \tag{5.30}$$

Since

$$\eta_b^+ < \alpha^* < \epsilon_a < \beta^*, \tag{5.31}$$

it follows that

$$\beta_i^m(\epsilon_a, \epsilon_b) = \beta_i^m(\beta^*, \epsilon_b) \text{ for } i \le \rho_2^m(\beta^*, \epsilon_b). \tag{5.32}$$

In particular,

$$\beta_i^m(\epsilon_a, \epsilon_b) \in t \text{ for } i < \rho_2^m(\beta^*, \epsilon_b), \tag{5.33}$$

and

$$\beta^m_{\rho_2^m(\beta^*, \epsilon_b)}(\epsilon_a, \epsilon_b) = \beta^*. \tag{5.34}$$

Given our choice of α (see (5.24) and (5.25)), we obtain

$$\beta^*(\epsilon_a, \epsilon_b) = c_m(\epsilon_a, \epsilon_b) = \beta^*, \tag{5.35}$$

as required. □

Note that the preceding claim tells us

$$\eta^*(\epsilon_a, \epsilon_b) = \eta^m(\beta^*, \epsilon_b) \tag{5.36}$$

as well.

Claim 4. $\alpha^*(\epsilon_a, \epsilon_b) = \alpha^*$.

Proof. Our choice of $\{\eta_b, \eta_b^+\}$ tells us that

$$\eta_a \leq \eta_a^+ < \eta_b \leq \eta^*(\epsilon_a, \epsilon_b) = \eta^m(\beta^*, \epsilon_b) \leq \eta_b^+ < \alpha^* < \epsilon_a,$$

and therefore

$$\beta_i^m(\eta^*(\epsilon_a, \epsilon_b), \epsilon_a) = \beta_i^m(\alpha^*, \epsilon_a) \text{ for all } i \leq \rho_2^m(\alpha^*, \epsilon_a). \tag{5.37}$$

After looking back to the definition of $\psi_{k,m}$, we see that (5.36) implies

$$f_{\eta_b} \restriction [k, \mathrm{cf}(\mu)) \leq f_{\eta^*(\epsilon_a, \epsilon_b)} \restriction [k, \mathrm{cf}(\mu)) \leq f_{\eta_b^+} \restriction [k, \mathrm{cf}(\mu)). \tag{5.38}$$

This has the following trivial consequences:

$$f_{\eta_b}(i_a) \leq f_{\eta^*(\epsilon_a, \epsilon_b)}(i_a) \tag{5.39}$$

$$f_{\eta_b}(i_a + 1) \leq f_{\eta^*(\epsilon_a, \epsilon_b)}(i_a + 1) \tag{5.40}$$

and

$$f_{\eta^*(\epsilon_a, \epsilon_b)} \restriction [i_a + 1, \mathrm{cf}(\mu)) \leq f_{\eta_b^+} \restriction [i_a + 1, \mathrm{cf}(\mu)). \tag{5.41}$$

For $i < \rho_2^m(\alpha^*, \epsilon_a)$, we know that $\{\eta_b, \eta_b^+\}$ works for $\{\alpha^*, \alpha\}$ at i_a, and so

$$\Gamma^+\big(\eta_b, \beta_i^m(\alpha^*, \epsilon_a)\big) = \Gamma^+\big(\eta_b^+, \beta_i^m(\alpha^*, \epsilon_a)\big) = i_a, \tag{5.42}$$

and

$$f_{\alpha^*}(i_a + 1) < f_{\eta^b}(i_a + 1). \tag{5.43}$$

From (5.37), (5.38), (5.39), (5.41), and (5.42) we conclude

$$\Gamma\big(\eta^*(\epsilon_a, \epsilon_b), \beta_i^m(\eta^*(\epsilon_a, \epsilon_b), \epsilon_a)\big) = i_a \text{ for all } i < \rho_2^m(\alpha^*, \epsilon_a). \tag{5.44}$$

