GETTING MORE COLORS I

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Abstract. We establish a coloring theorem for successors of singular cardinals, and use it prove that for any such cardinal μ, we have $\mu^+ \nrightarrow [\mu^+]^2_{\mu^+}$ if and only if $\mu^+ \nrightarrow [\mu^+]^2_\theta$ for arbitrarily large $\theta < \mu$.

1. Introduction

Our aim in this note is to prove a type of "negative stepping-up theorem" for square-brackets partition relations at successors of singular cardinals. In order to state our results precisely, we need to recall the following bit of notation due originally to Erdős, Hajnal, and Rado [7]:

Definition 1.1. $\kappa \rightarrow [\lambda]^\mu_\theta$ means that for any function $F : [\kappa]^\mu \rightarrow \theta$, (to which we refer as a *coloring*) we can find a set $H \subseteq \kappa$ of cardinality λ for which

$$\mathrm{ran}(F \upharpoonright [H]^\mu) \subsetneq \theta.$$

The negation of a square-brackets partition relation asserts the existence of a coloring which exhibits complicated behavior on every large subset of the domain. We will be concerned with relations of the form $\kappa \nrightarrow [\kappa]^2_\theta$, which states that one may color the pairs of ordinals from κ with θ colors in such a way that $f \upharpoonright [A]^2$ assumes all colors for any set $A \in [\kappa]^\kappa$. (We will usually identify $[\kappa]^2$ with those pairs $\langle \alpha, \beta \rangle \in \kappa \times \kappa$ with $\alpha < \beta$.) We refer the reader to Chapter XI of [8] for a more comprehensive introduction to square-brackets partition relations and their negations.

We mentioned in the opening sentence that we aim to prove a sort of negative stepping-up theorem. The terminology "negative stepping-up theorem" usually refers to results which increase the cardinal appearing on the left side of a given negative partition relation. This is not quite what we are after — we assume the existence of certain colorings on a cardinal κ and prove that the number of colors can automatically be upgraded while keeping the "domain" κ fixed. The following simple proposition provides our motivation:

Proposition 1.2. The following two statements are equivalent for a cardinal μ:

(1) $\mu^+ \nrightarrow [\mu^+]^2_{\mu^+}$

(2) $\mu^+ \nrightarrow [\mu^+]^2_\mu$

Proof. It is clear that (1) implies (2), so assume we have a function $c : [\mu^+]^2 \rightarrow \mu$ witnessing that $\mu^+ \nrightarrow [\mu^+]^2_\mu$. For each $\beta < \mu^+$, fix a function g_β mapping μ onto β, and define

$$c^*(\alpha, \beta) = g_\beta(c(\alpha, \beta)). \tag{1.1}$$

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We will show that c^* serves as a witness for $\mu^+ \nrightarrow [\mu^+]^2_{\mu^+}$.

To see this, suppose $A \subseteq \mu^+$ is of size μ^+, and let $\varsigma < \mu^+$ be arbitrary. Our goal is to produce $\alpha < \beta$ in A for which $c^*(\alpha, \beta) = \varsigma$, so without loss of generality we may assume that $\varsigma < \min(A)$.

Given $\epsilon < \mu$, define

$$A_\epsilon := \{\beta \in A : g_\beta(\epsilon) = \varsigma\}. \tag{1.2}$$

Since $\varsigma < \min(A)$, it follows that $A = \bigcup_{\epsilon<\mu} A_\epsilon$. In particular, we can choose a single $\epsilon < \mu$ for which A_ϵ has size μ^+. It follows that we can find $\alpha < \beta$ in A_ϵ (hence in A) for which $c(\alpha, \beta) = \epsilon$, and so

$$c^*(\alpha, \beta) = g_\beta(c(\alpha, \beta)) = g_\beta(\epsilon) = \varsigma, \tag{1.3}$$

where the last equality holds because $\beta \in A_\epsilon$. □

This simple little argument applies in many other situations. For example, one easily obtains by the same method the equivalence of $\mu^+ \nrightarrow [\mu^+]^n_{\mu^+}$ and $\mu^+ \nrightarrow [\mu^+]^n_\mu$ for any finite n.

These results, however, are of no interest in the case where μ is a regular cardinal, as a celebrated result of Todorčević [10] establishes $\mu^+ \nrightarrow [\mu^+]^2_{\mu^+}$ always holds when μ is a regular cardinal. On the other hand, the case where μ is singular is a much different story because it is still unknown whether $\mu^+ \nrightarrow [\mu^+]^2_{\mu^+}$ (or even the much weaker $\mu^+ \nrightarrow [\mu^+]^{<\omega}_{\mu^+}$) must hold. If we assume that μ is singular, then there is a natural way to attempt to strengthen Proposition 1.2 — one may ask if $\mu^+ \rightarrow [\mu^+]^2_{\mu^+}$ follows only by assuming that $\mu^+ \nrightarrow [\mu^+]^2_\theta$ holds for arbitrarily large $\theta < \mu$. Experience suggests that the answer should be yes, and that the result should follow by one of the standard "patching arguments" common in this area of set theory. Unfortunately, a naive attempt at this yields only the following weak result:

Proposition 1.3. Suppose μ is a singular cardinal. If $\mu^+ \nrightarrow [\mu^+]^2_\theta$ for arbitrarily large $\theta < \mu$, then

$$\mu^+ \nrightarrow [\mu^+]^4_{\mu^+}.$$

Proof. We give only a sketch. Note the same argument given in Proposition 1 tells us it suffices to establish

$$\mu^+ \nrightarrow [\mu^+]^4_\mu.$$

Let $\langle \theta_i : i < \mathrm{cf}(\mu)\rangle$ be an increasing sequence of cardinals unbounded in μ. For each $i < \mathrm{cf}(\mu)$, let d_i be a coloring witnessing $\mu^+ \nrightarrow [\mu^+]^2_{\theta_i}$ (note that our assumptions imply such colorings exist for *every* $\theta < \mu$). Now $\mu^+ \nrightarrow [\mu^+]^2_{\mathrm{cf}(\mu)}$ (a result of Shelah — see Conclusion 4.1 on page 67 of [9]), so we can fix a coloring c witnessing this.

