COUNTABLE COMPACTNESS, HEREDITARY π–CHARACTER, AND THE CONTINUUM HYPOTHESIS

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Abstract. We prove that the Continuum Hypothesis is consistent with the statement that countably compact regular spaces that are hereditarily of countable π–character are either compact or contain an uncountable free sequence. As a corollary we solve a well–known open question by showing that the existence of a compact S–space of size greater than $\aleph_1$ does not follow from the Continuum Hypothesis.

1. Introduction

S–spaces — regular topological spaces that are hereditarily separable but not hereditarily Lindelöf — have long been objects of interest in set–theoretic topology. For evidence of this, one need only take a quick glance at Roitman's 1984 survey [14] to see how often such spaces arise. It is well–known that the Continuum Hypothesis implies that many types of S–spaces exist, and the stronger axiom $\diamondsuit$ implies the existence of S–spaces with exotic properties — Ostaszewski's space [13] and Fedorčuk's space [6] are two examples of this phenomenon.

Fedorčuk's space is particularly interesting. It is a compact S–space in which every infinite closed subset has cardinality $2^{\aleph_1}$, and thus it contains no non–trivial convergent sequences. One of the goals of this paper is to prove that the Continuum Hypothesis alone is not enough to construct a space like this.

We do this by proving a more general theorem illuminating the relationship between compactness and countable compactness in regular spaces. More precisely, we establish that the Continuum Hypothesis (CH) is consistent with the following statement (abbreviated $\otimes$):

$\otimes$ A regular countably compact space that is hereditarily of countable π–character is either compact or contains an uncountable free sequence.

From this, we deduce that it is consistent with CH that all compact S–spaces are sequential. (Recall that a space X is sequential if a subset $A \subseteq X$ is closed if and only if A is closed under limits of convergent sequences.) As a corollary, we solve a well–known question (see [1]) by showing that the Continuum Hypothesis does not imply the existence of a compact S–space of size greater than $\aleph_1$. In particular, the construction of spaces like that of Fedorčuk requires something stronger than just the Continuum Hypothesis.

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We point out that Hajnal and Juhász [7] have constructed a countably compact, non–compact S–space from the Continuum Hypothesis. This highlights the role that hereditary π–character plays in our results, for their example shows that the Continuum Hypothesis implies the failure of the version of $\otimes$ obtained by replacing $h\pi\chi(X) = \aleph_0$ with the weaker condition of countable tightness. To see this, we note that since their space is hereditarily separable, it is countably tight and cannot contain an uncountable free sequence.)

We obtain our result by utilizing a proper forcing that destroys certain countably compact, non–compact S–spaces without any new reals appearing in the generic extension. We prove that the forcings of interest to us can be iterated without introducing new reals, and then show that the model obtained satisfies the desired conclusions.

The work presented here extends work of the author in [2]. In that paper, we showed that the Continuum Hypothesis is consistent with the following statement:

> A regular first countable, countably compact space is either compact or contains an uncountable free sequence.

A first countable space is hereditarily of countable π–character, so the model presented here subsumes that presented in [2]. In particular, this paper presents another model of CH with no Ostaszewski spaces.

Our notation is standard. We use the techniques of totally proper forcing developed in [5] and [2], and we assume that the reader has some familiarity with the milieu of proper forcing. Good references for the topological tools we use are [8], [10], and [16].

2. Preliminaries

In the introductory section we freely used many terms from set–theoretic topology, so before we proceed any further, we will take some time to define the most important of these. Hodel's article [8] is a standard source for this material.

Definition 2.1. Let X be a topological space. A sequence $\{x_\alpha : \alpha < \kappa\}$ is a free sequence (of length κ) if for each $\alpha < \kappa$,

$$\overline{\{x_\beta : \beta < \alpha\}} \cap \overline{\{x_\beta : \beta \geq \alpha\}} = \emptyset. \tag{2.1}$$

Clearly any space that contains an uncountable free sequence is not hereditarily separable, and therefore not an S–space.

Definition 2.2. Let A be a subset of the topological space X. A family $\mathcal{B}$ of subsets of X is a π–network for A in X if every open neighborhood of A contains some $B \in \mathcal{B}$. If $\mathcal{B}$ consists of open subsets of X, we say $\mathcal{B}$ is a π–base for A in X. If A is a singleton $\{x\}$, we call $\mathcal{B}$ a π–network, respectively π–base, for x in X.

Definition 2.3. We say a point x has countable π–character in X ($\pi\chi(x, X) = \aleph_0$) if x has a countable π–base in X. If $\pi\chi(x, X) = \aleph_0$ for every $x \in X$, then we say X has countable π–character and denote this by $\pi\chi(X) = \aleph_0$. We say that $X$is hereditarily of countable π–character ($h\pi\chi(X) = \aleph_0$) if $\pi\chi(Y) = \aleph_0$ for every subspace Y of X.

Closely related to the notion of π–character is the idea of tightness in topological spaces. We need only the following special case.

Definition 2.4. A space X is countably tight (written $t(X) = \aleph_0$) if whenever A is a subset of X and x is in the closure of A, then there is a countable $A_0 \subseteq A$ such that $x \in \mathrm{cl}_X(A)$.

Thus, the closure operator in a countably tight space is determined by its action on countable sets. Notice as well that an elementary argument shows that a space that is hereditarily of countable π–character is also countably tight.

We are now ready to commence with the prove that the Continuum Hypothesis is consistent with the statement $\otimes$. The idea is that given a that given a countably compact, non–compact regular space X with $h\pi\chi(X) = \aleph_0$, there is a "nice" notion of forcing that will shoot an uncountable free sequence through X.

In the remainder of this section, we lay the groundwork for the construction of the notion of forcing referred to which we referred in the previous paragraph. The fact that X is not compact means that there is a maximal filter $\mathcal{F}$ of closed subsets of X that is not fixed, i.e., such that $\bigcap \mathcal{F} = \emptyset$. Fix such a filter $\mathcal{F}$; throughout the rest of the paper, we shall use the phrase "*for almost all x*" to mean "*the set of such x is in $\mathcal{F}$*".

The space X is countably compact, and this means that the filter $\mathcal{F}$ is closed under countable intersections. With this in mind, let us say that a subset A of X is *large* if it meets every set in $\mathcal{F}$, otherwise we say that A is small. Since $\mathcal{F}$ is closed under countable intersections, it follows that a countable union of small sets is small; this is crucial to our arguments.

We need one more fact about the filter $\mathcal{F}$ — it is generated by separable sets, i.e., if $A \in \mathcal{F}$ then there is a countable $A_0 \subseteq A$ whose closure is again in $\mathcal{F}$. This is easily shown, because if it fails then we can use the fact that $\mathcal{F}$ is closed under countable intersections to build an uncountable free sequence in X.

The next batch of definitions is related to the way in which our space X interacts with countable elementary submodels. They have appeared in various guises in earlier work of the author, e.g., [5] and [2].

Definition 2.5. If N is a countable elementary submodel of $H(\chi)$ containing X and $\mathcal{F}$, then we define the trace of N, denoted $\mathrm{Tr}(N)$, by the formula

$$\mathrm{Tr}(N) = \cap\{A : A \in N \cap \mathcal{F}\}. \tag{2.2}$$

Note that $\mathrm{Tr}(N)$ is a countable intersection of elements of $\mathcal{F}$, and therefore $\mathrm{Tr}(N)$ is always an element of $\mathcal{F}$.

Definition 2.6. A *promise* is a function f whose domain is a large subset of X such that for $x \in \mathrm{dom}\, f$, $f(x)$ is an open neighborhood of x, i.e., f is a neighborhood assignment for a large subset of X. If f is a promise, then we say a point y is *banned by* f if the set of $x \in \mathrm{dom}\, f$ with $y \in f(x)$ is large. We let $\mathrm{Ban}\, f$ be the set of all $y \in X$ that are banned by f.

The next theorem tells us that a promise cannot ban too many points. In fact, if f is a promise then $\mathrm{Ban}(f)$ is a small closed set.

Theorem 2.7. *If f is a promise, then* $\mathrm{Ban}\, f$ *is closed set that is not in $\mathcal{F}$.*

Proof. We start by proving that $\mathrm{Ban}\, f$ is closed. The proof is by contradiction, so suppose this is not the case and let y be a limit point of $\mathrm{Ban}\, f$ that is not banned by f. Since X is countably tight, there is a countable set $A = \{y_n : n \in \omega\} \subseteq \mathrm{Ban}\, f$

such that $y \in \overline{A}$. Now let

$$B = \{x \in \operatorname{dom} f : y \in f(x)\}. \tag{2.3}$$

Note that B is large as y is not banned by f.

For $n \in \omega$, we let

$$B_n = \{x \in B : y_n \in f(x)\}. \tag{2.4}$$

Each B_n is small as y_n is banned by f, but since $y \in \overline{A}$, we have

$$B = \bigcup_{n \in \omega} B_n, \tag{2.5}$$

which is a contradiction and therefore $\operatorname{Ban} f$ is a closed set.

Now we suppose $\operatorname{Ban} f \in \mathcal{F}$. There is a separable set $A \subseteq \operatorname{Ban} f$ such that $A \in \mathcal{F}$, say $A = \overline{\{y_n : n \in \omega\}}$. Let $B = A \cap \operatorname{dom} f$. Since $\operatorname{dom} f$ is large and $A \in \mathcal{F}$, we have that B is large as well.