Given $i < \rho_2^m(\alpha^*, \epsilon_a)$, we see

$$\begin{aligned} f_{\beta_i^m(\eta^*(\epsilon_a, \epsilon_b))}(i_a) &= f_{\beta_i^m(\alpha^*, \epsilon_a)}(i_a) && \text{by (5.37)} \\ &< f_{\eta_b}(i_a) && \text{by (5.42)} \\ &\leq f_{\eta^*(\epsilon_a, \epsilon_b)}(i_a) && \text{by (5.39)} \end{aligned}$$

and

$$\begin{aligned} f_{\eta^*(\epsilon_a, \epsilon_b)} \restriction [i_a + 1, \mathrm{cf}(\mu) &\leq f_{\eta_b^+} \restriction [i_a + 1, \mathrm{cf}(\mu)) && \text{by (5.41)} \\ &\leq f_{\beta_i^m(\alpha^*, \epsilon_a)} \restriction [i_a + 1, \mathrm{cf}(\mu)) && \text{by (5.42)} \\ &= f_{\beta_i^m(\eta^*(\epsilon_a, \epsilon_b))} \restriction [i_a + 1, \mathrm{cf}(\mu)) && \text{by (5.37)} \end{aligned}$$

so

$$\Gamma\left(\eta^*(\epsilon_a, \epsilon_b), \beta_i^m(\eta^*(\epsilon_a, \epsilon_b), \epsilon_a)\right) = i_a \text{ for all } i < \rho_2^m(\alpha^*, \epsilon_a). \tag{5.45}$$

On the other hand, since

$$\beta^m_{\rho^2_m(\alpha^*, \epsilon_a)}(\eta^*(\epsilon_a, \epsilon_b), \epsilon_a) = \alpha^*, \tag{5.46}$$

it follows from (5.40), (5.43), and (5.38) that

$$\Gamma\big(\eta^*(\epsilon_a, \epsilon_b), \beta^m_{\rho^2_m(\alpha^*, \epsilon_a)}(\eta^*(\epsilon_a, \epsilon_b), \epsilon_a)\big) \neq i_a. \tag{5.47}$$

From (5.44), (5.47), and the definition of $c_{m(\epsilon_a, \epsilon_b)}$, we conclude that

$$\alpha^*(\epsilon_a, \epsilon_b) = c_{m(\epsilon_a, \epsilon_b)}(\eta^*(\epsilon_a, \epsilon_b), \epsilon_a) = \beta^m_{\rho^2_m(\alpha^*, \epsilon_a)}(\eta^*(\epsilon_a, \epsilon_b), \epsilon_a) = \alpha^*, \tag{5.48}$$

as required. □

Putting Claim 3, Claim 4, and (5.29) together, we see

$$D(\alpha,\beta)=\langle\alpha^*,\beta^*,\delta\rangle \tag{5.49}$$

and the proof is complete. □

6. Consequences

We turn our attention now to consequences of Theorem 1 and pick up the discussion of the introduction once more. Our first result gives the promised equivalence of (1.9) and (1.10), and also establishes an even stronger fact.

Theorem 2. *The following are equivalent for a singular cardinal μ:*

(6.1) $\mathrm{Pr}_1(\mu^+,\mu^+,\mu^+,\mathrm{cf}(\mu))$

(6.2) $\mathrm{Pr}_1(\mu^+,\mu^+,\mu,\mathrm{cf}(\mu))$

(6.3) $\mathrm{Pr}_1(\mu^+,\mu^+,\theta,\mathrm{cf}(\mu))$ *for arbitrarily large $\theta<\mu$.*

Proof. It is easy to see that each statement implies the one following, so we prove that (6.3) implies (6.1). Fix an increasing sequence of cardinals $\langle\theta_i : i<\mathrm{cf}(\mu)\rangle$ cofinal in μ such that $\mathrm{Pr}_1(\mu^+,\mu^+,\theta_i,\mathrm{cf}(\mu))$ holds for each $i<\mathrm{cf}(\mu)$, and let c_i be a coloring witnessing this for θ_i. Also, we fix for each $\beta<\mu^+$ a function g_β mapping μ onto β.