We now use c to patch together the colorings d_i, that is, we define a function f on $[\mu^+]^4$ by setting

$$f(\alpha, \beta, \gamma, \delta) = d_{c(\gamma,\delta)}(\alpha, \beta). \tag{1.4}$$

Given $\varsigma < \mu$ and unbounded $A \subseteq \mu^+$, we must find $\alpha < \beta < \gamma < \delta$ in A for which

$$f(\alpha, \beta, \gamma, \delta) = \varsigma.$$

This is, however, quite straightforward and the result follows. □

The rest of this paper is essentially concerned with turning the “4” in Proposition 1.3 into a “2”. We accomplish this by proving the following theorem (in ZFC) from which the desired result can be deduced as an easy corollary.

Main Theorem. Suppose μ is a singular cardinal. There is a function

$$D : [\mu^+]^2 \to [\mu^+]^2 \times \mathrm{cf}(\mu) \tag{1.5}$$

such that for any unbounded $A \subseteq \mu^+$, there is a stationary $S \subseteq \mu^+$ such that

$$[S]^2 \times \mathrm{cf}(\mu) \subseteq \mathrm{ran}(D \restriction [A]^2). \tag{1.6}$$

2. Background material

Minimal Walks

Recall that $\bar{e} = \langle e_\alpha : \alpha < \lambda\rangle$ *is a C-sequence for the cardinal* λ if e_α is closed unbounded in α for each $\alpha < \lambda$. Following Todorčević, given $\alpha < \beta < \lambda$ the *minimal walk from β to α along $\bar{e}$* is defined to be the sequence $\beta = \beta_0 > \cdots > \beta_n = \alpha$ obtained by setting

$$\beta_{i+1} = \min(e_{\beta_i} \setminus \alpha). \tag{2.1}$$

The *trace* of the walk from β to α is defined by

$$\mathrm{Tr}(\alpha, \beta) = \{\beta = \beta_0 > \beta_1 > \cdots > \beta_n = \alpha\}, \tag{2.2}$$

that is, $\mathrm{Tr}(\alpha, \beta)$ is the set of all ordinals visited on the walk from β down to α along $\bar{e}$.

There are other standard parameters associated with minimal walks that are quite important for our purposes. For example, we need the function $\rho_2 : [\lambda]^2 \to \omega$ giving the length of the walk from β to α, that is,

$$\rho_2(\alpha, \beta) = \text{ least } i \text{ for which } \beta_i(\alpha, \beta) = \alpha. \tag{2.3}$$

For $i \leq \rho_2(\alpha, \beta)$, we set

$$\beta_i^-(\alpha, \beta) = \begin{cases} 0 & \text{if } i = 0, \\ \sup(e_{\beta_j(\alpha,\beta)} \cap \alpha) & \text{if } i = j + 1 \text{ for } j < \rho_2(\alpha, \beta). \end{cases}$$

Note that if $0 < i < \rho_2(\alpha, \beta)$ then

- $\beta_i^-(\alpha, \beta) = \max(e_{\beta_{i-1}(\alpha,\beta)} \cap \alpha)$ (as opposed to “sup”),
- $\beta_i^-(\alpha, \beta) < \alpha < \beta_i(\alpha, \beta)$, and
- $\beta_i(\alpha, \beta) = \min(e_{\beta_{i-1}(\alpha,\beta)} \setminus \beta_i^-(\alpha, \beta) + 1)$.

Thus, for $0 < i < \rho_2(\alpha, \beta)$, the ordinals $\beta_i^-(\alpha, \beta)$ and $\beta_i(\alpha, \beta)$ are the two consecutive elements in $e_{\beta_{i-1}(\alpha,\beta)}$ which “bracket” α.

For our purposes, we need to analyze what happens in the case where $i = \rho_2(\alpha, \beta)$. In this situation, we have

- $\beta^-_{\rho_2(\alpha,\beta)}(\alpha, \beta) \leq \alpha = \beta_{\rho_2(\alpha,\beta)}(\alpha, \beta)$, and
- $\beta^-_{\rho_2(\alpha,\beta)}(\alpha, \beta) < \alpha$ if and only if $\alpha \in \mathrm{nacc}(e_{\beta_{\rho_2(\alpha,\beta)-1}(\alpha,\beta)})$.

Notice that α must be an element of $e_{\beta_{\rho_2(\alpha,\beta)-1}(\alpha,\beta)}$ by definition, and $\beta^-_{\rho_2(\alpha,\beta)}(\alpha, \beta)$ is less than α precisely when α fails to be an accumulation point of $e_{\beta_{\rho_2(\alpha,\beta)-1}(\alpha,\beta)}$.

We are going to make use of some standard patterns of argument using minimal walks, and this is going to require a couple of more bits of notation. To wit, we define

$$\gamma(\alpha,\beta) = \beta_{\rho_2(\alpha,\beta)-1}(\alpha,\beta), \tag{2.4}$$

$$\gamma^-(\alpha,\beta) = \max\{\beta_i^-(\alpha,\beta) : i < \rho_2(\alpha,\beta)\}, \tag{2.5}$$

and

$$\eta(\alpha,\beta) = \max\{\beta_i^-(\alpha,\beta) : i \leq \rho_2(\alpha,\beta)\}. \tag{2.6}$$

The following proposition captures some standard facts about minimal walks; the proof is an easy induction.

Proposition 2.1. Suppose $\alpha < \beta$.

(1) $\gamma^-(\alpha,\beta) < \alpha$, and if $\gamma^-(\alpha,\beta) < \alpha^* \leq \alpha$ then

$$\beta_i(\alpha,\beta) = \beta_i(\alpha^*,\beta) \text{ for } i < \rho_2(\alpha,\beta), \tag{2.7}$$

and

$$\beta_i^-(\alpha,\beta) = \beta_i^-(\alpha^*,\beta) \text{ for } i < \rho_2(\alpha,\beta). \tag{2.8}$$

(2) $\eta(\alpha,\beta) \leq \alpha$, and if it happens that $\eta(\alpha,\beta) < \alpha^* \leq \alpha$, then

$$\beta_i(\alpha,\beta) = \beta_i(\alpha^*,\beta) \text{ for } i \leq \rho_2(\alpha,\beta). \tag{2.9}$$

In particular,

$$\beta_{\rho_2(\alpha,\beta)}(\alpha^*,\beta) = \alpha. \tag{2.10}$$

Note that part (2) of the above proposition is of no interest unless we can guarantee $\eta(\alpha,\beta) < \alpha$ (or equivalently, guarantee $\alpha \in \operatorname{nacc}(e_{\gamma(\alpha,\beta)})$); this will be one of our concerns in the sequel.