Now let $B_n = \{x \in B : y_n \in f(x)\}$. Each B_n is small as y_n is banned by f, but since $B \subseteq \overline{\{y_n : n \in \omega\}}$, we have

$$B = \bigcup_{n \in \omega} B_n, \tag{2.6}$$

a contradiction. □

3. A notion of forcing

Let us now fix a countably compact non–compact regular space X with $h\pi\chi(X) = \aleph_0$, and let $\mathcal{F}$ be a maximal filter of closed subsets of X such that $\bigcap \mathcal{F} = \emptyset$. Our goal in this section is to define a totally proper (= proper and adds no new reals to the ground model) notion of forcing that adjoins a sequence $\{x_\alpha : \alpha < \omega_1\}$ with the property that every initial segment of the sequence has small closure, but for every $A \in \mathcal{F}$, there is an $\alpha < \omega_1$ such that

$$\{x_\beta : \beta \geq \alpha\} \subseteq A. \tag{3.1}$$

The sequence we adjoin may not be a free sequence, but such a sequence can easily be refined to give a free sequence of length ω_1.

Definition 3.1. A forcing condition p is a triple (σ_p, A_p, Φ_p), where

(1) σ_p is a one–to–one function from some countable ordinal into X, and we define $[p] := \operatorname{ran}(\sigma_p)$
(2) $A_p \in \mathcal{F}$
(3) Φ_p is a countable set of promises
(4) $\operatorname{cl}_X([p]) \cap A_p = \emptyset$

and a condition q extends p (written $q \leq p$) if

(5) $\sigma_q \supseteq \sigma_p$
(6) $A_q \subseteq A_p$
(7) $\Phi_q \supseteq \Phi_p$
(8) $[q] \setminus [p] \subseteq A_p$
(9) if $f \in \Phi_p$, then the set
$$Y(f, q, p) := \{x \in \operatorname{dom} f : [q] \setminus [p] \subseteq f(x)\}$$
is large, and $f \restriction Y(f, q, p) \in \Phi_q$.

A forcing condition $p = (\sigma_p, A_p, \Phi_p)$ explicitly fixes an initial segment of the sequence we are adjoining, and the definition of extension commits the remainder of the sequence to be in A_p. The promises in Φ_p provide some further restrictions on how our sequence can grow.

It is straightforward to verify that the preceding definition does indeed yield a partially–ordered set. We now lay the groundwork for proving that the forcing defined above is totally proper (= proper and adds no new reals). We remark that since $\mathcal{F}$ is generated by separable sets, the set of conditions $p \in P$ for which A_p is separable is dense in P. This becomes relevant later on when we show that P satisfies a strong version of the $\aleph_2$–chain condition.

Definition 3.2. Let $p \in P$ be a condition. A point $z \in X$ is *eligible* for p if there is a condition $q \leq p$ such that $z \in [q]$.

The definition of our partial order tells us that not every point in X is eligible for p — for example, if $f \in \Phi_p$ and $z \in \operatorname{Ban} f \setminus [p]$, then no extension of p can "pick up" the point z. However, the fact that $\mathcal{F}$ is closed under countable intersections lets us prove that almost every point in X is eligible for p, as demonstrated by the following lemma.

Lemma 3.3. Let $p \in P$ be a condition. Then there is a set $A \in \mathcal{F}$ such that every point in A is eligible for p.

Proof. Let us define B to be the union of sets of the form $\operatorname{Ban} f$ for $f \in \Phi_p$. By Theorem 2.7 and the fact that Φ_p is countable, we know that B is a small set. Thus there is a set $A \in \mathcal{F}$ such that $A \cap B = \emptyset$ and furthermore, without loss of generality $A \subseteq A_p$.

Take a point $x \in A$, and let A' be a subset of A in $\mathcal{F}$ that does not contain x. For each $f \in \Phi_p$, let $Y_f = \{y \in \operatorname{dom} f : x \in f(y)\}$. Each Y_f is a large set by the definitions involved. Let $\alpha = \operatorname{dom} \sigma_p$. We define a condition q by setting

$$\sigma_q = \sigma_p \cup \{\langle \alpha, x \rangle\},$$

$$A_q = A',$$

and

$$\Phi_q = \Phi_p \cup \{f \restriction Y_f : f \in \Phi_p\}.$$

It is straightforward to verify that q is a condition in P that extends p with $x \in [q]$. □

The following corollary is immediate; when combined with a density argument it shows that the sequence adjoined by our notion of forcing is uncountable.

Corollary 3.4. If $p \in P$, then there is a condition $q \leq p$ such that $[q] \setminus [p]$ is non–empty.

We now need to bring in some topology; recall that π–networks were defined in Definition 2.2.

Lemma 3.5. Let $p \in P$ be arbitrary, and let $D \subseteq P$ be dense open. For almost every $x \in X$, the collection

$$\mathcal{D} = \{[q] \setminus [p] : q \leq p \text{ and } q \in D\} \tag{3.2}$$

is a π–network for $x \in X$.

Proof. First, we point out that the set of such points is closed, so it suffices to prove that the complement of this set is small. Let us define

$$E = \{x \in X : \mathcal{D} \text{ is not a } \pi\text{–network for } x \text{ in } X\ \},$$

and assume by way of contradiction that E is large.

For each $x \in E$, there is an open set U_x such that $x \in U_x$ and there is no $q \leq p$ such that $q \in D$ and $[q] \setminus [p]$ is a non–empty subset of U_x. The function f with domain E defined by $f(x) = U_x$ is a promise (as E is large), and

$$p' = (\sigma_p, A_p, \Phi_p \cup \{f\})$$

is a condition in P. Since D is dense in P, we can find an extension q of p' that lies in D. By Corollary 3.4, without loss of generality $[q] \setminus [p']$ is non–empty.

By the definition of extension, the set

$$Y(f, q, p') = \{x \in \operatorname{dom} f : [q] \setminus [p'] \subseteq f(x)\}$$

is large, hence non–empty. Choose $x \in Y(f, q, p')$. For this particular x, we have

$$[q] \setminus [p] = [q] \setminus [p'] \subseteq f(x) = U_x, \tag{3.3}$$

and this contradicts the choice of U_x. □

Our next task is to sharpen the preceding lemma; we start with an *ad hoc* definition.

Definition 3.6. Assume $p \in P$, $D \subseteq P$ is dense open, and $A \in \mathcal{F}$. We say that a point x is good to p, D, and A if the set

$$\mathcal{D}_A = \{[q] \setminus [p] : q \leq p, q \in D, \text{ and } [q] \setminus [p] \subseteq A\}$$

is a π–network for $x \in X$. We let $\mathrm{Good}(p, D, A)$ denote the set of points that are good to p, D, and A.

Lemma 3.7. Given $p \in P$, $D \subseteq P$ dense open, and $A \in \mathcal{F}$, then almost every point is good to p, D, and A.

Proof. Again, the set of points that are good to p, D, and A is closed, so it suffices to prove that the set of such points is in $\mathcal{F}$. Suppose this fails, and fix a set $B \in \mathcal{F}$ such that no $x \in B$ is good to p, D, and A. Let N be a countable elementary submodel of $H(\chi)$ that contains the objects X, $\mathcal{F}$, P, p, D, A, and B.

Fix a point $x \in \mathrm{Tr}(N)$. Since $\mathrm{Tr}(N) \subseteq B$ we know that x has a neighborhood U such that for no $q \leq p$ in D is $[q] \setminus [p]$ is a non–empty subset of $U \cap A$. Let us define

$$p' = (\sigma_p, A_p \cap A, \Phi_p).$$

Now p' is a condition in P that extends p and, more importantly for our purposes, the condition p' is in the model N. The set of points $y \in X$ for which $\{[q] \setminus [p'] : q \leq p' \text{ and } q \in D\}$ forms a π–network for y in X is an element of $\mathcal{F}$, and since all parameters required to define this set are in N, it is a set in $N \cap \mathcal{F}$. Since $x \in \mathrm{Tr}(N)$, this means that there is a $q \leq p'$ in D such that $[q] \setminus [p']$ is a non–empty subset of U. This is a contradiction as q is an extension of p in D, and

$$[q] \setminus [p] = [q] \setminus [p'] \subseteq A_p \cap A \subseteq A.$$

□

We now begin to take advantage of our assumption that X is hereditarily of countable π–character — this assumption is crucial in the proof of the next theorem.

Definition 3.8. Assume $p \in P$, $D \subseteq P$ is dense open, and $A \in \mathcal{F}$. A point $x \in X$ is nice to p, D, and A if there is a countable family of conditions $\{q_n : n \in \omega\}$ such that each q_n extends p, each q_n is in D, and the set $\{[q_n] \setminus [p] : n \in \omega\}$ forms a π–network for x in A.

The fact that there are only countably many conditions q_n is what gives Definition 3.8 and the following theorem their power. We shall see this during the proof that P is totally proper.

Theorem 3.9. *If p, D, and A are as in the previous definition, then almost every point x is nice to p, D, and A.*

We will prove this theorem shortly, but first we need a key lemma on π–networks in countably compact spaces.

Lemma 3.10. Let X be a countably compact space, and let $\{A_n : n \in \omega\}$ be a decreasing family of closed sets. Let U be an open set that meets $K := \bigcap_{n\in\omega} A_n$. Then $\{U \cap A_n : n \in \omega\}$ is a π–network for $\overline{U} \cap K$ in X.

Proof. Let V be an open neighborhood of $\overline{U} \cap K$. It suffices to show that there is an n such that $(U \cap A_n) \setminus V$ is finite, as if $y \in (U \cap A_n) \setminus V$ then $y \notin K$ and hence (since the sequence is decreasing) there is an $m > n$ such that $y \notin A_m$. Given that $(U \cap A_n) \setminus V$ is finite, we can simply increase n to ensure that $(U \cap A_n) \setminus V$ is empty, i.e., $U \cap A_n \subseteq V$.

By way of contradiction, suppose that no such n exists. We can then choose distinct points x_n for $n \in \omega$ such that $x_n \in (U \cap A_n) \setminus V$. Since X is countably compact, the infinite set $\{x_n : n \in \omega\}$ has a limit point x^*.