Let D be a coloring as in our main theorem, and let α^*, β^*, and δ denote functions defined by the recipe

$$D(\alpha,\beta)=\langle\alpha^*(\alpha,\beta),\beta^*(\alpha,\beta),\delta(\alpha,\beta)\rangle. \tag{6.4}$$

Finally, define the function $c:[\mu^+]^2\to\mu^+$ as follows:

$$c(\alpha,\beta):=g_{\beta^*(\alpha,\beta)}\left(c_{\delta(\alpha,\beta)}\left(\alpha^*(\alpha,\beta),\beta^*(\alpha,\beta)\right)\right). \tag{6.5}$$

Now suppose $\langle t_\beta : \beta<\mu^+\rangle$ is a sequence of pairwise disjoint elements of $[\mu^+]^{<\mathrm{cf}(\mu)}$, and let $\varsigma<\mu^+$ be arbitrary. We must find $\alpha<\beta$ such that $c\restriction t_\alpha\times t_\beta$ is constant with value ς.

Fix stationary sets S and T as in the conclusion of our main theorem. An application of Fodor's Theorem allows us to find $\epsilon<\mu$ and stationary $T^*\subseteq T$ such that $g_{\beta^*}(\epsilon)=\varsigma$ for all $\beta^*\in T^*$.

Next, we construct a sequence $\langle s_\gamma : \gamma<\mu^+\rangle$ of pairwise disjoint elements of $S\circledast T^*$ such that $\max(s_\zeta)<\min(s_\eta)$ whenever $\zeta<\eta$. This is easily done, as both S and T^* are unbounded in μ^+. Now let $\delta<\mathrm{cf}(\mu)$ be chosen so that $\epsilon<\theta_\delta$.

Our choice of c_δ provides us with $\zeta<\eta<\mu^+$ such that

$$c_\delta\restriction s_\zeta\times s_\eta \text{ is constant with value } \epsilon. \tag{6.6}$$

Now supposing

$$s_\zeta=\{\alpha^*_\zeta,\beta^*_\zeta\} \tag{6.7}$$

and

$$s_\eta=\{\alpha^*_\eta,\beta^*_\eta\}, \tag{6.8}$$

we define

$$\alpha^*=\alpha^*_\zeta \tag{6.9}$$

and

$$\beta^* = \beta^*_\eta \tag{6.10}$$

It should be clear that $\langle \alpha^*, \beta^* \rangle \in S \circledast T^*$.

Our assumptions about D now give us $\alpha < \beta$ such that

$$D \upharpoonright t_\alpha \times t_\beta \text{ is constant with value } \langle \alpha^*, \beta^*, \delta \rangle. \tag{6.11}$$

Clearly we can also demand that $\sup(t_\alpha) < \min(t_\beta)$, and now we show

$$c \upharpoonright t_\alpha \times t_\beta \text{ is constant with value } \varsigma. \tag{6.12}$$

Given $\epsilon_a \in t_\alpha$ and $\epsilon_b \in t_\beta$, we know

$$\alpha^*(\epsilon_a, \epsilon_b) = \alpha^*, \tag{6.13}$$

$$\beta^*(\epsilon_a, \epsilon_b) = \beta^*, \tag{6.14}$$

and

$$\delta(\epsilon_a, \epsilon_b) = \delta. \tag{6.15}$$

A glance at (6.5) tells us

$$c(\epsilon_a, \epsilon_\beta) = g_{\beta^*}(c_\delta(\alpha^*, \beta^*)), \tag{6.16}$$

and now the result follows immediately as $\beta^* \in T^*$ and $c_\delta(\alpha^*, \beta^*) = \epsilon$. □

The main theorem of our paper [3] established, among other things, that if μ is singular of uncountable cofinality, then $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$ holds unless the stationary subsets of μ^+ possess many instances of stationary reflection. In the case where the cofinality of μ is countable, we were only able to get the analogous result with $\mathrm{Pr}_1(\mu^+, \mu^+, \mu, \mathrm{cf}(\mu))$, but this defect is repaired now by the equivalence of (6.2) and (6.1). This allows to state the following theorem without restrictions on the cofinality of μ:[1]