The content of Proposition 2.1 is essentially the only property of minimal walks we need. A discussion of more sophisticated applications is beyond the scope of this paper.We refer the reader to [10] or [11] for more information.

We will, however, need one a generalization of the minimal walks machinery in order to handle some issues that arise when dealing with successors of singular cardinals of countable cofinality. These techniques were introduced in [6], and used again in [4].

Definition 2.2. Let λ be a cardinal. A *generalized C-sequence* is a family

$$\bar{e} = \langle e_\alpha^m : \alpha < \lambda, m < \omega \rangle$$

such that for each $\alpha < \lambda$ and $m < \omega$,

- e_α^m is closed unbounded in α, and
- $e_\alpha^m \subseteq e_\alpha^{m+1}$.

One can think of a generalized C-sequence as a countable family of C-sequences which are increasing in a sense. One can also utilize generalized C-sequences in the context of minimal walks. In this paper, we do this in the simplest fashion — given $m < \omega$ and $\alpha < \beta < \lambda$, we let the *m-walk from β to α along $\bar{e}$* consist of the minimal walk from β to α using the C-sequence $\langle e_\gamma^m : \gamma < \lambda \rangle$. Such walks have their associated parameters, and we use the superscript m to indicate which part of the generalized C-sequence is being used in computations. So, for example, the m-walk from β to α along $\bar{e}$ will have length $\rho_2^m(\alpha,\beta)$, and consist of ordinals denoted $\beta_i^m(\alpha,\beta)$ for $i \leq \rho_2^m(\alpha,\beta)$.

The requirement that $e_\alpha^m \subseteq e_\alpha^{m+1}$ is relevant for the following reason. Given $\alpha < \beta$, we note that the sequence $\langle \min(e_\beta^m \setminus \alpha) : m < \omega \rangle$ is non-increasing, and therefore eventually constant. From this it follows easily that the m-walk from β to α along $\bar{e}$ is exactly the same for all sufficiently large m.

Club-guessing

Our arguments are going to make use of generalized C-sequences that have been carefully selected to interact with certain club-guessing sequences. The type of club-guessing sequence we use depends on whether or not the cofinality of our singular cardinal μ is uncountable, so we deal with each case separately. In both cases, we will be defining a stationary set S, a club-guessing sequence $\bar{C}$, and a generalized C-sequence $\bar{e}$.

If $\mathrm{cf}(\mu) > \aleph_0$, then we define

$$S := S^{\mu^+}_{\mathrm{cf}(\mu)} = \{\delta < \mu^+ : \mathrm{cf}(\delta) = \mathrm{cf}(\mu)\}. \tag{2.11}$$

By Claim 2.6 on page 127 of [9] (or see Theorem 2 of [6]), there is a sequence $\langle C_\delta : \delta \in S \rangle$ such that

- C_δ is club in δ,
- $\mathrm{otp}(C_\delta) = \mathrm{cf}(\mu)$,
- $\langle \mathrm{cf}(\alpha) : \alpha \in \mathrm{nacc}(C_\delta) \rangle$ increases to μ, and
- whenever E is club in μ^+, there are stationarily many $\delta \in S$ for which $C_\delta \subseteq E$.

Here "$\mathrm{nacc}(C_\delta)$" refers to the non-accumulation points of C_δ, that is, those elements of C_δ that are not limits of points in C_δ.

We now use the "ladder swallowing" trick (see Lemma 13 of [3]) to build a C-sequence $\langle e_\alpha : \alpha < \mu^+ \rangle$ such that for each $\alpha < \mu^+$,

$$|e_\alpha| < \mu, \tag{2.12}$$

and

$$\delta \in S \cap e_\alpha \Longrightarrow C_\delta \subseteq e_\alpha. \tag{2.13}$$

We then construct a "silly" generalized C-sequence $\bar{e} = \langle e_\alpha^m : m < \omega, \alpha < \mu^+ \rangle$ by setting $e_\alpha^m = e_\alpha$ for all $m < \omega$.

In the case where μ is of countable cofinality, our definition of S, $\bar{C}$, and $\bar{e}$ is a little more involved as it is an open question whether one can find club-guessing sequences analogous to those above. Our argument will rely on technology developed in [4].

In this case, we start by setting

$$S := S^{\mu^+}_{\aleph_1} = \{\delta < \mu^+ : \mathrm{cf}(\delta) = \aleph_1\}, \tag{2.14}$$

and assume $\langle \mu_i : i < \omega \rangle$ is an increasing sequence of uncountable cardinals cofinal in μ.

We are going to present a simplified version of the conclusion of Theorem 4 of [4]; the reader can consult that paper for a detailed proof (Proposition 5.8 is particularly relevant).

This earlier work tells us that there is a sequence $\langle C_\delta : \delta \in S\rangle$ such that each C_δ is club in δ, and $C_\delta = \bigcup_{m<\omega} C_\delta[m]$ where

- $C_\delta[m]$ is club in δ,
- $|C_\delta[m]| \leq \mu_m^+$,

and such that for every club $E \subseteq \mu^+$, there are stationarily many $\delta \in S$ such that for each $m < \omega$, $\mathrm{nacc}(C_\delta[m]) \cap E$ contains unboundedly many ordinals of cofinality greater than μ_m^+. (Note that the use of "nacc" is redundant as the cofinality assumption guarantees such an ordinal cannot be a limit point of $C_\delta[m]$.)

Lemma 5.10 of [4] provides a generalized C-sequence $\bar{e} = \langle e_\alpha^m : \alpha < \mu^+, m < \omega\rangle$ such that

$$|e_\alpha^m| \leq \mathrm{cf}(\alpha) + \mu_m^+ \tag{2.15}$$

and

$$\delta \in S \cap e_\alpha^m \Longrightarrow C_\delta[m] \subseteq e_\alpha^m. \tag{2.16}$$

In either case, the phrase "choose $\delta \in S$ such that C_δ guesses E" should be given the obvious meaning.

Scales

The next ingredient we need for our theorem is the concept of a scale for a singular cardinal.

Definition 2.3. Let μ be a singular cardinal. A *scale for* μ is a pair $(\vec{\mu}, \vec{f})$ satisfying

(1) $\vec{\mu} = \langle \mu_i : i < \mathrm{cf}(\mu)\rangle$ is an increasing sequence of regular cardinals such that $\sup_{i<\mathrm{cf}(\mu)} \mu_i = \mu$ and $\mathrm{cf}(\mu) < \mu_0$.

