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Near coherence and filter games

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Abstract. We investigate a two-player game involving pairs of filters on ω. Our results generalize a result of Shelah ([7] Chapter VI) dealing with applications of game theory in the study of ultrafilters.

1. Introduction

In this note, we investigate a game between two players I and II that involves a given pair of filters on the natural numbers. Our notation is fairly standard, but we give a quick review before plunging ahead.

The natural numbers will be denoted by ω, and by ω^ω we mean the set of all functions from the natural numbers to the natural numbers. By 'filter', we mean a collection of non-empty subsets of ω that contains all the cofinite sets and is closed under finite intersections and taking supersets, i.e., we only consider non-principal filters on the natural numbers. A filter $\mathscr{F}$ is said to be a P-filter if, given a countable set $\{X_n : n \in \omega\}$ of members of $\mathscr{F}$, we can find a set $Y \in \mathscr{F}$ such that $Y \setminus X_n$ is finite for each n. We denote this by $Y \subseteq^* X_n$, or say that Y is almost a subset of each X_n.

We also need the following standard notion.

Definition 1.1. *Let $\mathscr{F}$ be a filter, and let f be a function in ω^ω. The filter $f(\mathscr{F})$ is defined by*

$$X \in f(\mathscr{F}) \iff f^{-1}(X) \in \mathscr{F}. \tag{1.1}$$

It is easily seen that if f is finite-to-one, then $f(\mathscr{F})$ is a filter as well. (If we are willing to allow our filters to contain finite sets, then we do not need the finite-to-one restriction on f.) The following definitions are again standard ones.

Definition 1.2. *A filter $\mathscr{F}$ is an ultrafilter if it is not properly included in any filter. An ultrafilter that is also a P-filter is called a P-point. If $\mathscr{U}$ and $\mathscr{V}$ are ultrafilters, and there is a function f such that $f(\mathscr{U}) = \mathscr{V}$, then we say that $\mathscr{V} \leq_{\mathrm{RK}} \mathscr{U}$. This relation (the Rudin–Keisler ordering) partially orders the set of all ultrafilters on ω.*

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We need a concept related to the Rudin-Keisler ordering.

Definition 1.3. *Let $\mathscr{F}_0$ and $\mathscr{F}_1$ be filters. We say that $\mathscr{F}_0$ and $\mathscr{F}_1$ are nearly coherent if there is a finite-to-one function f such that*

$$f(\mathscr{F}_0) \cup f(\mathscr{F}_1) \text{ has the finite intersection property,} \tag{1.2}$$

by which we mean the intersection of finitely many members of $f(\mathscr{F}_0) \cup f(\mathscr{F}_1)$ is non-empty.

It is readily seen that if $\mathscr{F}_0$ and $\mathscr{F}_1$ are nearly coherent and f is finite-to-one, then in fact the intersection of finitely many members of $f(\mathscr{F}_0) \cup f(\mathscr{F}_1)$ must be infinite. The notion of near coherence is due to Blass [1], and it is consistent with the usual axioms of set theory that all pairs of filters on ω are nearly coherent [3].

2. A Characterization of near coherence

We begin with a simple characterization of when two filters fail to be nearly coherent; it follows immediately from the definitions.

Proposition 2.1. *Two filters $\mathscr{F}_0$ and $\mathscr{F}_1$ are not nearly coherent if and only if for every partition $\Pi = \{a_n : n \in \omega\}$ of ω into finite sets, we can find disjoint A_0 and A_1 so that*

$$\bigcup_{n\in A_0} a_n \in \mathscr{F}_0 \quad \text{and} \quad \bigcup_{n\in A_1} a_n \in \mathscr{F}_1. \tag{2.1}$$

Notice that this is equivalent to the existence of a single A for which

$$\bigcup_{n\in A} a_n \in \mathscr{F}_0 \quad \text{and} \quad \bigcup_{n\notin A} a_n \in \mathscr{F}_1. \tag{2.2}$$

Notice also that if $\mathscr{F}_0$ and $\mathscr{F}_1$ are not nearly coherent, and $\Pi = \{I_n : n \in \omega\}$ is a partition of ω into finite itervals, we can merge blocks of consecutive intervals of Π to obtain a new partition $\Pi' = \{J_n : n \in \omega\}$ that satisfies

$$\bigcup_{n\in \omega} J_{2n} \in \mathscr{F}_0 \quad \text{and} \quad \bigcup_{n\in \omega} J_{2n+1} \in \mathscr{F}_1. \tag{2.3}$$

The following proposition is the key to several applications of near coherence. In some sense, it says that if $\mathscr{F}_0$ and $\mathscr{F}_1$ are not nearly coherent, we can find sets in each filter that stay far away from each other.