Since each x_n is in U, we know that x^* is in the closure of U. Furthermore, because $x_n \in A_n$ and the sequence $\langle A_n : n < \omega\rangle$ is decreasing, we know that x^* is an element of K as well. Since $\overline{U} \cap K \subseteq V$, we conclude that

$$x^* \in \mathrm{cl}_X(\{x_n : n < \omega\}) \cap V. \tag{3.4}$$

On the other hand, we made sure that no x_n is in V, and therefore

$$\mathrm{cl}(\{x_n : n \in \omega\}) \cap V = \emptyset. \tag{3.5}$$

The contradictory nature of (3.4) and (3.5) finishes the proof. □

Now that we have established Lemma 3.10, we can turn to the proof of Theorem 3.9.

Proof of Theorem 3.9. We begin by proving that the set of points in x that are nice to p, D, and A is closed. Let B be the set of such points, and suppose that x is in the closure of B. Since X is countably tight, we know that there are countably many points $\{x_n : n < \omega\}$ in B that pick up x in their closure. Each of those has a relevant countable π–network, and the union of these countably many π–networks yields a π–network that certifies x's membership in B.

To finish the proof of Theorem 3.9, we fix a countable elementary submodel N of $H(\chi)$ that contains p, D, and A, as well as the other relevant parameters and show that every point in $\mathrm{Tr}(N)$ is nice to p, D, and A.

Choose $x \in \mathrm{Tr}(N)$. Our assumptions on the space X imply that x has countable π–character in $\mathrm{Tr}(N)$. Taken together with the fact that X is regular, this implies the existence of a family $\{U_m : m \in \omega\}$ of open sets in X such that

- $U_m \cap \mathrm{Tr}(N) \neq \emptyset$, and
- for every open neighborhood V of X, there is an m for which

$$\mathrm{cl}_X(U_m) \cap \mathrm{Tr}(N) \subseteq V. \tag{3.6}$$

Let $\{B_i : i \in \omega\}$ be a decreasing family of open sets in $N \cap \mathcal{F}$ generating $N \cap \mathcal{F}$ and also satisfying $B_0 \subseteq A$; clearly $\mathrm{Tr}(N) = \bigcap_{i<\omega} B_i$.

For each $m < \omega$, we can apply Lemma 3.10 to conclude that

$$\mathcal{B}_m := \{B_i \cap U_m : i \in \omega\} \tag{3.7}$$

is a π–network for $\overline{U}_m \cap \mathrm{Tr}(N)$.

Now fix i and m. The set B_i is in $N \cap \mathcal{F}$, so by Lemma 3.7 applied in N, there is an $i^* > i$ such that $B_{i^*} \subseteq \mathrm{Good}(p, D, B_i)$. Since U_m meets $\mathrm{Tr}(N)$ and $\mathrm{Tr}(N) \subseteq B_{i^*}$, we can find a point $y_{m,i}$ in $U_m \cap \mathrm{Good}(p, D, B_i)$. By definition, this means that there is some condition $q_{m,i}$ such that

- $q_{m,i} \leq p$,
- $[q_{m,i}] \setminus [p] \neq \emptyset$,
- $q_{m,i} \in D$, and
- $[q_{m,i}] \setminus [p] \subseteq U_m \cap B_i$.

To finish, we show that the family $\{q_{m,i} : m, i < \omega\}$ witnesses that x is nice to p, D, and A. For this, we must take an arbitrary neighborhood V of x and show that for some m and i,

$$[q_{m,i}] \setminus [p] \subseteq V \cap A. \tag{3.8}$$

Given V, there is an m such that $\overline{U}_m \cap \mathrm{Tr}(N) \subseteq V$. Our definition of $\mathcal{B}_m$ implies that there is an i such that $U_m \cap B_i \subseteq V$. Our choice of $q_{m,i}$ means

$$[q_{m,i}] \setminus [p] \subseteq B_i \cap U_m \subseteq V, \tag{3.9}$$

and since $B_i \subseteq A$, the condition $q_{m,i}$ is as required. □

Now at last we can prove that our notion of forcing is totally proper. We have referred to this concept in passing in previous portions of the paper, but since this is the first place where the actual definition is needed, we take a moment to recall it.

Definition 3.11. A notion of forcing P is totally proper if whenever we are given $N \prec H(\chi)$ countable (with χ "large enough") such that $P \in N$, and $p \in N \cap P$, we can find $q \leq p$ such that for every dense open subset D of P that is in N, there is some $p' \geq q$ such that $p' \in N \cap D$. Such a q is said to be totally (N, P)–generic.

We now come to the main theorem of this section of the paper.

Theorem 3.12. *Let X be a countably compact, non–compact regular space with $h\pi\chi(X) = \aleph_0$, and let $\mathcal{F}$ be a maximal filter of closed subsets of X with $\bigcap \mathcal{F} = \emptyset$. Then the notion of forcing P from Definition 3.1 is totally proper.*

Proof. Let N be a countable elementary submodel of $H(\chi)$ containing all parameters needed to define P. The crux of the proof consists of two lemmas which we have christened the Extension Lemma (Lemma 3.13) and the Focus Lemma (Lemma 3.14). Variations of these two lemmas will be used throughout the remainder of the paper, so they are quite important for our argument.

Lemma 3.13 (**Extension Lemma**). Let $p \in N \cap P$, and let D be a dense subset of P that is an element of N. Given $A \in N \cap \mathcal{F}$ and an open set U (not necessarily in N!) such that $U \cap \mathrm{Tr}(N) \neq \emptyset$, we can find an extension q of p such that $q \in N \cap D$ and $[q] \setminus [p] \subseteq U \cap A$.

Proof. We know by previous work that the set of all points that are nice to p, D, and A is in the filter $\mathcal{F}$, and since p, D, and A are all in N, it follows that this set is in N as well. We know as well that $\mathcal{F}$ is generated by separable sets, and therefore there is a separable set B in $N \cap \mathcal{F}$ such that every point in B is nice to p, D, and A.

Inside the model N we can find a countable set B_0 that is dense in B. Since B_0 is countable, it follows that every element of B_0 is an element of N. Also notice that $U \cap B_0 \neq \emptyset$ because U is an open set that meets $\mathrm{Tr}(N)$.

Fix a point y in $U \cap B_0$. Since y is nice to p, D, and A, there is a family $\{q_n : n \in \omega\}$ that witnesses this. Because y is in the model N, we may assume that $\{q_n : n \in \omega\}$ is in N as well, and therefore $\{q_n : n \in \omega\} \subseteq N$. By elementarity, we know that $\{q_n : n \in \omega\}$ is a π–network for x in A back in the real world, so in particular there must be an $n < \omega$ such that

$$[q_n] \setminus [p] \subseteq U \cap A, \tag{3.10}$$

as required. □

In the discussion following Definition 3.8, we mentioned that the countability present in that definition would prove crucial. The preceding lemma is a prime example of this — because the π–network for y is countable, we know that every member of it lies in N, and elementarity tells us that these objects in N "work" even though the set U is not in N. The requirement that $h\pi\chi(X) = \aleph_0$ is what allows this argument to work, and allows objects inside the model M to "communicate" with objects that are outside of N.

We now turn to the other crucial lemma — the Focus Lemma.

Lemma 3.14 (**Focus Lemma**). Let $f \in N$ be a promise, and let U be an open set that meets $\mathrm{Tr}(N)$. There is a set $A \in N \cap \mathcal{F}$ and an open $V \subseteq U$ such that

$$V \cap \mathrm{Tr}(N) \neq \emptyset, \tag{3.11}$$

and

$$\{x \in \operatorname{dom} f : V \cap A \subseteq f(x)\} \text{ is large.} \tag{3.12}$$

Proof. Choose $z \in U \cap \mathrm{Tr}(N)$. Since X is regular and $\pi\chi(x, \mathrm{Tr}(N)) = \aleph_0$, we can find a family of open sets $\{U_n : n \in \omega\}$ such that

- $U_n \subseteq U$
- $U_n \cap \mathrm{Tr}(N) \neq \emptyset$, and
- if W is an open neighborhood of x, then there is an n such that
$$\overline{U}_n \cap \mathrm{Tr}(N) \subseteq W.$$

Since $\mathrm{Tr}(N) \cap \operatorname{Ban} f = \emptyset$, we know that $E_0 := \{x \in \operatorname{dom} f : z \in f(x)\}$ is large. If $x \in E_0$, there is an n such that $\overline{U}_n \cap \mathrm{Tr}(N) \subseteq f(x)\}$. Since a countably union of small sets is small, there must be an n for which

$$E_1 := \{x \in E_0 : \overline{U}_n \cap \mathrm{Tr}(N) \subseteq f(x)\} \text{ is large.}$$

Choose such an n, and define $V = U_n$.

Now let A_i be a decreasing family in $N \cap \mathcal{F}$ that generates $N \cap \mathcal{F}$; note that $\mathrm{Tr}(N) = \bigcap_{i<\omega} A_i$. By Lemma 3.10, the sets $\{V \cap A_i : i \in \omega\}$ form a π–network for $\overline{V} \cap \mathrm{Tr}(N)$. Thus if $x \in E_1$, there is an i such that $V \cap A_i \subseteq f(x)$. This means there exists a single i such that

$$E_2 := \{x \in E_1 : V \cap A_i \subseteq f(x)\} \text{ is large.}$$

If we let $A = A_i$, then

$$E_2 \subseteq \{x \in \mathrm{dom}\, f : V \cap A \subseteq f(x)\}, \tag{3.13}$$

as required. □

As will become clear shortly, the Extension Lemma and the Focus Lemma tell us that sets of the form $U \cap A$, where U is an open set meeting $\mathrm{Tr}(N)$ and A is an element of $N \cap \mathcal{F}$ can serve as "targets" in the proof of the total properness of P. The Extension Lemma says that such sets are large enough for required extension to be found, and the Focus Lemma tells us that such sets can be made small enough that they can "take care of" promises. The reader should keep this in mind when reading the following proof of the total properness of P.