(2) $\vec{f} = \langle f_\alpha : \alpha < \mu^+\rangle$ is a sequence of functions such that
   (a) $f_\alpha \in \prod_{i<\mathrm{cf}(\mu)} \mu_i$.
   (b) If $\gamma < \delta < \beta$ then $f_\gamma <^* f_\beta$, where the notation $f <^* g$ means that $\{i < \mathrm{cf}(\mu) : g(i) \leq f(i)\}$ is bounded in $\mathrm{cf}(\mu)$.
   (c) If $f \in \prod_{i<\mathrm{cf}(\mu)} \mu_i$ then there is an $\alpha < \beta$ such that $f <^* f_\alpha$.

It is an important theorem of Shelah (see page Main Claim 1.3 on page 46 of [9]) that scales exist for any singular μ; readers seeking a gentler exposition of this and related topics can consult [2], or [5]. If μ is singular and $(\vec{\mu}, \vec{f})$ is a scale for μ, then there is a natural way to color the pairs of ordinals $\alpha < \beta < \mu^+$ using $\mathrm{cf}(\mu)$ colors, namely

$$\Gamma(\alpha, \beta) = \sup(\{i < \mathrm{cf}(\mu) : f_\beta(i) \leq f_\alpha(i)\}) \tag{2.17}$$

The coloring Γ is the critical ingredient in Shelah's proof of $\mu^+ \nrightarrow [\mu^+]^2_{\mathrm{cf}(\mu)}$ for singular μ, and it plays a central role in the sequel as well. One can consult Conclusion 4.1(a) on page 67 of [9])or Section 5 of [5] (among many other places) for an exposition of this.

We need one standard fact about scales in our proof. We remind the reader that notation of the form "$(\exists^*\beta < \lambda)\psi(\beta)$" means $\{\beta < \lambda : \psi(\beta) \text{ holds}\}$ is unbounded below λ, while "$(\forall^*\beta < \lambda)\psi(\beta)$" means that $\{\beta < \lambda : \psi(\beta) \text{ fails}\}$ is bounded below λ.

Lemma 2.4. Let $(\vec{\mu}, \vec{f})$ be a scale for μ. Then

$$(\forall^* i < \mathrm{cf}(\mu))(\forall \eta < \mu_i)(\exists^* \alpha < \mu^+)[\eta < f_\alpha(i)]. \tag{2.18}$$

Proof. If not, one easily obtains a contradiction to $(\vec{\mu}, \vec{f})$ being a scale. □

Elementary Submodels

We have the usual conventions when dealing with elementary submodels. In brief, we always assume that χ is a regular cardinal much larger than anything relevant to our theorem, and we let $\mathfrak{A}$ denote the structure $\langle H(\chi), \in, <_\chi\rangle$ where $H(\chi)$ is the collection of sets hereditarily of cardinality less than χ, and $<_\chi$ is some well-order of $H(\chi)$. The use of $<_\chi$ means that our structure $\mathfrak{A}$ has definable Skolem functions and it makes sense to talk about Skolem hulls. In general, if $B \subseteq H(\chi)$, then we denote the Skolem hull of B in $\mathfrak{A}$ by $\mathrm{Sk}_{\mathfrak{A}}(B)$.

The following technical lemma due originally to Baumgartner [1] (see the last section of [5], or [3] for a proof) is crucial for our work.

Lemma 2.5. Assume that $M \prec \mathfrak{A}$ and let $\sigma \in M$ be a cardinal. If we define $N = \mathrm{Sk}_{\mathfrak{A}}(M \cup \sigma)$ then for all regular cardinals $\tau \in M$ greater than σ, we have

$$\sup(M \cap \tau) = \sup(N \cap \tau).$$

As a corollary to the above, we can deduce an important fact about *characteristic functions* of models, which we define next.

Definition 2.6. Let μ be a singular cardinal of cofinality κ, and let $\vec{\mu} = \langle \mu_i : i < \kappa\rangle$ be an increasing sequence of regular cardinals cofinal in μ. If M is an elementary submodel of $\mathfrak{A}$ such that

- $|M| < \mu$,
- $\langle \mu_i : i < \mathrm{cf}(\mu)\rangle \in M$, and
- $\kappa + 1 \subseteq M$,

then the *characteristic function of M on $\vec{\mu}$* (denoted $\mathrm{Ch}^{\vec{\mu}}_M$) is the function with domain κ defined by

$$\mathrm{Ch}^{\vec{\mu}}_M(i) := \begin{cases} \sup(M \cap \mu_i) & \text{if } \sup(M \cap \mu_i) < \mu_i, \\ 0 & \text{otherwise.} \end{cases}$$

If $\vec{\mu}$ is clear from context, then we suppress reference to it in the notation.

In the situation of Definition 2.6, it is clear that $\mathrm{Ch}^{\vec{\mu}}_M$ is an element of the product $\prod_{i<\kappa} \mu_i$, and furthermore, $\mathrm{Ch}^{\vec{\mu}}_M(i) = \sup(M \cap \mu_i)$ for all sufficiently large $i < \kappa$. We can now see that the following corollary follows immediately from Lemma 2.5.

Corollary 2.7. Let μ, κ, $\vec{\mu}$, and M be as in Definition 2.6. If $i^* < \kappa$ and we define N to be $\mathrm{Sk}_{\mathfrak{A}}(M \cup \mu_{i^*})$, then

$$\mathrm{Ch}_M \restriction [i^* + 1, \kappa) = \mathrm{Ch}_N \restriction [i^* + 1, \kappa). \tag{2.19}$$

We introduce one more bit of notation concerning elementary submodels.

Definition 2.8. Let λ be a regular cardinal. A *λ-approximating sequence* is a continuous $\in$-chain $\mathfrak{M} = \langle M_i : i < \lambda\rangle$ of elementary submodels of $\mathfrak{A}$ such that

1. $\lambda \in M_0$,
2. $|M_i| < \lambda$,
3. $\langle M_j : j \leq i\rangle \in M_{i+1}$, and
4. $M_i \cap \lambda$ is a proper initial segment of λ.

If $x \in H(\chi)$, then we say that $\mathfrak{M}$ is a λ-approximating sequence over x if $x \in M_0$.

Note that if $\mathfrak{M}$ is a λ-approximating sequence and $\lambda = \mu^+$, then $\mu + 1 \subseteq M_0$ because of condition (4) and the fact that μ is an element of each M_i.