Theorem 3. *If μ is singular and $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$ fails then there is a $\theta < \mu$ such that for any sequence $\langle S_\alpha : \alpha < \sigma \rangle$ of stationary subsets of $S^{\mu^+}_{\geq\theta}$ of length $\sigma < \mathrm{cf}(\mu)$, there is an ordinal $\delta < \mu^+$ such that $S_\alpha \cap \delta$ is stationary in δ for all $\alpha < \sigma$.*

Proof. This is restatement the contrapositive of parts (2) and (3) of the main theorem of [3], in light of the equivalence of (6.2) and (6.1). □

It is still open whether $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$ can fail for a singular cardinal. The above theorem tells us that obtaining such a consistency result will necessarily involve some considerable large cardinals. In light of the implications in (1.8), we see that this is true for the consistency of μ^+ being Jonsson, or the consistency of $\mu^+ \to [\mu^+]^2_{\mu^+}$ as well.

Our next result is a relative of one of the conclusions derived in [4].

Theorem 4. *The following are equivalent for a singular cardinal μ and cardinal $\theta \leq \mu^+$:*

(1) $\mathrm{Pr}_1(\mu^+, \mu^+, \theta, \mathrm{cf}(\mu))$.

[1] The result has been improved considerably in more recent work [6] of the author to give simultaneous reflection for *any* collection of fewer than $\mathrm{cf}(\mu)$ stationary subsets of μ^+.

(2) *There is a function* $c : [\mu^+]^2 \to \theta$ *such that for any unbounded subsets* A *and* B *of* μ^+,

$$\theta \subseteq \operatorname{ran}(c \restriction A \circledast B). \tag{6.17}$$

(3) *There is a function* $d : [\mu^+]^2 \to \theta$ *such that for any stationary subsets* S *and* T *of* μ^+,

$$\theta \subseteq \operatorname{ran}(d \restriction S \circledast T). \tag{6.18}$$

We are abusing notation a little bit in (6.17) and (6.18), as elements of $S \circledast T$ are technically ordered pairs and not pairs of ordinals, but the meaning should be clear. Also note that (2) is relative of the relation

$$\mu^+ \nrightarrow [(\mu^+ : \mu^+)]^2_\theta \tag{6.19}$$

(see [9]), which states that there is a function $f : [\mu^+]^2 \to \theta$ such that for any unbounded subsets A and B of μ^+ and any $\varsigma < \theta$, there are $\alpha \in A$ and $\beta \in B$ with $f(\alpha, \beta) = \varsigma$. The difference between (2) and (6.19) is very slight — in (6.19), it is not required that α is less than β, while we need this in order to apply Theorem 1.

Proof. The fact that (1) implies (2) is well-known (we did something similar in the proof of Theorem 2), but we give it for completeness. We show something a little bit stronger, namely that any function witnessing (1) also works for (2).

Thus, let c witness that $\mathrm{Pr}_1(\mu^+, \mu^+, \theta, \mathrm{cf}(\mu))$ holds, let $\varsigma < \theta$ be given, and let S and T be stationary subsets of μ^+. Construct a sequence $\langle t_\alpha : \alpha < \mu^+\rangle$ of pairwise disjoint elements of $S \circledast T$ such that $\max(t_\alpha) < \min(t_\beta)$ whenever $\alpha < \beta$, and then fix $\alpha < \beta$ such that $c \restriction t_\alpha \times t_\beta$ is constant with value ς. Let $\alpha^* = \min(t_\alpha)$ and $\beta^* = \max(t_\beta)$. Then $\langle \alpha^*, \beta^*\rangle \in S \circledast T$ is as required.