Let $\{D_m : m \in \omega\}$ list the dense subsets of P that are elements of N, and suppose p is an arbitrary element of $N \cap P$. Our task is to construct a decreasing sequence of conditions $\langle p_n : n < \omega\rangle$ such that $p_0 = p$ and $p_{n+1} \in N \cap D_n$ in such a way that the sequence has a lower bound q, which will necessarily be totally (N, P)–generic. The requirement that the sequence has a lower bound is what causes the difficulty, and this is the reason why the following construction looks so complicated.

By induction on $n \in \omega$, we construct objects p_n, U_n, and A_n such that

(1) $p_0 = p$, $A_0 = X$
(2) U_0 is some open set that meets $\mathrm{Tr}(N)$ and satisfies $\overline{U}_0 \notin \mathcal{F}$
(3) p_{n+1} is an extension of p_n that is in $N \cap D_n$
(4) U_n is an open set that meets $\mathrm{Tr}(N)$
(5) A_n is a member of $N \cap \mathcal{F}$
(6) the sequences $\{U_n : n \in \omega\}$ and $\{A_n : n \in \omega\}$ are $\subseteq$–decreasing
(7) $[p_{n+1}] \setminus [p_n] \subseteq U_{n+1} \cap A_{n+1}$
(8) for each n and $f \in \Phi_{p_n}$, there is a stage $m \geq n$ for which

$$\{x \in Y(f, p_m, p_n) : A_{m+1} \cap U_{m+1} \subseteq f(x)\} \text{ is large.} \tag{3.14}$$

We say that the promise f is *taken care of at stage* $m + 1$.

At stage $n + 1$, we are handed p_n, D_n, U_n, and A_n as well as some promise f appearing in some earlier Φ_{p_i} that must be taken care of at this stage. By the definition of extension, we know that $f' = f \restriction Y(f, p_n, p_i)$ is an element of Φ_{p_n}. Since Φ_{p_n} is countable, we know $f' \in N$ as well. By the Focus Lemma (Lemma 3.14), we can find an open $U_{n+1} \subseteq U_n$ that meets $\mathrm{Tr}(N)$ and an $A_{n+1} \subseteq A_n$ in $N \cap \mathcal{F}$ such that

$$\{x \in \mathrm{dom}\, f' : A_{n+1} \cap U_{n+1} \subseteq f(x)\} \text{ is large.} \tag{3.15}$$

By the Extension Lemma (Lemma 3.13) we can find $p_{n+1} \leq p_n$ in $N \cap D_n$ such that

$$[p_{n+1}] \setminus [p_n] \subseteq A_{n+1} \cap U_{n+1}. \tag{3.16}$$

To finish our proof, we need only prove that the sequence $\{p_n : n \in \omega\}$ has a lower bound q in P. First, we let $\sigma_q := \bigcup_{n\in\omega} \sigma_{p_n}$. It is clear that σ_q is a one–to–one function from a countable ordinal into X, and we let $[q]$ denote the range of σ_q. We selected U_0 so that $\overline{U}_0 \notin \mathcal{F}$ and our construction ensures $[q] \setminus [p] \subseteq U_0$. This tells us that the closure of $[q]$ is not in $\mathcal{F}$.

Now letA_q be some member of $\mathcal{F}$ that is a subset of $\mathrm{Tr}(N)$ and disjoint to the closure of $[q]$. Clearly A_q is then a subset of A_{p_n} for each n.

If f is a promise appearing in Φ_{p_n} for some n, there is a stage $m \geq n$ where we take care of f at stage $m+1$. Recall that this means we ensure

$$E := \{x \in Y(f, p_m, p_n) : U_{m+1} \cap A_{m+1} \subseteq f(x)\} \text{ is large.} \tag{3.17}$$

Our construction guarantees that $[q] \setminus [p_m]$ is a subset of $U_{m+1} \cap A_{m+1}$, and this means

$$Y(f, q, p_n) := \{x \in \operatorname{dom} f : [q] \setminus [p_n] \subseteq f(x)\} \text{ is large.} \tag{3.18}$$

Thus if we define

$$\Phi_q = \bigcup_{n \in \omega} \Phi_{p_n} \cup \bigcup_{n \in \omega} \{f \upharpoonright Y(f, q, p_n) : f \in \Phi_{p_n}\}, \tag{3.19}$$

it is straightforward to verify that $q = (\sigma_q, A_q, \Phi_q)$ is a condition in P that is a lower bound for the sequence $\{p_n : n \in \omega\}$. Thus q is a totally (N, P)–generic extension of p, and hence P is totally proper. □

Note that the proof actually establishes the following slightly stronger result.

Corollary 3.15. Let N be a countable elementary submodel of $H(\chi)$ containing P, X, and $\mathcal{F}$. Let p be a condition in $N \cap P$, let A be a set in $N \cap \mathcal{F}$, and let U be an open subset of X that meets $\mathrm{Tr}(N)$. Then we can find a totally (N, P)–generic $q \leq p$ such that

$$[q] \setminus [p] \subseteq U \cap A. \tag{3.20}$$

4. Totally proper iterations and weak $< \omega_1$–properness

Recall that our goal is to construct a model of the Continuum Hypothesis in which the following principle holds:

⊗ A regular countably compact space that is hereditarily of countable π–character is either compact or contains an uncountable free sequence.

In the previous section, we showed that if we are given a countably compact, non–compact regular space X satisfying $h\pi\chi(X) = \aleph_0$ and containing no uncountable free sequence, there is a totally proper notion of forcing P that will shoot an uncountable free sequence through X. If one is going to prove the consistency of $\mathsf{CH}+\otimes$ using a countable support iteration of notions of forcing, then this is certainly critical. However, the journey from the existence of such a notion of forcing to constructing a model of $\mathsf{CH}+\otimes$ is a long one.

In particular, the question of whether the limit of a countable support iteration of totally proper forcings is quite delicate. To show that this is indeed the case when the iterands are defined as in the previous section, we need the following theorem from [4].

Theorem 4.1 (Theorem 4 of [4]). *Let $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha : \alpha < \epsilon\rangle$ be a countable support iteration such that*

(1) $\Vdash_{P_\alpha} \dot{Q}_\alpha$ *is totally proper*

(2) $\Vdash_{P_\alpha} \dot{Q}_\alpha$ *is weakly* $< \omega_1$*–proper*

(3) *For each* α, *if* N_0, N_1, $\dot{q}$, $\bar{G}$, *and* $\langle G^\ell : \ell < k\rangle$ *satisfy*
   - N_0 *and* N_1 *are countable elementary submodels of* $H(\chi)$
   - $N_0 \in N_1$
   - $\{\mathbb{P}, \alpha, \dot{q}\} \in N_0$
   - $\Vdash_{P_\alpha} \dot{q} \in \dot{Q}_\alpha$
   - $\bar{G} \in \mathrm{Gen}^+(N_0, P_\alpha, p) \cap N_1$
   - *for* $\ell < k$, $G^\ell \in \mathrm{Gen}(N_1, P_\alpha)$
   - *for* $\ell < k$, $\bar{G} \subseteq G^\ell$

   then there is a sequence $\langle \dot{q}_n : n \in \omega\rangle$ *in* $N_1 \cap \mathrm{Gen}(N_0[\bar{G}], \dot{Q}, \dot{q})$ *such that for all* $\ell < k$,

$$G^\ell \Vdash \langle \dot{q}_n : n \in \omega\rangle \text{ has a lower bound in } \dot{Q}.$$

Then P_ϵ *is totally proper.*

In our proof of the consistency of $\mathsf{CH} + \otimes$, we use forcings as in the previous section at successor stages of the iteration. We deal now with showing our iteration satisfies the last two conditions of the theorem. We start with the following definition from [4].

Definition 4.2. A totally proper notion of forcing P is weakly $< \omega_1$–proper if whenever we are given $\alpha < \omega_1$ and a tower of models $\mathfrak{N} = \langle N_\beta : \beta < \alpha\rangle$ such that

- $P \in N_0$
- $\mathfrak{N}$ is continuous at limits
- $\langle N_\gamma : \gamma \leq \beta\rangle \in N_{\beta+1}$

then for each $p \in N_0 \cap P$ we can find $q \leq p$ such that

$$\{\beta < \alpha : q \text{ is totally } (N_\beta, P)\text{–generic}\} \text{ has order–type } \alpha.$$

We will shortly prove that the notion of forcing considered in the previous section is weakly $< \omega_1$–proper, but we first need the following *ad hoc* definition.

Definition 4.3. Let α be a countable ordinal ordinal, and let X, $\mathcal{F}$, and P be as in the previous section. We say that α is *good* if whenever we are given objects $\mathfrak{N}$, p, A, and U such that

- $\mathfrak{N} = \langle N_\beta : \beta \leq \alpha\rangle$ is a continuous $\in$–tower of countable elementary submodels of $H(\chi)$ for some large regular χ,
- $\langle N_\beta : \beta \leq \alpha_0\rangle \in N_{\alpha_0+1}$ for $\alpha_0 < \alpha$,
- $\{X, P, \mathcal{F}\} \in N_0$,
- $p \in N_0 \cap P$,
- $A \in N_0 \cap \mathcal{F}$, and
- U is an open subset of X such that $U \cap \mathrm{Tr}(N_\alpha) \neq \emptyset$,

we can find a condition $q \leq p$ such that

- $[q] \setminus [p] \subseteq U \cap A$, and
- $\{\beta < \alpha : q \text{ is totally } (N_\beta, P)\text{–generic}\}$ has order–type α

Theorem 4.4. *If* $\alpha = \omega^\gamma$ *for some countable* γ, *then* α *is good.*

Proof. The key point here is that ordinals of the form ω^γ are indecomposable, i.e., closed under ordinal addition. We prove this proposition by induction with the case of $\alpha = \omega^0 = 1$ taken care of by Corollary 3.15.