3. Main Lemma

In this section we prove a lemma which shows that the generalized C-sequences isolated in the preceding section have some very nice properties. The following *ad hoc* definition is key; note that the terminology implicitly assumes the presence of a generalized C-sequence in the background.

Definition 3.1. Suppose k and m are natural numbers, and $\eta < \mu^+$. The formula $\varphi_{k,m,\eta}(\beta^*, \beta)$ says

(1) $\beta^* < \beta$,
(2) $\rho_2^m(\beta^*, \beta) = k$,
(3) $\eta = \eta^m(\beta^*, \beta)$, and
(4) $\eta < \beta^*$.

The formula $\varphi_{k,m,\eta}$ isolates a particular configuration of ordinals, a configuration whose importance can be glimpsed in the following lemma:

Lemma 3.2. If $\varphi_{k,m,\eta}(\beta^*, \beta)$ holds, then

$$\eta < \alpha \leq \beta^* \Longrightarrow \beta_i^m(\alpha, \beta) = \beta_i^m(\beta^*, \beta) \text{ for } i \leq k. \tag{3.1}$$

Given the role of k, we see that if $\varphi_{m,k,\eta}(\beta^*, \beta)$ holds, then

$$\eta < \alpha \leq \beta^* \Longrightarrow \beta_k^m(\alpha, \beta) = \beta^*. \tag{3.2}$$

Proof. This follows immediately from Proposition 2.1. □

We come now to the main lemma of this paper:

Lemma 3.3. Let μ be a singular cardinal, and let $\bar{e}$ be a generalized C-sequence as in the preceding section. Then for any unbounded $A \subseteq \mu^+$, there are k, m, and η for which

$$(\exists^{\text{stat}} \beta^* < \mu^+)(\exists^* \beta \in A)[\varphi_{k,m,\eta}(\beta^*, \beta)]. \tag{3.3}$$

Proof. Let S, $\bar{C}$, and $\bar{e}$ be as in previous section's discussion, and let $A \subseteq \mu^+$ be unbounded. We set

$$x := \{\mu, S, \bar{C}, \bar{e}, A\} \tag{3.4}$$

and let $\langle M_\alpha : \alpha < \mu^+ \rangle$ be a μ^+-approximating sequence over x. Define

$$E := \{\delta < \mu^+ : M_\delta \cap \mu^+ = \delta\}, \tag{3.5}$$

and fix $\delta \in S$ such that C_δ guesses E in the appropriate sense.

Now let β be any member of A greater than δ. To prove our lemma, we are first going to find a specific β^* (together with its associated k, m, and η) for which the required formula holds, and then use elementary submodel tricks to conclude that in fact there are *many* pairs β^*, β as needed. To accomplish this, we need to treat two cases, depending on whether the cofinality of μ is uncountable or not.

Case $\mathrm{cf}(\mu) > \aleph_0$:

This case is somewhat simpler than the other because $\bar{C}$ has stronger club-guessing properties (that is, we know $C_\delta \subseteq E$), and the "generalized" part of $\bar{e}$ is trivial: the superscripts don't matter. In the interest of readability, we will therefore eliminate the superscripts associated with generalized C-sequences from our notation in our consideration of this case.

Also, since we have fixed δ and β, we can simplify our notation even more and define

$$\gamma := \gamma(\delta, \beta), \tag{3.6}$$

$$\gamma^- := \gamma^-(\delta, \beta), \tag{3.7}$$

and finally

$$k := \rho_2(\delta, \beta). \tag{3.8}$$

(See the discussion surrounding (2.4) and (2.5) for a reminder of what these symbols mean.)

Next, fix β^* such that

$$\beta^* \in \operatorname{nacc}(C_\delta) \cap E, \tag{3.9}$$

$$\gamma^- < \max(C_\delta \cap \beta^*), \tag{3.10}$$

and

$$\operatorname{cf}(\beta^*) > |e_\gamma|. \tag{3.11}$$

Notice that these conditions are satisfied by all sufficiently large $\beta^* \in \operatorname{nacc}(C_\delta)$ because of our assumptions on $\bar{C}$ and $\bar{e}$. We now define

$$\eta := \sup(e_{\beta_{k-1}(\beta^*, \beta)} \cap \beta^*), \tag{3.12}$$

and we claim $\varphi_{k,m,\eta}(\beta^*, \beta)$ holds.

Clearly $\beta^* < \beta$, so the first requirement is of no concern. Since

$$\gamma^- < \beta^* < \delta, \tag{3.13}$$

we know that

$$\beta_i(\beta^*, \beta) = \beta_i(\delta, \beta), \tag{3.14}$$

and

$$\beta_i^-(\beta^*, \beta) = \beta_i^-(\delta, \beta) \tag{3.15}$$

for all $i < k$.

By definition, we have

$$\delta \in e_\gamma = e_{\beta_{k-1}(\delta, \beta)}, \tag{3.16}$$

and so by our choice of $\bar{e}$ we obtain

$$\beta^* \in C_\delta \subseteq e_\gamma = e_{\beta_{k-1}(\delta, \beta)} = e_{\beta_{k-1}(\beta^*, \beta)}. \tag{3.17}$$

It follows immediately that

$$\rho_2(\beta^*, \beta) = k, \tag{3.18}$$

and so we have obtained the second requirement of Definition 3.1.

Next, we note that for $i < k$ we have

$$\beta_i^-(\beta^*, \beta) = \beta_i^-(\delta, \beta) \leq \gamma^- < \max(C_\delta \cap \beta^*) \leq \sup(e_{\beta_{k-1}(\beta^*, \beta)} \cap \beta^*) = \eta.$$

From this, we see

$$\eta = \eta(\beta^*, \beta), \tag{3.19}$$

and we have met the third demand of Definition 3.1.