It is clear that (2) implies (3), so assume d be as in (3), and let D be the function from Theorem 1, with

$$D(\alpha, \beta) = \langle \alpha^*(\alpha, \beta), \beta^*(\alpha, \beta), \delta(\alpha, \beta)\rangle. \tag{6.20}$$

We define a function $f : [\mu^+]^2 \to \theta$ by

$$f(\alpha, \beta) = c(\alpha^*(\alpha, \beta), \beta^*(\alpha, \beta)), \tag{6.21}$$

and the verification that f has the required properties is straightforward. □

We will present only one application of the preceding theorem here, but we note that we can use Theorem 4 to greatly simplify the proof of the main result of [7]. It also allows us to solve the main problem left open by [8]. We intend to present this work elsewhere, as it is joint with Shelah.

We start with a lemma, proved by a standard argument

Lemma 6.1. Let $(\vec{\mu}, \vec{f})$ be a scale for the singular cardinal μ, and suppose $A \subseteq \mu^+$ is unbounded. Then

$$(\forall^* i < \mathrm{cf}(\mu))(\exists^* \xi < \mu_i)(\exists^* \alpha \in A)[f_\alpha(i) = \xi]. \tag{6.22}$$

Proof. The proof is by contradiction, so suppose the conclusion fails for some unbounded $A \subseteq \mu^+$. Parsing what this means, we find that there are unboundedly many $i < \mathrm{cf}(\mu)$, for all sufficiently large $\xi < \mu_i$, the set of $\alpha \in A$ with $f_\alpha(i) = \xi$ is bounded in μ^+.

Let I consist of those $i < \mathrm{cf}(\mu)$ for which the above is true, and for $i \in I$ choose ξ_i such that

$$\xi_i \leq \xi < \mu_i \Longrightarrow |\{\alpha \in A : f_\alpha(i) = \xi\}| < \mu^+. \tag{6.23}$$

Given $i \in I$ and ξ with $\xi_i \leq \xi < \mu_i$, we can fix an ordinal $\alpha(\xi, i) < \mu^+$ such that

$$\{\alpha \in A : f_\alpha(i) = \xi\} \subseteq \alpha(\xi, i), \tag{6.24}$$

and then define

$$\alpha^* := \sup\{\alpha(\xi, i) : i < \mathrm{cf}(\mu), \xi_i \leq \xi < \mu_i\}. \tag{6.25}$$

It is clear that $\alpha^* < \mu^+$ as there are only μ possibilities for ξ and i.

After the dust has settled, we see that if $\alpha \in A$ is greater than α^*, then

$$i \in I \Longrightarrow f_\alpha(i) < \xi_i, \tag{6.26}$$

and this easily contradicts our assumption that $(\vec{\mu}, \vec{f})$ is a scale. □

The theorem we prove below has many antecedents in the literature, but our result seems to be the first in which the partition relation holding at the successor of the singular cardinal represents an upgrade over those assumed to hold at the smaller cardinals.

Theorem 5. *Suppose μ is singular, and there is a scale $(\vec{\mu}, \vec{f})$ for μ such that*

$$\mu_i \nrightarrow [(\mu_i : \mu_i)]^2_{\mu_i} \tag{6.27}$$

for all $i < \mathrm{cf}(\mu)$. Then $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$ holds.

Proof. By Conclusion 4.1A on page 67 of [11], we know that $\mathrm{Pr}_1(\mu^+, \mu^+, \mathrm{cf}(\mu), \mathrm{cf}(\mu))$ holds, so we can fix a function $c : [\mu^+]^2 \to \mathrm{cf}(\mu)$ as in part (2) of Theorem 4 with $\mathrm{cf}(\mu)$ standing in for θ.

For each $i < \mathrm{cf}(\mu)$, let $d_i : [\mu_i]^2 \to \mu_i$ be a witness for (6.27), and define a function $d : [\mu^+]^2 \to \mu$ by

$$d(\alpha, \beta) = d_{c(\alpha,\beta)}(f_\alpha(c(\alpha, \beta)), f_\beta(c(\alpha, \beta))). \tag{6.28}$$

Given $\varsigma < \mu$ and unbounded subsets A and B of μ^+, we will find $\langle \alpha, \beta \rangle$ in $A \circledast B$ with $d(\alpha, \beta) = \varsigma$. This implies $\mathrm{Pr}_1(\mu^+, \mu^+, \mu, \mathrm{cf}(\mu))$ by Theorem 4, which in turn gives us $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$ by Theorem 2.