Suppose now that $\alpha = \omega^\gamma$ for some $\gamma > 0$. We define a sequence of ordinals $\langle \alpha_n : n < \omega \rangle$ according to the following recipe:
Case 1: $\alpha = \omega^\gamma$ for γ a successor ordinal.

In this case, α is equal to $\beta \cdot \omega$ for some indecomposable β; we let each α_n equal β.
Case 2: $\alpha = \omega^\gamma$ for γ a limit ordinal.

In this case, we let $\langle \alpha_n : n < \omega \rangle$ be a strictly increasing sequence of indecomposable ordinals cofinal in α; such a sequence can be found because γ is a (countable) limit ordinal.

Let $\mathfrak{N} = \langle N_i : i \leq \alpha \rangle$, p, A, and U be given. By induction on $n < \omega$, we will construct objects p_n, A_n, β_n, and U_n such that

(1) $p_0 = p$, $A_0 = A$, $\beta_0 = 0$, and $U_0 = U$
(2) $p_{n+1} \leq p_n$, $A_{n+1} \subseteq A_n$, $\beta_{n+1} > \beta_n$, and $U_{n+1} \subseteq U_n$
(3) $p_{n+1} \in N_{\beta_{n+1}+1} \cap P$
(4) $A_{n+1} \in N_{\beta_{n+1}} \cap \mathcal{F}$
(5) U_{n+1} is an open set that meets $\mathrm{Tr}(N_\alpha)$
(6) p_{n+1} is totally $(N_{i_{n+1}}, P)$–generic
(7) $[p_{n+1}] \setminus [p_n] \subseteq U_{n+1} \cap A_{n+1}$
(8) $\{i \in (\beta_n, \beta_{n+1}) :\ p_{n+1}$ is totally (N_i, P)–generic $\}$ has order–type α_{n+1}
(9) if f is a promise appearing in Φ_{p_i} for some i, then there is a stage $n \geq i$ such that

$$\{x \in Y(f, p_n, p_i) : U_{n+1} \cap A_{n+1} \subseteq f(x)\} \text{ is large.} \tag{4.1}$$

How is this accomplished? Assume we are given p_n, β_n, A_n, and U_n, and that our bookkeeping has given us a promise f (from some earlier Φ_{p_i}) to take care of. Clearly the promise $f' := f \upharpoonright Y(f, p_n, p_i)$ is an element of N_α, so by the Focus Lemma (Lemma 3.14) applied to the objects f, U_n, and N_α, there are U_{n+1} and A' such that

- U_{n+1} is an open set that meets $\mathrm{Tr}(N_\alpha)$,
- $U_{n+1} \subseteq U_n$,
- $A' \in N_\alpha \cap \mathcal{F}$,
- $A' \subseteq A_n$, and
- $\{x \in Y(f, p_n, p_i) : U_{n+1} \cap A' \subseteq f(x)\}$ is large.

We know that $N_\alpha = \cup_{i<\alpha} N_i$ because α is a limit ordinal. Let β' be the least ordinal $> \beta_n$ such that $A' \in N_{\beta'}$, and define $\beta_{n+1} = \beta' + \alpha_{n+1}$. Note that p_n is an element of $N_{\beta_{n+1}}$ as well.

Lemma 4.5. We can find an open set $V \in N_{\beta_{n+1}+1}$ such that

- $V \cap \mathrm{Tr}(N_{\beta_{n+1}}) \neq \emptyset$, and
- $\overline{V} \cap \mathrm{Tr}(N_{\beta_{n+1}}) \subseteq U_{n+1}$.

Proof. We know that $\mathrm{Tr}(N_{\beta_{n+1}}) \in N_{\beta_{n+1}+1} \cap \mathcal{F}$, and since $\mathcal{F}$ is generated by separable sets there is a countable $B \subseteq \mathrm{Tr}(N_{\beta_{n+1}})$ in $N_{\beta_{n+1}+1}$ with $\overline{B} \in \mathcal{F}$. Furthermore, the countability of B implies that $B \subseteq N_{i_{n+1}+1}$. Now U_{n+1} is an open set that meets $\mathrm{Tr}(N_\alpha) \subseteq \overline{B}$, and so there exists a point

$$z \in U_{n+1} \cap \mathrm{Tr}(N_{\beta_{n+1}}) \cap N_{\beta_{n+1}+1}. \tag{4.2}$$

Our assumptions about the space X imply that $\pi\chi(z, \mathrm{Tr}(N_{\beta_{n+1}})) = \aleph_0$, and this together with the regularity of X will give us the required $V \in N_{\beta_{n+1}+1}$. □

By Lemma 3.10, we can find A_{n+1} such that

- $A_{n+1} \subseteq A'$,
- $A_{n+1} \in N_{\beta_{n+1}} \cap \mathcal{F}$, and
- $V \cap A_{n+1} \subseteq U_{n+1}$.

Note that since A_{n+1} is a subset of A', we have ensured that

$$\{x \in Y(f, p_n, p_i) : U_{n+1} \cap A_{n+1} \subseteq f(x)\} \text{ is large.} \tag{4.3}$$

Assume for a moment that β_{n+1} is a limit ordinal, i.e., that $\alpha_n \neq 1$, so we can find some ordinal β'' in the interval $[\beta', \beta_{n+1})$ with $A_{n+1} \in N_{\beta''}$. Now step inside the model $N_{\beta_{n+1}}$. The tower

$$\mathfrak{N}' := \langle N_i : \beta'' < i \leq \beta_{n+1} \rangle \tag{4.4}$$

is an element of $N_{\beta_{n+1}}$, and the order–type of the interval (β'', β_{n+1}) is α_{n+1} because α_{n+1} is indecomposable.

The objects p_n and A_{n+1} are in $N_{\beta''}$, so we may apply our induction hypothesis inside the model $N_{\beta_{n+1}+1}$ to $\mathfrak{N}'$, p_n, V, and A_{n+1} to obtain a condition $p_{n+1} \leq p_n$ satisfying

- $[p_{n+1}] \setminus [p_n] \subseteq V \cap A_{n+1} \subseteq U_{n+1} \cap A_{n+1}$, and
- $\{i \in (\beta'', \beta_{n+1+1}) :\ p_{n+1}$ is totally (N_i, P)–generic$\}$ has order–type α_{n+1}.

If it happens that β_{n+1} is a successor ordinal (so $\alpha = \omega$), then our task is easier — we are guaranteed that A_{n+1} and p_{n+1} are in $N_{\beta_{n+1}}$, so we just need to apply Corollary 3.15 again.

In either case, we have managed to produce the required U_{n+1}, β_{n+1}, A_{n+1}, and p_{n+1} while simultaneously taking care of the promise f. To finish our proof, note that the sequence $\langle p_n : n < \omega \rangle$ has a least upper bound q with $[q] \setminus [p] \subseteq U \cap A$ — this follows from the proof of Theorem 3.12. Thus α is good. □

Corollary 4.6. The notion of forcing P is weakly $< \omega_1$–proper.

Proof. The proof is by induction on $\alpha < \omega_1$ with the case $\alpha = 1$ taken care of by Theorem 3.12. If α is indecomposable, then what we need follows immediately from the fact that α is good. The final case is where $\alpha = \beta + \gamma$, where β and γ are both $< \alpha$, and this follows easily from the induction hypothesis. □

In order to prove that our iteration satisfies the third hypothesis of Theorem 4.1, we will have to do a bit of work refining the proof that the forcing notions defined as in Section 3 are totally proper.

5. Depth and N–spines

This section is quite technical; our goal is to develop a framework for proving that our notions of forcing are totally proper that requires as little information as possible from outside the given countable model N. It might be helpful to consider a very loose description of the proof of total properness.

The proof of Theorem 3.12 utilizes sets of the form $U \cap A$ where A is in $N \cap \mathcal{F}$ and U is an open set for which $U \cap \mathrm{Tr}(N) \neq \emptyset$. In order to prove that our notions of forcing can be iterated without adding reals, we will need to replace the set $U \cap A$ by an object that admits a more explicit description depending solely on things available in N. The problem is that our "targets" must be large enough so that a version of the Extension Lemma (Lemma 3.13) remains true, and yet small enough so that we can prove a version of the Focus Lemma (Lemma 3.14). Such "targets" can be found (we call them N–*spines*), and the subject of this section is to show this.

In our previous work, we saw that given $p \in P$, $A \in \mathcal{F}$, and a dense set $D \subseteq P$ that almost every point in X is nice to p, D, and A (in the sense of Definition 3.8). We are going to generalize this result quite a bit through the introduction of (p, D, A)–depth.

Definition 5.1. Let $\mathfrak{D}$ be the collection of all sets $\mathcal{D}$ consisting of finitely many pairs (p, D) where $p \in P$ and D is a dense subset of P.

Definition 5.2. Suppose $\mathcal{D} \in \mathfrak{D}$ and $A \in N \cap \mathcal{F}$.

(1) A point x is said to have $(\mathcal{D}, A)$–depth ≥ 0 (denoted $\mathrm{Dp}(\mathcal{D}, A)[x] \geq 0$) if x is nice to p, D, and A for all $(p, D) \in \mathcal{D}$.
(2) $\Delta^0(\mathcal{D}, A) = \{x \in X : \mathrm{Dp}(\mathcal{D}, A)[x] \geq 0\}$.
(3) $E \subseteq X$ is relatively open of $\mathcal{D}$–level ≥ 0 (written $\mathrm{Lev}_{\mathcal{D}}(E) \geq 0$) if there are an open set U and a set $A \in N \cap \mathcal{F}$ such that
   - $E = U \cap A$, and
   - $E \cap \Delta^0(\mathcal{D}, A) \neq \emptyset$.
(4) $\mathrm{Dp}(\mathcal{D}, A)[x] \geq n+1$ if x has a countable π–network in $\Delta^n(\mathcal{D}, A)$ consisting of sets that are relatively open of $\mathcal{D}$–level $\geq n$
(5) $\Delta^{n+1}(\mathcal{D}, A) = \{x \in X : \mathrm{Dp}(\mathcal{D}, A)[x] \geq n + 1\}$
(6) $E \subseteq X$ is relatively open of $\mathcal{D}$–level $\geq n + 1$ (written $\mathrm{Lev}_{\mathcal{D}}(E) \geq n + 1$) if there are an open set U and a set $A \in N \cap \mathcal{F}$ such that
   - $E = U \cap A$, and
   - $E \cap \Delta^{n+1}(\mathcal{D}, A) \neq \emptyset$.