Finally, our requirement (3.11) taken together with (3.17) lets us conclude

$$\beta^* \in \mathrm{nacc}(e_{\beta_{k-1}(\beta^*,\beta)}), \tag{3.20}$$

and therefore

$$\eta = \sup(e_{\beta_{k-1}(\beta^*,\beta)} \cap \beta^*) < \beta^*. \tag{3.21}$$

Case $\mathrm{cf}(\mu) = \aleph_0$:

In this situation we must work a little harder. First, we define

$$\gamma^* := \sup\{\gamma^{m,-}(\delta, \beta) : m < \omega\}. \tag{3.22}$$

Since $\mathrm{cf}(\delta) > \aleph_0$, we know that $\gamma^* < \delta$, and

$$\beta_i^{m,-}(\delta, \beta) \leq \gamma^* \text{ for all } m < \omega \text{ and } i < \rho_2^m(\delta, \beta). \tag{3.23}$$

Next (see the discussion after Definition 2.2) we fix $m^* < \omega$ so that

$$m^* \leq m < \omega \Rightarrow \langle \beta_i^m(\delta, \beta) : i < \rho_2^m(\delta, \beta)\rangle = \langle \beta_i^{m^*}(\delta, \beta) : i < \rho_2^{m^*}(\delta, \beta)\rangle, \tag{3.24}$$

and define

$$k := \rho_2^{m^*}(\delta, \beta). \tag{3.25}$$

We then let $m \geq m^*$ be the least natural number for which

$$\mathrm{cf}(\gamma^{m^*}(\delta, \beta)) \leq \mu_m^+. \tag{3.26}$$

Taking this together with (2.15), we conclude

$$|e^m_{\gamma^m(\delta,\beta)}| = |e^m_{\gamma^{m^*}(\delta,\beta)}| \leq \mu_m^+. \tag{3.27}$$

Note as well that (2.16) tells us

$$C_\delta[m] \subseteq e^m_{\gamma^m(\delta,\beta)} \tag{3.28}$$

as well.

Our assumptions on $\bar{C}$ now allow us to find β^* satisfying the following:

- $\beta^* \in \mathrm{nacc}(C_\delta[m]) \cap E$,
- $\mathrm{cf}(\beta^*) > \mu_m^+$, and
- $\gamma^* < \max(C_\delta[m] \cap \beta^*)$.

Note that the last requirement can be achieved because the set of candidates satisfying the first two demands is unbounded in δ.

We now define

$$\eta := \sup(e^m_{\gamma^m(\delta,\beta)} \cap \beta^*). \tag{3.29}$$

The verification that $\varphi_{k,m,\eta}(\beta^*, \beta)$ holds follows the same broad outline as we saw in the preceding case. Once again, since $\beta^* \in C_\delta$ it is immediate that $\beta^* < \beta$.

Since $\gamma^* < \beta^* < \delta$ and $m \geq m^*$, it follows that for $i < k$ we have

$$\beta_i^m(\beta^*, \beta) = \beta_i^m(\delta, \beta) = \beta_i^{m^*}(\delta, \beta), \tag{3.30}$$

and in particular

$$\beta^m_{k-1}(\beta^*, \beta) = \gamma^m(\delta, \beta). \tag{3.31}$$

By (3.28), it follows that

$$\beta^m_k(\beta^*,\beta) = \min(e^m_{\beta^m_{k-1}(\beta^*,\beta)} \setminus \beta^*) = \beta^* \tag{3.32}$$

and we conclude

$$\rho^m_2(\beta^*,\beta) = k. \tag{3.33}$$

The fact that $\eta = \eta^m(\beta^*,\beta)$ also follows easily as we have ensured

$$\gamma^* < \max(C_\delta[m] \cap \beta^*) \le \sup(e^m_{\gamma^m(\beta^*,\beta)} \cap \beta^*) = \sup(e^m_{\gamma^m(\delta,\beta)} \cap \beta^*) = \eta. \tag{3.34}$$

Finally, since

$$\mathrm{cf}(\beta^*) > \mu^+_m \ge |e^m_{\gamma^m(\delta,\beta)}| = |e^m_{\beta^m_{k-1}(\delta,\beta)}|, \tag{3.35}$$

it follows that

$$\eta := \sup(e^m_{\gamma^m(\delta,\beta)} \cap \beta^*) < \beta^*, \tag{3.36}$$

and therefore $\varphi_{k,m,\eta}(\beta^*,\beta)$ holds, giving us what we need for the case $\mathrm{cf}(\mu) = \aleph_0$.

Combining the two cases, we find that we have k, m, η, β^*, δ, and β such that

- $\varphi_{k,m,\eta}(\beta^*,\beta)$ holds,
- $\beta^* < \delta < \beta$, and
- both β^* and δ are in E.

We finish the proof using standard elementary submodel arguments. Since the model M_δ contains x together with k, m, η, and β^*, but $M_\delta \cap \mu^+ = \delta \le \beta$, it follows that

$$(\exists^* \beta \in A)[\varphi_{k,m,\eta}(\beta^*,\beta)]. \tag{3.37}$$

Since x, k, m, and η are in M_{β^*} and $\beta^* = M_{\beta^*} \cap \mu^+$, we obtain

$$(\exists^{\mathrm{stat}} \beta^* < \mu^+)(\exists^* \beta \in A)[\varphi_{k,m,\eta}(\beta^*,\beta)], \tag{3.38}$$

as required. □

4. Main Theorem

Theorem 1. *Suppose μ is a singular cardinal. There is a function*

$$D : [\mu^+]^2 \to [\mu^+]^2 \times \mathrm{cf}(\mu) \tag{4.1}$$

such that for any unbounded $A \subseteq \mu^+$, there is a stationary $S \subseteq \mu^+$ such that

$$[S]^2 \times \mathrm{cf}(\mu) \subseteq \mathrm{ran}(D \restriction [A]^2). \tag{4.2}$$

Proof. Let $(\vec{\mu},\vec{f})$ be a scale for μ, and let $\bar{e}$ be a generalized C-sequence as in Lemma 3.3. Choose a function

$$\iota : \mathrm{cf}(\mu) \to \omega \times \omega \times \mathrm{cf}(\mu) \tag{4.3}$$

such that for any natural numbers k and m, and any $\delta < \mathrm{cf}(\mu)$, there are unboundedly many $\gamma < \mathrm{cf}(\mu)$ such that $\iota(\gamma) = \langle k, m, \delta\rangle$. Let $\Gamma : [\mu^+]^2 \to \mathrm{cf}(\mu)$ be the function from (2.17).