Proposition 2.2. *Suppose $\mathscr{F}_0$ and $\mathscr{F}_1$ are not nearly coherent, and let $\Pi = \{I_n : n \in \omega\}$ be a partition of ω into finite intervals. Then we can find disjoint $X_0 \in \mathscr{F}_0$ and $X_1 \in \mathscr{F}_1$ such that between intervals of Π met by X_0 and those met by X_1, there is at least one "buffer" interval that meets neither.*

Proof. To begin, find $A \subseteq \omega$ such that

$$\bigcup_{n \in A} I_n \in \mathscr{F}_0 \quad \text{and} \quad \bigcup_{n \notin A} I_n \in \mathscr{F}_1. \tag{2.4}$$

By merging blocks of consecutive Π intervals, we can form a partition $\Pi' = \{J_n : n \in \omega\}$ such that

$$\bigcup_{n \in \omega} J_{2n} \in \mathscr{F}_0 \quad \text{and} \quad \bigcup_{n \in \omega} J_{2n+1} \in \mathscr{F}_1. \tag{2.5}$$

Since each interval of Π' is a union of intervals from Π, if we establish the result for Π' then we have the result for Π as well, so we might as well forget about Π' and assume that Π satisfies

$$\bigcup_{n \in \omega} I_{2n} \in \mathscr{F}_0 \quad \text{and} \quad \bigcup_{n \in \omega} I_{2n+1} \in \mathscr{F}_1. \tag{2.6}$$

Now let $f : \omega \to \omega$ be defined by

$$f^{-1}(\{n\}) = I_{2n} \cup I_{2n+1}. \tag{2.7}$$

Since $\mathscr{F}_0$ and $\mathscr{F}_1$ are not nearly coherent, we can find disjoint A_0 and A_1 such that $f^{-1}(A_0) \in \mathscr{F}_0$ and $f^{-1}(A_1) \in \mathscr{F}_1$.

Next let g be defined by setting $g^{-1}(\{0\}) = I_0 \cup I_1 \cup I_2$, and

$$g^{-1}(\{n+1\}) = I_{2n+3} \cup I_{2n+4}. \tag{2.8}$$

Again, we can find disjoint B_0 and B_1 such that

$$g^{-1}(B_i) \in \mathscr{F}_i. \tag{2.9}$$

For technical reasons, assume that 0 is not in $B_0 \cup B_1$.

Define sets X_0 and X_1 by

$$X_0 = f^{-1}(A_0) \cap g^{-1}(B_0) \cap \bigcup_{n \in \omega} I_{2n} \in \mathscr{F}_0, \tag{2.10}$$

$$X_1 = f^{-1}(A_1) \cap g^{-1}(B_1) \cap \bigcup_{n \in \omega} I_{2n+1} \in \mathscr{F}_1. \tag{2.11}$$

Now assume that $X_0 \cap I_n \neq \emptyset$. We arranged things so that $n \neq 0$ and n is even. Since no interval of Π meets both X_0 and X_1, we know that $X_1 \cap I_n = \emptyset$. Since $X_0 \subseteq f^{-1}(A_0)$ and $X_1 \subseteq f^{-1}(A_1)$, we know that

$$X_1 \cap I_{n+1} = \emptyset. \tag{2.12}$$

Since $X_0 \subseteq g^{-1}(B_0)$ and $X_1 \subseteq g^{-1}(B_1)$, we know that

$$X_1 \cap I_{n-1} = \emptyset. \tag{2.13}$$

Thus if $X_0 \cap I_n \neq \emptyset$, then

$$X_1 \cap (I_{n-1} \cup I_n \cup I_{n+1}) = \emptyset, \tag{2.14}$$

as desired. □

We can merge blocks of consecutive intervals to obtain the following result.

Corollary 2.3. *If $\mathscr{F}_0$ and $\mathscr{F}_1$ are not nearly coherent, and $\Pi = \{I_n : n \in \omega\}$ is a partition of ω into finite intervals, then we can merge blocks of consecutive intervals from Π to obtain a partition $\Pi' = \{J_n : n \in \omega\}$ that satisfies*

$$\bigcup_{n\in\omega} J_{4n+1} \in \mathscr{F}_0 \quad \text{and} \quad \bigcup_{n\in\omega} J_{4n+3} \in \mathscr{F}_1. \tag{2.15}$$