We choose $i^* < \mathrm{cf}(\mu)$ so that

$$\varsigma < \mu_{i^*}, \tag{6.29}$$

$$(\exists^* \zeta < \mu_{i^*})(\exists^* \alpha \in A)[f_\alpha(i^*) = \zeta], \tag{6.30}$$

and

$$(\exists^* \eta < \mu_{i^*})(\exists^* \beta \in B)[f_\alpha(i^*) = \eta]. \tag{6.31}$$

Clearly this is possible by Lemma 6.1.

Next, we define

$$A^* := \{\zeta < \mu_{i^*} : (\exists^* \alpha \in A)[f_\alpha(i^*) = \zeta]\} \tag{6.32}$$

and

$$B^* := \{\eta < \mu_{i^*} : (\exists^* \beta \in B)[f_\alpha(i^*) = \eta]\}. \tag{6.33}$$

Both of these sets are unbounded in μ_{i^*}, and so we can find $\alpha^* \in A^*$ and $\beta^* \in B^*$ with

$$d_{i^*}(\alpha^*, \beta^*) = \varsigma. \tag{6.34}$$

Now define

$$A^\dagger = \{\alpha \in A : f_\alpha(i^*) = \alpha^*\} \tag{6.35}$$

and

$$B^\dagger = \{\beta \in B : f_\beta(i^*) = \beta^*\}. \tag{6.36}$$

Both of these sets are unbounded in μ^+, and so we can find $\langle \alpha, \beta \rangle$ in $A \circledast B$ with $c(\alpha, \beta) = i^*$.

We find now that

$$\begin{aligned} d(\alpha,\beta) &= d_{c(\alpha,\beta)}(f_\alpha(c(\alpha,\beta)), f_\beta(c(\alpha,\beta))) \\ &= d_{i^*}(f_\alpha(i^*), f_\beta(i^*)) = d_{i^*}(\alpha^*, \beta^*) = \varsigma, \end{aligned} \tag{6.37}$$

as required. □

We come now to a result promised at the end of the introduction. The hypothesis $\mathrm{pp}(\mu) = \mu^+$ refers to Shelah's pseudo-power function, but we will not elaborate as we need only one easily understood consequence of this assumption. The reader can consult [11] for the definition of pp, and the author's [2] contains the proof of the relevant facts.

Corollary 6.2. $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$ holds for any singular μ with $\mathrm{pp}(\mu) = \mu^+$.

Proof. Since $\mathrm{pp}(\mu) = \mu^+$, there is a scale $(\vec{\mu}, \vec{f})$ such that for each $i < \mathrm{cf}(\mu)$, $\mu_i = \kappa_i^{++}$ for some uncountable regular cardinal κ_i. By [12], it follows that

$$\mu_i \nrightarrow [(\mu_i : \mu_i)]^2_{\mu_i} \tag{6.38}$$

for each $i < \mathrm{cf}(\mu)$, and so we get what we need by way of Theorem 5. □

The proof of the preceding is deceptively short, for [12] is quite a difficult paper. Shelah shows there that

$$\mathrm{Pr}_1(\kappa^{++}, \kappa^{++}, \kappa^{++}, \kappa) \tag{6.39}$$

holds for regular κ, but the argument for the implication "(1) $\Longrightarrow$ (2)" in Theorem 4 tells us that

$$\kappa^{++} \nrightarrow [(\kappa^{++} : \kappa^{++})]^2_{\kappa^{++}} \tag{6.40}$$

for every regular κ as well.

Corollary 6.2 can also be proved in the following manner. Claim 4.1E on page 70 of [11] implies, when suitably interpreted and combined with the result quoted in (6.39), that $\mathrm{Pr}_1(\mu^+, \mu^+, \mu, \mathrm{cf}(\mu))$ holds. This fact was noted by Shelah in a personal communication with the author, but Theorem 2 is still necessary to obtain a coloring with μ^+ colors. We chose our approach because Theorem 5 is of independent interest and of more general applicability.

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