The definition of D will require several other auxiliary functions defined on $[\mu^+]^2$. First, we let k, m, and δ be the two-place functions defined by the recipe

$$\iota(\Gamma(\alpha,\beta)) = \langle k(\alpha,\beta), m(\alpha,\beta), \delta(\alpha,\beta)\rangle. \tag{4.4}$$

We then define

$$\beta^*(\alpha,\beta) = \beta^{m(\alpha,\beta)}_{k(\alpha,\beta)}(\alpha,\beta), \tag{4.5}$$

$$\eta^*(\alpha,\beta) = \eta^{m(\alpha,\beta)}(\beta^*(\alpha,\beta),\beta), \tag{4.6}$$

$$\alpha^*(\alpha,\beta) = \beta^{m(\alpha,\beta)}_{k(\alpha,\beta)}(\eta^*(\alpha,\beta)+1,\alpha), \tag{4.7}$$

and

$$D(\alpha,\beta) = \langle\{\alpha^*(\alpha,\beta),\beta^*(\alpha,\beta)\},\delta(\alpha,\beta)\rangle \tag{4.8}$$

whenever these make sense. If one of the above functions isn't actually defined for a particular α and β, we can just set it to some default value; the point is the above definitions make sense often enough to push our argument through.

The computation of $D(\alpha,\beta)$ can be described in English as follows. Given $\alpha<\beta$, we use ι and Γ to obtain k, m, and δ. The ordinal β^* is the k-th step in the m-walk from β to α, and η^* is the corresponding value of $\eta^m(\beta^*,\beta)$ computed from this walk. The ordinal α^* is then the k-th step in the m-walk from α down to η^*+1, and D returns the value $\langle\{\alpha^*,\beta^*\},\delta\rangle$.

Given an unbounded $A\subseteq\mu^+$, we fix k, m, and η as in Lemma 3.3. Now define

$$S^* := \{\beta^*<\mu^+ : (\exists^*\beta\in A)\varphi_{k,m,\eta}(\beta^*,\beta)\} \tag{4.9}$$

and

$$x := \{\bar{e},(\vec{\mu},\vec{f}),A,\eta\}. \tag{4.10}$$

Let $\langle M_\alpha : \alpha<\mu^+\rangle$ be a μ^+-approximating sequence over x, and define

$$S := \{\delta\in S^* : M_\delta\cap\mu^+ = \delta\}. \tag{4.11}$$

We claim that the stationary set S satisfies the conclusion of the theorem. Thus, given $\alpha^*<\beta^*$ in S and $\delta<\mathrm{cf}(\mu)$, we must find $\alpha<\beta$ in A such that $D(\alpha,\beta)=\langle\{\alpha^*,\beta^*\},\delta\rangle$. We will do this by striving for the following goal:

Goal: Find $\alpha<\beta$ in A such that

(1) $\alpha^*<\alpha<\beta^*<\beta$,

(2) $\varphi_{k,m,\eta}(\beta^*,\beta)$,

(3) $\varphi_{k,m,\eta}(\alpha^*,\alpha)$, and

(4) $\iota(\Gamma(\alpha,\beta))=\langle k,m,\delta\rangle$.

Proposition 4.1. If α and β are as above, then $D(\alpha,\beta)=\langle\{\alpha^*,\beta^*\},\delta\rangle$.

Proof. By (4), we know

$$k(\alpha,\beta)=k, \tag{4.12}$$

$$m(\alpha,\beta)=m, \tag{4.13}$$

and

$$\delta(\alpha,\beta)=\delta. \tag{4.14}$$

Since $\varphi_{k,m,\eta}(\beta^*,\beta)$ holds and $\eta<\alpha<\beta^*$, an application of Lemma 3.2 tells us

$$\beta^*(\alpha,\beta)=\beta^m_k(\alpha,\beta)=\beta^*. \tag{4.15}$$

The definition of η^* together with the fact that $\varphi_{k,m,\eta}(\beta^*, \beta)$ holds informs us that

$$\eta^*(\alpha, \beta) = \eta^m(\beta^*, \beta) = \eta, \tag{4.16}$$

and we can once again apply Lemma 3.2 (this time using $\varphi_{k,m,\eta}(\alpha^*, \alpha)$) to conclude that

$$\alpha^*(\alpha, \beta) = \beta^m_k(\eta + 1, \alpha) = \alpha^*, \tag{4.17}$$

as required.

□

So how do we go about obtaining our goal? We start by choosing $\beta \in A$ for which $\varphi_{k,m,\eta}(\beta^*, \beta)$ is true. We set

$$y := x \cup \{\alpha^*\}, \tag{4.18}$$

and define

$$\mathcal{M} := \mathrm{Sk}_{\mathfrak{A}}(y \cup \mathrm{cf}(\mu)). \tag{4.19}$$

Since $y \in M_{\alpha^*+1}$, the construction of $\mathcal{M}$ can be done in the model M_{α^*+2} by taking the Skolem hull of $y \cup \mathrm{cf}(\mu)$ in the model M_{α^*+1}. Thus,

$$\mathcal{M} \in M_{\alpha^*+1} \subseteq M_{\beta^*}. \tag{4.20}$$

From this it follows that

$$\mathrm{Ch}^{\vec{\mu}}_{\mathcal{M}} \in M_{\beta^*} \cap \prod_{i<\mathrm{cf}(\mu)} \mu_i.$$

Since $\beta^* = M_{\beta^*} \cap \mu^+ < \beta$ and $(\vec{\mu}, \vec{f})$ is a scale, we conclude that there is an $i_0 < \mathrm{cf}(\mu)$ such that

$$\mathrm{Ch}^{\vec{\mu}}_{\mathcal{M}}(i) < f_\beta(i) \text{ whenever } i_0 \leq i < \mathrm{cf}(\delta). \tag{4.21}$$

Our next move is to note that since $I := \{\alpha \in A : \varphi_{k,m,\eta}(\alpha^*, \alpha)\}$ is unbounded, the sequence $\langle f_\alpha : \alpha \in I\rangle$ together with $\vec{\mu}$ forms (modulo re-indexing) a scale for μ. Thus we can apply Lemma 2.4 and fix a value i_1 such that whenever $i_1 \leq i < \mathrm{cf}(\delta)$,

$$(\forall \zeta < \mu_i)(\exists^* \alpha \in A)[\varphi_{k,m,\eta}(\alpha^*, \alpha) \wedge \zeta < f_\alpha(i)]. \tag{4.22}$$

Given our choice of the function ι, it follows that we can choose $i^* < \mathrm{cf}(\mu)$ such that $\max\{i_0, i_1\} < i^*$ and $\iota(i^*) = \langle k, m, \delta\rangle$. In particular, for this choice of i^* we have

$$\mathrm{Ch}^{\vec{\mu}}_{\mathcal{M}} \restriction [i^*, \mathrm{cf}(\mu)) < f_\beta \restriction [i^*, \mathrm{cf}(\mu)), \tag{4.23}$$

$$(\forall \zeta < \mu_{i^*})(\exists^* \alpha \in A)[\varphi_{k,m,\eta}(\alpha^*, \alpha) \wedge \zeta < f_\alpha(i^*)], \tag{4.24}$$

and

$$\iota(i^*) = \langle k, m, \delta\rangle. \tag{4.25}$$

We now define

$$\mathcal{N} = \mathrm{Sk}_{\mathfrak{A}}(\mathcal{M} \cup \mu_{i^*}). \tag{4.26}$$