One should picture Δ as a type of Cantor-Bendixson derivative; in fact, the next proposition shows that the analogy makes sense as the "derivative" of a closed set is closed. In Theorem 5.4 we show that the filter $\mathcal{F}$ is closed under these derivatives, but first we collect a few easy observations.

Proposition 5.3.

(1) $\Delta^n(\mathcal{D}, A)$ is a closed subset of X
(2) $\Delta^{n+1}(\mathcal{D}, A) \subseteq \Delta^n(\mathcal{D}, A)$
(3) If $B \subseteq A$ in $\mathcal{F}$, then $\Delta^n(\mathcal{D}, B) \subseteq \Delta^n(\mathcal{D}, A)$.
(4) If $\mathcal{E} \supseteq \mathcal{D}$ in $\mathfrak{D}$, then $\Delta^n(\mathcal{E}, A) \subseteq \Delta^n(\mathcal{D}, A)$.

We leave the proof of the above proposition to the reader, and concentrate instead on the most important property of δ.

Theorem 5.4. *For each $n < \omega$, if $\mathcal{D} \in \mathfrak{D}$ and $A \in \mathcal{F}$, then $\Delta^n(\mathcal{D}, A) \in \mathcal{F}$.*

Proof. Theorem 3.9 implies that $\Delta^0(\mathcal{D}, A) \in \mathcal{F}$ for all choices of $\mathcal{D}$ and A. Assume now that the theorem holds for n; and suppose by way of contradiction that we have found $\mathcal{D}$ and A such that $\Delta^{n+1}(\mathcal{D}, A) \notin \mathcal{F}$.

Since $\Delta^{n+1}(\mathcal{D}, A)$ is closed, it must be the case that $\Delta^{n+1}(\mathcal{D}, A)$ is disjoint to a set C in $\mathcal{F}$. Let N be a countable elementary submodel of $H(\chi)$ that contains all of the objects X, $\mathcal{F}$, P, $\mathcal{D}$, A, and C. Since $\mathrm{Tr}(N)$ is a subset of B, we will reach a contradiction provided we establish that $\mathrm{Tr}(N) \cap \Delta^{n+1}(\mathcal{D}, A) \neq \emptyset$.

Let $\vec{U} = \langle U_i : i < \omega \rangle$ be a family of open sets such that

- U_i is an open set in N,
- the closure of U_i is not in $\mathcal{F}$,

- if $i \neq j$, then U_i and U_j have disjoint closures, and
- if $B \in N \cap \mathcal{F}$, then $U_i \cap B \neq \emptyset$ for all but finitely many i.

It should be clear that a sequence of the preceding form can be found. Now we define $\Lambda(\vec{U})$ to be the set of points z for which there exists an infinite set $I \subseteq \omega$ and points $\langle x_i : i \in I\rangle$ such that

- $x_i \in N \cap U_i$, and
- if $B \in N \cap \mathcal{F}$, then $x_i \in B$ for all but finitely many i.

We say that such a sequence $\vec{U}$*–diagonalizes* $N \cap \mathcal{F}$.) Because X is countably compact, it is clear that $\Lambda(\vec{U})$ is a non–empty subset of $\mathrm{Tr}(N)$.

Lemma 5.5. Let V be an open set that meets $\Lambda(\vec{U})$, and let B be a set in $N \cap \mathcal{F}$. Then there are infinitely many i such that $(V \cap U_i) \cap B$ is relatively open of $\mathcal{D}$–level $\geq n$. In fact, for all but finitely many i we have

$$[(V \cap U_i) \cap B] \neq \emptyset \Longrightarrow \mathrm{Lev}_{\mathcal{D}}(V \cap U_i \cap B) \geq n. \tag{5.1}$$

Proof. By our induction hypothesis, the set $\Delta^n(\mathcal{D}, B)$ is an element of $\mathcal{F}$, and clearly this set is in N as well. Since V meets $\Lambda(\vec{U})$, there is an infinite set $I \subseteq \omega$ and sequence $\langle x_i : i \in I\rangle$ of points in in $N \cap V$ that $\vec{U}$–diagonalizes $N \cap \mathcal{F}$.

By throwing away finitely many members of I, we may assume that for each $i \in I$, the point x_i is in $\Delta^n(\mathcal{D}, B)$. Thus x_i witnesses that $(V \cap U_i) \cap B$ is relatively open of $\mathcal{D}$–level at least n. □

Fix a point in $\Lambda(\vec{U})$. We finish the proof of Theorem 5.4 by establishing that x has a countable π–network in the set $\Delta^n(\mathcal{D}, A)$ of the required form.

Once again we need our assumption that X is hereditarily of countable π–character. Because the π–character of x in $\Lambda(\vec{U})$ is countable, there is a sequence of open sets $\langle V_i : i < \omega\rangle$ such that

- each V_i is an open set that meets $\Lambda(\vec{U})$, and
- if U is an open neighborhood of x, then there is an i such that

$$\overline{V}_i \cap \Lambda(\vec{U}) \subseteq U. \tag{5.2}$$

Given $B \in N \cap \mathcal{F}$ and $i < \omega$, we define $\mathcal{B}(B, i)$ to be the collection of sets of the form $[(V_i \cap U_j) \cap B]$ satisfying $\mathrm{Lev}_{\mathcal{D}}(V_i \cap U_j \cap B) \geq n$. Lemma 5.5 tells us that this will be the case for all but finitely many of the infinitely many j such that $[(V_i \cap U_j) \cap B]$ is non–empty.

We define

$$\mathcal{B} = \bigcup_{B \in N \cap \mathcal{F}} \bigcup_{i < \omega} \mathcal{B}(B, i). \tag{5.3}$$

Note that $\mathcal{B}$ consists of sets that are relatively open of $\mathcal{D}$–level $\geq n$; all that remains to show is that it is a π–network for x in $\Delta^n(\mathcal{D}, A)$. The following lemma is the key ingredient for the proof.

Lemma 5.6. If U is a neighborhood of x, then there is a natural number m and a set B in $N \cap \mathcal{F}$ such that

$$\bigcup_{i < \omega} [(V_m \cap U_i) \cap B] \subseteq U. \tag{5.4}$$

Proof. Let U be a neighborhood of x. By our assumptions on $\{V_m : m < \omega\}$, there is an m such that $\overline{V}_m \cap \Lambda(\vec{U}) \subseteq U$. If the proposition is false, then for every $B \in N \cap \mathcal{F}$ there exists an i such that $[(V_m \cap U_i) \cap B] \setminus U$ is non–empty. In fact, since each U_i has small closure, we know that for each $B \in N \cap \mathcal{F}$, there are infinitely many i with $[(V_m \cap U_i) \cap B] \setminus U] \neq \emptyset$.

This means it is possible to find an infinite set I and points $x_i \in (V_m \cap U_i) \setminus U$ for $i \in I$ such that for all $B \in N \cap \mathcal{F}$,

$$\{x_i : i \in I\} \setminus B \text{ is finite.} \tag{5.5}$$

Since X is countably compact (and the points x_i are all distinct), the infinite set $\{x_i : i \in I\}$ has a point of accumulation z. It follows that

$$z \in \overline{V}_m \cap \Lambda(\vec{U}) \setminus U, \tag{5.6}$$

and this is a contradiction because

$$\overline{V}_m \cap \Lambda(\vec{U}) \subseteq U. \tag{5.7}$$

□

Now let U be any open neighborhood of x. We must produce a set E in $\mathcal{B}$ such that

$$E \subseteq U \cap \Delta^n(\mathcal{D}, A). \tag{5.8}$$

Our choice of $\{V_m : m < \omega\}$ implies that there must be an m for which

$$\overline{V}_m \cap \Lambda(\vec{U}) \subseteq U \tag{5.9}$$

By Lemma 5.6, there is a set $B \in N \cap \mathcal{F}$ such that

$$\bigcup_{j<\omega} [(V_m \cap U_j) \cap B] \subseteq U. \tag{5.10}$$

Shrinking B in $N \cap \mathcal{F}$ causes no harm, so we may assume that $B \subseteq \Delta^n(\mathcal{D}, A)$. By Lemma 5.6, there is an i such that $[(V_m \cap U_i) \cap B]$ is relatively open of $\mathcal{D}$–depth at least n. Thus

$$[(V_m \cap U_i) \cap B] \in \mathcal{B} \tag{5.11}$$

and

$$[(V_m \cap U_i) \cap B] \subseteq U \cap \Delta^n(\mathcal{D}, A), \tag{5.12}$$

as required. □

We now come to the objects that will be crucial in the proof that our notions of forcing can be iterated without new reals appearing at limit stages of the iteration.

Definition 5.7.

(1) An N–spine is a sequence $\vec{U} = \langle U_i \cap A_i : i < \omega \rangle$ such that
- U_i is an open set in N
- $i \neq j \Longrightarrow \overline{U}_i \cap \overline{U}_j = \emptyset$
- $A_i \in N \cap \mathcal{F}$
- $j < i \Longrightarrow A_i \subseteq A_j$
- $\langle A_i : i < \omega \rangle$ generates $N \cap \mathcal{F}$
- given $\mathcal{D} \in N \cap \mathfrak{D}$ and $n < \omega$, $U_i \cap A_i$ is relatively open of $\mathcal{D}$–level at least n for all but finitely many i.