Proof. The proof is easy. Given Π, we can get sets $X_0 \in \mathscr{F}_0$ and $X_1 \in \mathscr{F}_1$ satisfying the conclusion of Proposition 2.2. Let $\{y_n : n \in \omega\}$ be the increasing enumeration of $X_0 \cup X_1$. If y_n and y_{n+1} are from different X_i's, then there is an interval of Π contained between y_n and y_{n+1}. For every such pair of y's, choose a buffer interval from Π, and then merge all intervals that lie between consecutive buffer intervals. Finally, merge together an initial segment of the intervals we have built to obtain the property (2.15). □

Arguments similar to those of Blass [2] show that in fact the preceding proposition characterizes near coherence. Although in this paper we only consider a single application of Proposition 2.2, the conclusion of the proposition is a very useful hypothesis in combinatorial arguments, e.g. [4].

Definition 2.4. *Let $\mathscr{F}_0$ and $\mathscr{F}_1$ be filters on ω. The game $\mathscr{G}(\mathscr{F}_0, \mathscr{F}_1)$ has two players and is of length ω. At stage $2n$, Player I selects a finite set $a_{2n} \subseteq \omega$ disjoint from either player's previous even moves, and then Player II responds by choosing a finite set b_{2n} disjoint from a_{2n} as well as either player's previous even moves. On the odd stages, we do the same thing – I chooses a_{2n+1} and II chooses b_{2n+1} – except now both players stay disjoint from the odd moves selected previously by any player. After ω stages, Player II is declared the winner if*

$$\bigcup_{n\in\omega} b_{2n} \in \mathscr{F}_0 \quad \text{and} \quad \bigcup_{n\in\omega} b_{2n+1} \in \mathscr{F}_1. \tag{2.16}$$

Since it is in both players' advantage to choose as large a set as possible, we can assume that each move by either player is in fact an interval, and that there are no gaps between moves. From this perspective, the game can be viewed as building two partitions of ω, one interval at a time. Player II wins if the union of her pieces of each partition lies in the appropriate filter. Notice also that if $\mathscr{F}_0$ and $\mathscr{F}_1$ are the same filter, then Player I can win by mirroring II's strategy. The following theorem can be viewed as an analysis of when this 'mirror' strategy will fail.

Theorem 1. *Let $\mathscr{F}_0$ and $\mathscr{F}_1$ be two filters on ω. Player I has a winning strategy in the game $\mathscr{G}(\mathscr{F}_0, \mathscr{F}_1)$ if and only if $\mathscr{F}_0$ and $\mathscr{F}_1$ are nearly coherent.*

Proof. First, assume that $\mathscr{F}_0$ and $\mathscr{F}_1$ are nearly coherent, and fix a finite-to-one function f so that

$$f(\mathscr{F}_0) \cup f(\mathscr{F}_1) \text{ has the finite intersection property.} \tag{2.17}$$

We adopt the point of view that Player I is building two (as yet unspecified) sets A_0 and A_1, and Player II is building two sets B_0 and B_1, subject to the constraints that A_i and B_i must be disjoint. The goal of Player I 's strategy will be to ensure that $f(B_0) \cap f(B_1) = \emptyset$; condition (2.17) tells that that one of B_0 and B_1 fails to be in the corresponding filter. To realize this goal, Player I will tailor his strategy to achieve

$$f^{-1}(f[B_1]) \subseteq A_0 \quad \text{and} \quad f^{-1}(f[B_0]) \subseteq A_1. \tag{2.18}$$

Player I should choose $a_0 = \{0\}$ as his initial move, to which Player II responds by choosing b_0. At stage 1, Player I can select $f^{-1}(f[b_0])$ as his a_1, and Player II will respond with b_1.

We claim that $f^{-1}(f[b_1]) \cap b_0 = \emptyset$. To see this, note that because

$$a_1 \cap b_1 = f^{-1}(f[b_0]) \cap b_1 = \emptyset, \tag{2.19}$$

we know that $f[b_0]$ and $f[b_1]$ are disjoint. This implies that $f^{-1}(f[b_0])$ and $f^{-1}(f[b_1])$ are disjoint, and since b_0 is a subset of $f^{-1}(f[b_0])$, the claim is established.

Thus Player I can choose a_2 so that $f^{-1}(f[b_1]) \subseteq a_0 \cup a_2$.