Notice that (4.24) holds in $\mathcal{N}$ as this model contains i^* and all parameters relevant to the formula. We have also ensured that the ordinal $f_\beta(i^*)$ is in $\mathcal{N}$ too. Thus, we can find an ordinal α such that

$$\alpha \in \mathcal{N} \cap A, \tag{4.27}$$

$$\varphi_{k,m,\eta}(\alpha^*, \alpha), \tag{4.28}$$

and

$$f_\beta(i^*) < f_\alpha(i^*). \tag{4.29}$$

We claim now that α and β are as required. The following statements are immediate from our preceding work:

- α and β are in A,
- $\alpha^* < \alpha < \beta^* < \beta$,
- $\varphi_{k,m,\eta}(\alpha^*, \alpha)$, and
- $\varphi_{k,m,\eta}(\beta^*, \beta)$,

and so we will achieve our goal provided we can show $\Gamma(\alpha, \beta) = i^*$.

This, however, follows almost immediately by a standard argument. Since $f_\beta(i^*) < f_\alpha(i^*)$, it is clear that $i^* \leq \Gamma(\alpha, \beta)$. By Lemma 2.7, we know

$$\mathrm{Ch}^{\vec{\mu}}_{\mathcal{N}} \upharpoonright [i^* + 1, \mathrm{cf}(\delta)) = \mathrm{Ch}^{\vec{\mu}}_{\mathcal{M}} \upharpoonright [i^* + 1, \mathrm{cf}(\delta)), \tag{4.30}$$

and so (4.23) in combination with (4.27) implies $\Gamma(\alpha, \beta) \leq i^*$ as well. Thus $\Gamma(\alpha, \beta) = i^*$, and we have achieved our goal. As noted before, this is enough to finish the proof of the theorem.

□

5. Conclusions

In this last section we will deduce several results as corollaries of Theorem 1, including those results mentioned in our introduction.

Proposition 5.1. Suppose μ is a singular cardinal, and let $\langle \theta_i : i < \mathrm{cf}(\mu) \rangle$ be a sequence of cardinals with supremum θ^*. If $\mu^+ \nrightarrow [\mu^+]^2_{\theta_i}$ for each i, then

$$\mu^+ \nrightarrow [\mu^+]^2_{\theta^*}. \tag{5.1}$$

Proof. For each $i < \mathrm{cf}(\mu)$, let $c_i : [\mu^+]^2 \to \theta_i$ witness $\mu^+ \nrightarrow [\mu^+]^2_{\theta_i}$. Define the coloring $d : [\mu^+]^2 \to \theta^*$ by

$$d(\alpha, \beta) = c_{m^*(\alpha,\beta)}(\alpha^*(\alpha, \beta), \beta^*(\alpha, \beta)). \tag{5.2}$$

Given an unbounded $A \subseteq \mu^+$, let S be the stationary set provided by the second conclusion of Theorem 1. Given $\varsigma < \theta^*$, we choose m^* such that $\varsigma < \theta_{m^*}$, and then select $\alpha^* < \beta^*$ in S such that $c_{m^*}(\alpha^*, \beta^*) = \varsigma$. Theorem 1 tells us that there are $\alpha < \beta$ in A for which $D(\alpha, \beta) = \langle \alpha^*, \beta^*, m^* \rangle$, and so we have

$$d(\alpha, \beta) = c_{m^*(\alpha,\beta)}(\alpha^*(\alpha, \beta), \beta^*(\alpha, \beta)) = c_{m^*}(\alpha^*, \beta^*) = \varsigma, \tag{5.3}$$

as required. □

Corollary 5.2. Let μ be a singular cardinal, and let θ be the least cardinal for which $\mu \to [\mu^+]^2_\theta$. Then $\mathrm{cf}(\mu) < \mathrm{cf}(\theta)$.

The following proposition yields the result from the abstract as an immediate corollary.

Proposition 5.3. The following statements are equivalent for a singular cardinal μ:

(1) $\mu^+ \nrightarrow [\mu^+]^2_{\mu^+}$

(2) $\mu^+ \nrightarrow [\mu^+]^2_{\mu}$

(3) $\mu^+ \nrightarrow [\mu^+]^2_\theta$ for all $\theta < \mu$
(4) $\mu^+ \nrightarrow [\mu^+]^2_\theta$ for arbitrarily large $\theta < \mu$.

Proof. The equivalence of (1) and (2) is given by Proposition 1.2. Each statement on the list implies the next, we need only establish that (4) implies (2), but this is immediate from Proposition 5.1.

□

Our final proposition also seems to be of interest.

Proposition 5.4. Let μ be a singular cardinal, and let $\theta \leq \mu^+$. Then the following statements are equivalent:

(1) $\mu^+ \nrightarrow [\mu^+]^2_\theta$
(2) There is a function $c : [\mu^+]^2 \to \theta$ such that whenever $T \subseteq \mu^+$ is stationary and $\varsigma < \theta$, there are $\alpha < \beta$ in T with $c(\alpha, \beta) = \varsigma$.

Proof. It is clear that (1) implies (2), so let us assume that c is as in (2). Define the function $d : [\mu^+]^2 \to \theta$ by

$$d(\alpha, \beta) = c(\alpha^*(\alpha, \beta), \beta^*(\alpha, \beta)). \tag{5.4}$$

Suppose now that we are given an unbounded $A \subseteq \mu^+$ and $\varsigma < \theta$. Let S be the stationary set guaranteed to exist by Theorem 1, and choose $\alpha^* < \beta^*$ in S^* with $c(\alpha^*, \beta^*) = \varsigma$. Then there are $\alpha < \beta$ in A for which $\alpha^*(\alpha, \beta) = \alpha^*$ and $\beta^*(\alpha, \beta) = \beta^*$, and

$$d(\alpha, \beta) = c(\alpha^*(\alpha, \beta), \beta^*(\alpha, \beta)) = c(\alpha^*, \beta^*) = \varsigma \tag{5.5}$$

as required. □

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