(2) If $\vec{U} = \langle U_i \cap A_i : i < \omega\rangle$ is an N–spine, then we define the *top of* $\vec{U}$ (denoted $\top(\vec{U})$) to be the set of all points z for which there exists an infinite $I \subseteq \omega$ and points $\langle x_i : i \in I\rangle$ such that
   - $x_i \in U_i \cap A_i$
   - if $\mathcal{D} \in N \cap \mathfrak{D}$, then

$$\lim_{i \in I} \mathrm{Dp}(\mathcal{D}, A_i)[x_i] = \infty. \tag{5.13}$$

   We say the sequence $\langle x_i : i \in I\rangle$ *strongly* $\vec{U}$*–diagonalizes* $N \cap \mathcal{F}$.

(3) If $\vec{U} = \langle U_i \cap A_i : i < \omega\rangle$ and $\vec{V} = \langle V_j \cap B_j : j < \omega\rangle$ are N–spines, then we say that $\vec{V}$ refines $\vec{U}$ if
   - for each j there exists an i with $V_j \cap B_j \subseteq U_i \cap A_i$, and
   - for each i there exists at most one j with $V_j \cap B_j \subseteq U_i \cap A_i$.

Proposition 5.8.
(1) N–spines exist.
(2) If $\vec{U}$ is an N–spine, then $\top(\vec{U})$ is a non–empty subset of $\mathrm{Tr}(N)$.

Proof. For the first part, we let $\langle A_n : n < \omega\rangle$ be a $\subseteq$–decreasing sequence that generates $N \cap \mathcal{F}$ and let $\langle \mathcal{D}_n : n < \omega\rangle$ be a $\subseteq$–increasing enumeration of a $\subseteq$–cofinal subset of $N \cap \mathfrak{D}$. We construct our N–spine in ω stages, choosing at stage n an open set U_n and point x_n such that

- $U_n \in N$,
- $\overline{U}_n$ is small,
- $\overline{U}_n \cap \bigcup_{i<n} \overline{U}_i = \emptyset$,
- $x_n \in N \cap U_n$, and
- $\mathrm{Dp}(\mathcal{D}_n, A_n)[x_n] \geq n$.

There are no obstacles to be overcome in the construction, and it is straightforward to verify that $\langle U_n \cap A_n : n < \omega\rangle$ is an N–spine.

For the second part, note that by definition of N-spine one can find a sequence $\langle x_i : i \in I\rangle$ that strongly $\vec{U}$–diagonalizes $N \cap \mathcal{F}$. The countable compactness of X guarantees that the (infinite) set $\{x_i : i \in I\}$ has a limit point. Since $\langle A_i : i < \omega\rangle$ is a $\subseteq$–decreasing sequence generating $N \cap \mathcal{F}$, any limit point of $\langle x_i : i \in I\rangle$ is in $\mathrm{Tr}(N)$. □

The question of whether or not a sequence is an N–spine can be settled based on information available in N, even though the sequence is not necessarily an element of N. We now show that N–spines can be used as "targets" in the proof of total properness.

Lemma 5.9 (Extension Lemma for N–spines)**.** Suppose $p \in N \cap P$, $D \in P$ is dense in P, and $\vec{U} = \langle U_i \cap A_i : i < \omega\rangle$ be an N–spine. Then there is a $q \leq p$ in $N \cap D$ such that $[q] \setminus [p] \subseteq \bigcup_{i<\omega} U_i \cap A_i$.

Proof. Let $\mathcal{D} = \{(p, D)\}$. There is an i such that $U_i \cap A_i$ is relatively open of $\mathcal{D}$–depth greater than or equal to 0 by definition of N–spine, and this immediately provides us with the required condition q. □

Lemma 5.10 (Focus Lemma for N–spines)**.** Let $\vec{U}$ be an N–spine, and let f be a promise from the model N. Then $\vec{U}$ can be refined to an N–spine

$$\vec{V} = \langle V_j \cap B_j : j < \omega\rangle \tag{5.14}$$

such that

$$\{x \in \operatorname{dom} f : \bigcup_{j<\omega} V_j \cap B_j \subseteq f(x)\} \text{ is large.} \tag{5.15}$$

Proof. First, note that since $\pi\chi(\top(\vec{U})) = \aleph_0$, there is an open set V such that

$$V \cap \top(\vec{U}) \neq \emptyset, \tag{5.16}$$

and

$$\{x \in \operatorname{dom} f : \overline{V} \cap \top(\vec{U}) \subseteq f(x)\} \text{ is large.} \tag{5.17}$$

Given an open set U with

$$\overline{V} \cap \top(\vec{U}) \subseteq U, \tag{5.18}$$

there must exist $\mathcal{D} \in N \cap \mathfrak{D}$ and $n < \omega$ such that

$$V \cap \bigcup_{i<\omega} (U_i \cap \Delta^n(\mathcal{D}, A_i)) \subseteq U. \tag{5.19}$$

This is easy to see — if it fails, then we can build a sequence $\langle x_i : i \in I\rangle$ in $V \setminus U$ that strongly $\vec{U}$–diagonalizes $N \cap \mathcal{F}$. Any limit point of this sequence will contradict (5.18).

As a corollary to this observation, we deduce that there must exist $\mathcal{D} \in N \cap \mathfrak{D}$ and $n < \omega$ such that

$$\left\{x \in \operatorname{dom} f : \bigcup_{i<\omega} [V \cap U_i \cap \Delta^n(\mathcal{D}, A_i)] \subseteq f(x)\right\} \text{ is large.} \tag{5.20}$$

Fix such n and $\mathcal{D}$. For $i < \omega$, let us define

$$E_i = V \cap U_i \cap \Delta^n(\mathcal{D}, A_i), \tag{5.21}$$

so

$$\{x \in \operatorname{dom} f : \bigcup_{i<\omega} E_i \subseteq f(x)\} \text{ is large.} \tag{5.22}$$

Our goal is to build the N–spine $\vec{V} = \langle V_j \cap B_j : j < \omega\rangle$ so that for each j, there is an i with

$$V_j \cap B_j \subseteq E_i. \tag{5.23}$$

This suffices for (5.15) because of (5.22).

To begin, let $\langle \mathcal{D}_j : j < \omega\rangle$ satisfy

- $\mathcal{D}_j \in N \cap \mathfrak{D}$,
- $\mathcal{D}_j \subseteq \mathcal{D}_{j+1}$,
- $\mathcal{D}_0 = \mathcal{D}$, and
- if $\mathcal{E} \in N \cap \mathfrak{D}$, then $\mathcal{E} \subseteq \mathcal{D}_j$ for some j.

At stage j of our construction, we will choose objects i_j, V_j, and B_j such that

- $\langle i_j : j < \omega\rangle$ is a strictly increasing sequence of natural numbers.
- V_j is an open set in N
- $B_j \in N \cap \mathcal{F}$
- $B_j \subseteq A_{i_j}$
- $V_j \cap B_j$ is relatively open of $\mathcal{D}_j$–level $\geq n$
- $V_j \cap B_j \subseteq E_{i_j}$

The difficult part (and the reason for all the attention given to countable π–networks) is achieving that V_j and B_j are elements of N. The following proposition shows us how to complete the construction.

Proposition 5.11. Suppose $\mathfrak{D} \subseteq \mathcal{E} \in N \cap \mathfrak{D}$ and $n \leq m < \omega$. Then there are infinitely many $i < \omega$ for which we can find sets W and B such that

- W is an open set ***in*** N,
- $B \in N \cap \mathcal{F}$,
- $B \subseteq A_i$,
- $W \cap B \subseteq E_i$, and
- $W \cap B$ is relatively open of $\mathcal{E}$–level $\geq m$.

Proof. Since $V \cap \top(\vec{U}) \neq \emptyset$, we can find a sequence $\langle x_i : i \in I\rangle$ that strongly $\vec{U}$–diagonalizes $N \cap \mathcal{F}$ such that $x_i \in V$ for for $i \in I$. By definition, this means that for all but finitely many $i \in I$, we have

$$\mathrm{Dp}_{(\mathcal{E}, A_i)}(x_i) \geq m + 1. \tag{5.24}$$

Given such an i, we know that x_i has a countable π–network in

$$A_i(\mathcal{E}, m) := \{y \in A_i : \mathrm{Dp}_{(\mathcal{E}, A_i)}(y) \geq m\} \tag{5.25}$$

consisting of sets that are relatively open of $\mathcal{E}$–depth $\geq m$. Since x_i, A_i, and $\mathcal{E}$ are all elements of N, every member of this countable π–network is an element of N.

Since $\mathcal{D} \subseteq \mathcal{E}$ and $m \geq n$, we know that $E_i \cap A_i(\mathcal{E}, m) = V \cap U_i \cap A_i(\mathcal{E}, m)$ is an open neighborhood of x_i in $A_i(\mathcal{E}, m)$. Since every member of x_i's countable π–network is in N, this means we can find W and B in N such that

- $W \cap B$ is relatively open of $\mathcal{E}$–depth $\geq m$, and
- $W \cap B \subseteq E_i$,

as required.