The argument we just gave for stage 2 generalizes to higher stages – at stage $2n$, Player I chooses a_{2n} so that

$$a_0 \cup a_2 \cup \ldots \cup a_{2n} \supseteq f^{-1}(f[b_{2n-1}]), \tag{2.20}$$

and at stage $2n+1$, Player I chooses a_{2n+1} so that

$$a_1 \cup a_3 \cup \ldots \cup a_{2n+1} \supseteq f^{-1}(f[b_{2n}]). \tag{2.21}$$

After the game ends, Player I has achieved $f(B_0) \cap f(B_1) = \emptyset$ as desired. We can utilize some fixed well-ordering of $[\omega]^{<\omega}$ to convert the preceding argument into a well-defined winning strategy for Player I .

For the converse, assume that $\mathscr{F}_0$ and $\mathscr{F}_1$ are not nearly coherent, and fix a strategy $\mathscr{S}$ for Player I , i.e., $\mathscr{S}$ is a function that takes as input initial segments of the game that are empty or end with a move by Player II , and outputs a response for Player I . We show that Player II can defeat the strategy $\mathscr{S}$.

Notice that for each n there are only finitely many σ such that

1. σ is a finite legal sequences of moves
2. I's actions in σ have been dictated by $\mathscr{S}$
3. the last entry in σ is a move by Player II
4. all sets appearing in σ are subsets of n.

We call such a σ an $\mathscr{S}$–game of rank $\leq n$.

Fix a strictly increasing function f such that $f(0) = 0$, and if σ is an $\mathscr{S}$-game of rank $\leq f(n)$, then $\max \mathscr{S}(\sigma) < f(n+1)$. This is easily achieved, as there are only finitely many such σ to consider for each n.

Define a partition $\Pi = \{I_n : n \in \omega\}$ by setting $I_n = [f(n), f(n+1)]$. Since $\mathscr{F}_0$ and $\mathscr{F}_1$ are not nearly coherent, we can apply Corollary 2.3 to get a partition $\Pi' = \{J_n : n \in \omega\}$ such that each J_n is a union of intervals from Π, and

$$\bigcup_{n\in\omega} J_{4n+1} \in \mathscr{F}_0 \quad \text{and} \quad \bigcup_{n\in\omega} J_{4n+3} \in \mathscr{F}_1. \tag{2.22}$$

Since each interval from Π' is a union of intervals from Π, it follows that for each n, $f(\min J_n) \leq \min J_{n+1}$.

We claim that if Player I uses his strategy $\mathscr{S}$, at stage $2n$ Player II will be able to choose J_{4n+1}, and at stage $2n+1$ Player II will be able to choose J_{4n+3}.

Player I 's initial move, $a_0 = \mathscr{S}(\emptyset)$, is a subset of $f(1)$, and so $J_1 \cap a_0 = \emptyset$. This means that Player II is free to select $b_0 = J_1$ as her response. Since $\sigma = \langle a_0, b_0\rangle$ is an $\mathscr{S}$-game of rank $\leq \min J_2$, we know that $a_1 = \mathscr{S}(\sigma)$ will be a subset of J_2, and hence Player II is free to select $b_1 = J_3$.

Continuing in the fashion, we see that Player II can achieve

$$\bigcup_{n\in\omega} b_{2n} = \bigcup_{n\in\omega} J_{4n+1} \in \mathscr{F}_0 \tag{2.23}$$

and

$$\bigcup_{n\in\omega} b_{2n+1} = \bigcup_{n\in\omega} J_{4n+3} \in \mathscr{F}_1, \tag{2.24}$$

and therefore she has defeated the strategy $\mathscr{S}$. □

3. Variations and generalizations

In light of the results of the preceding section, it is natural to ask what happens if you forget about the second filter and consider the simplified version of the game that concerns itself only with the first filter; this is essentially the same thing as both players agreeing to forget about the odd-numbered moves in the original game. It is not difficult to see that Player I has a winning strategy in this game if and only if the filter has the Property of Baire – a fact noted by Laflamme [6], and that this game is undetermined when the filter is a non-principal ultrafilter. If we look at what happens when we restrict our attention to ultrafilters, we see that $\mathscr{U}$ and $\mathscr{V}$ nearly coherent means that there is a finite-to-one function f for which $f(\mathscr{U}) = f(\mathscr{V})$. This follows because $f(\mathscr{U})$ and $f(\mathscr{V})$ are ultrafilters, and by maximality they must be equal. This is the notion (when translated into the context of ultraproducts of models of arithmetic) of "cofinal equivalence" studied by Blass [1].

It is a well-known result of Galvin and Mackenzie [5] that P-points can be characterized in terms of two-player games. We consider here a slight modification of the game from the previous section; it is closely related to the characterization of P-points just mentioned. The result we obtain is also closely related to work of Shelah [7] on the Rudin–Keisler ordering of P-points.

Definition 3.1. *Let $\mathscr{F}_0$ and $\mathscr{F}_1$ be two filters on ω. The game $\mathscr{G}'(\mathscr{F}_0, \mathscr{F}_1)$ involves two players, I and II, who alternate in selecting subsets of ω. At an even stage, say $2n$, Player I chooses a set $A_n \in \mathscr{F}_0$, and Player II responds by choosing a finite subset a_n of A_n. At stage $2n+1$, I chooses $B_n \in \mathscr{F}_1$ and Player II responds by choosing $b_n \subseteq B_n$ finite. After ω stages, Player II is declared the winner if and only if*

$$\bigcup_{n\in\omega} a_n \in \mathscr{F}_0 \quad \text{and} \quad \bigcup_{n\in\omega} b_n \in \mathscr{F}_1 \tag{3.1}$$

The game $\mathscr{G}'(\mathscr{F}_0, \mathscr{F}_1)$ is closely related to the game $\mathscr{G}(\mathscr{F}_0, \mathscr{F}_1)$; the difference is that things are now easier for Player I . In $\mathscr{G}$, I is required to choose a finite set at each move, and II responds with something disjoint. If we shift our point of view, we can view this as Player I choosing a cofinite set, and Player II choosing a finite subset of Player I 's choice. Thus a legal move for Player I in $\mathscr{G}$ is essentially a legal move for him in $\mathscr{G}'$. Equivalently, we could view $\mathscr{G}'$ as allowing Player I to choose as his move something that is disjoint from a set in the appropriate filter, and Player II chooses a finite subset of whatever I has left behind.

Proposition 3.2. *If $\mathscr{F}_0$ and $\mathscr{F}_1$ are P-filters, then Player I has a winning strategy in $\mathscr{G}(\mathscr{F}_0, \mathscr{F}_1)$ if and only if he has a winning strategy in $\mathscr{G}'(\mathscr{F}_0, \mathscr{F}_1)$.*

Proof. We have seen how I can convert a winning $\mathscr{G}$ strategy into a winning $\mathscr{G}'$ strategy – he simply plays in $\mathscr{G}'$ the complement of the set he wants to play in $\mathscr{G}$.

If $\mathscr{S}$ is a strategy for Player I in the game $\mathscr{G}'(\mathscr{F}_0, \mathscr{F}_1)$, he can take advantage of the fact that the filters are P-filters to find sets $X_0 \in \mathscr{F}_0$ and $X_1 \in \mathscr{F}_1$ that are almost included in every set used by $\mathscr{S}$ at even and odd moves, respectively. This means that all of $\mathscr{S}$'s responses are co-finite when restricted to the appropriate X_i, and thus we can convert a winning strategy for I in $\mathscr{G}'(\mathscr{F}_0, \mathscr{F}_1)$ into a winning strategy for him in $\mathscr{G}(\mathscr{F}_0, \mathscr{F}_1)$. □

It is well-known that if $\mathscr{U}$ is a P-point, then any function $f \in \omega^\omega$ becomes finite-to-one or constant when restricted to a set in $\mathscr{U}$. The main portion of the proof of the following result is the same as the last part of the proof of the implication from (a) to (b) in Theorem 8 of [1].

Proposition 3.3. *Let $\mathscr{U}$ and $\mathscr{V}$ be P-points. Then Player I has a winning strategy in the game $\mathscr{G}(\mathscr{U}, \mathscr{V})$ (equivalently, the game $\mathscr{G}'(\mathscr{U}, \mathscr{V})$) if and only if $\mathscr{U}$ and $\mathscr{V}$ have a common lower bound in the Rudin–Keisler ordering.*

Proof. If $\mathscr{U}$ and $\mathscr{V}$ are nearly coherent, then clearly they have a common lower bound. Conversely, suppose we have functions f and g in ω^ω such that $f(\mathscr{U}) = g(\mathscr{V})$. If $\mathscr{U} = \mathscr{V}$, then we know they are nearly coherent for trivial reasons, so we can assume that there are disjoint sets $X \in \mathscr{U}$ and $Y \in \mathscr{V}$. Since $\mathscr{U}$ and $\mathscr{V}$ are both P-points, we may assume that f is finite-to-one when restricted to X, and g is finite-to-one when restricted to Y (recall that we are not considering principal ultrafilters in this paper, so neither f nor g is constant on a set in $\mathscr{U}$ or $\mathscr{V}$ respectively). If we define a function h by setting it equal to f on X and equal to g on Y (and arbitrary elsewhere), then h witnesses that $\mathscr{U}$ and $\mathscr{V}$ are nearly coherent. □

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