□

Now that we are armed with the preceding proposition, it is straightforward to construct the required $\vec{V}$. □

6. Back to the Iteration

We are now ready to prove that our proposed iteration satisfies the last condition of Theorem 4.1. Let us recall that we need to establish
Hypothesis 3 of Theorem 4.1:
For each α, if N_0, N_1, $\dot{q}$, $\bar{G}$, and $\langle G^\ell : \ell < k\rangle$ satisfy

- N_0 and N_1 are countable elementary submodels of $H(\chi)$
- $N_0 \in N_1$
- $\{\mathbb{P}, \alpha, \dot{q}\} \in N_0$
- $\Vdash_{P_\alpha} \dot{q} \in \dot{Q}_\alpha$
- $\bar{G} \in \mathrm{Gen}^+(N_0, P_\alpha, p) \cap N_1$
- for $\ell < k$, $G^\ell \in \mathrm{Gen}(N_1, P_\alpha)$
- for $\ell < k$, $\bar{G} \subseteq G^\ell$

then there is a sequence $\langle \dot{q}_n : n \in \omega\rangle$ in $N_1 \cap \mathrm{Gen}(N_0[\bar{G}], \dot{Q}, \dot{q})$ such that for all $\ell < k$,

$$G^\ell \Vdash \langle \dot{q}_n : n \in \omega\rangle \text{ has a lower bound in } \dot{Q}.$$

This looks much worse than it actually is, but before we discuss how to prove it we should make sure that the notation is understood. Let us begin in the following context:

- P is a totally proper notion of forcing
- $\dot{Q}$ is a P–name for a totally proper notion of forcing
- $N_0 \in N_1$ are countable elementary submodels of $H(\chi)$
- $\{P, \dot{Q}\} \in N_0$

The reader should keep in mind that the P referred to above is going to correspond to an initial segment of our iteration, while the $\dot{Q}$ will be the notion of forcing used to handle a particular space as in the preceding sections. We now bring in a few more definitions from the paper [4].

Definition 6.1. Let P be a notion of forcing, let N be a countable elementary submodel of some $H(\chi)$ with $P \in N$, and let $p \in N \cap P$. We define

(1) $N^P = \{\dot{\tau} \in N : \dot{\tau} \text{ is a } P\text{–name}\}$
(2) $\mathrm{Gen}(N, P) = \{G \subseteq N \cap P : G \text{ is an } N\text{-generic filter on } N \cap P\}$
(3) $\mathrm{Gen}^+(N, P) = \{G \in \mathrm{Gen}(N, P) : G \text{ has a lower bound in } P\}$
(4) $\mathrm{Gen}(N, P, p) = \{G \in \mathrm{Gen}(N, P) : p \in G\}$
(5) $\mathrm{Gen}^+(N, P, p) = \mathrm{Gen}(N, P, p) \cap \mathrm{Gen}^+(N, P)$.

The above definition gives us vocabulary to discuss the way in which elementary submodels interact with totally proper forcing. For example, a density argument establishes that if G is a generic subset of the totally proper notion of forcing P, then $N \cap G \in \mathrm{Gen}(N, P)$ if and only if G contains a totally (N, P)–generic condition. A member of $\mathrm{Gen}(N, P)$ contains enough information to decide the behavior in the generic extension of objects with names coming from N — if $\theta(x_0, \ldots, x_{n-1})$ is a formula, $G \in \mathrm{Gen}(N, P)$, and $\dot{\tau}_i$ is an element of N^P for $i < n$, then there is a $p \in G$ such that either

$$p \Vdash \theta(\dot{\tau}_0, \ldots, \dot{\tau}_n), \text{ or} \\ p \Vdash \neg\theta(\dot{\tau}_0, \ldots, \dot{\tau}_n). \tag{6.1}$$

We can summarize this by saying that an element of $\mathrm{Gen}(N, P)$ decides all sentences involving only P–names taken from N.[1] So, for example, an element of $\mathrm{Gen}(N, P)$ has enough information to decide whether or not a name $\tau \in N^P$ is going to be interpreted as an open subset of $\dot{X}$ in the generic extension.

We actually need a bit more than this. Let $\bar{G}$ be a member of $\mathrm{Gen}(N, P)$. We have already mentioned that $\bar{G}$ has enough information to determine what happens to objects named by names from N^P, but in fact $\bar{G}$ can say a lot about countable sequences of such objects.

Definition 6.2. Let P be a notion of forcing, let $A \subseteq P$, and let θ be a sentence of the forcing language. We define

$$A \Vdash \theta$$

to mean that whenever $G \subseteq P$ is a generic filter such that $A \subseteq G$, then θ holds in the generic extension $V[G]$.

[1]This is part of a more general phenomenon — if G is a generic subset of P for which $N \cap G$ is in $\mathrm{Gen}(N, P)$, then the structure $N[G]$ is isomorphic to a structure in the ground model.

If we adopt the convention that all notions of forcing are complete Boolean algebras, then "$A \Vdash \theta$" is equivalent to the condition of $\wedge A$ (the infimum of A) forcing θ to be true. We do not require the full generality of the above definition, as we are only interested in the special case where A is a member of $\mathrm{Gen}(N, P)$ for some N. The following definition shows us one way in which members of $\mathrm{Gen}(N, P)$ can speak about countable sequences of names from N.

Definition 6.3. Assume P is a totally proper notion of forcing, $\dot{Q}$ is a P–name for a totally proper notion of forcing, N is a countable elementary submodel of $H(\chi)$ that contains P and $\dot{Q}$, and $\bar{G} \in \mathrm{Gen}(N, P)$. A sequence $\langle \dot{q}_n : n < \omega \rangle$ is in $\mathrm{Gen}(N[\bar{G}], \dot{Q})$ if

(1) $\dot{q}_n \in N_0^P$
(2) $\bar{G} \Vdash \dot{q}_n \in \dot{Q}$
(3) $\bar{G} \Vdash \dot{q}_{n+1} \leq \dot{q}_n$ in $\dot{Q}$
(4) if $\dot{D} \in N^P$ and
$$\bar{G} \Vdash \dot{D} \text{ is a dense subset of } \dot{Q},$$
then there is an n such that
$$\bar{G} \Vdash \dot{q}_n \in \dot{D}.$$

If $\dot{q}$ is a name in N^P for a condition in $\dot{Q}$, then $\mathrm{Gen}(N[\bar{G}], \dot{Q}, \dot{q})$ consists of those members of $\mathrm{Gen}(N[\bar{G}], \dot{Q})$ for which $\dot{q}_0 = \dot{q}$.

Note that in the situation of the above definition we can determine whether or not a sequence of names belongs to $\mathrm{Gen}(N[\bar{G}], \dot{Q})$ simply by consulting $\bar{G}$. Before pursuing these ideas further, let us note that all terms in the statement of Hypothesis 3 of Theorem 4.1 have now been defined and so it behooves us to change to that specific context. Thus, we now assume

- P is a totally proper notion of forcing,
- $\dot{X}$ and $\dot{\mathcal{F}}$ are P–names for objects as in Section 2,
- $\dot{Q}$ is a P–name for the notion of forcing associated with $\dot{X}$ and $\dot{\mathcal{F}}$,
- $N_0 \in N_1$ are countable elementary submodels of $H(\chi)$,
- $\{P, \dot{Q}, \dot{X}, \dot{\mathcal{F}}\} \in N_0$,
- $\dot{q} \in N_0^P$,
- $\Vdash_P \dot{q} \in \dot{Q}$,
- $\bar{G} \in \mathrm{Gen}^+(N_0, P, p) \cap N_1$,
- $G^\ell \in \mathrm{Gen}(N_1, \cap P)$ for $\ell < k$, and
- $\bar{G} \subseteq G^\ell$ for $\ell < k$

We have seen already that set $\bar{G}$ "pins down" what the generic extension looks like as far as N_0 is concerned, but N_1 and the various G^ℓ are an added wrinkle to the discussion. One should adopt the point of view that each G^ℓ gives us a peek at what the generic extension potentially looks like with regard to N_1. All of them agree on what the generic extension looks like when localized to N_0 (as they all extend $\bar{G}$), but they may differ on their opinions about objects with names from N_1. What we need is to build a member of $\mathrm{Gen}(N_0[\bar{G}], \dot{Q}, \dot{q})$ that is forced to have a lower bound no matter which G^ℓ happens to be the the "real" $N_1 \cap \dot{G}_P$.[2] Note that even though $\bar{G}$ possesses enough information to tell whether a sequence of names is

[2]This strange condition is a crucial part of the proof of Theorem 4.1; in the parlance of Shelah's work these sorts of requirements are called "medicine against weak diamond".

in $\mathrm{Gen}(N_0[\bar{G}], \dot{Q})$ or not, the determination of whether such a sequence has a lower bound requires information that is not available in N_0. This is the reason for N_1 and the G^ℓ.

Our proof will make use of the technology developed in the previous section. Strictly speaking, we will not be using N–spines; rather we will be using sequences of names from N_0^P that bear roughly the same relation to N–spines as sequences in $\mathrm{Gen}(N[\bar{G}], \dot{Q})$ bear to $\mathrm{Gen}(N, P)$.

Definition 6.4. In the context of our discussion, we say that a sequence $\dot{\vec{U}} = \langle (\dot{U}_i, \dot{A}_i) : i < \omega \rangle$ is an $N_0[\bar{G}]$–spine if each $\dot{U}_i$ and $\dot{A}_i$ is an element of N_0^P, and

$$\bar{G} \Vdash \text{“}\vec{U} := \langle \dot{U}_i[\dot{G}_P] \cap \dot{A}_i[\dot{G}_P] : i < \omega \rangle \text{ is an } N_0[\dot{G}_P]\text{–spine.”}.$$

Thus our $N_0[\bar{G}]$–spines are sequences of pairs of names from N_0^P, and whenever G is a generic subset of P extending $\bar{G}$ the sequence arising after interpreting the names will form an $N[G]$–spine. Things are starting to get a bit complicated, so perhaps a bit of notation will make things easier: If $\dot{\vec{U}}$ is an $N_0[\bar{G}]$–spine and G is a generic subset of P then we let $\dot{\vec{U}}[G]$ denote the $N_0[G]$–spine obtained by evaluating names with G. Thus $\dot{\vec{U}}$ is an object in the ground model, and $\dot{\vec{U}}[G]$ is the corresponding object in the extension.[3]

Notice that it makes sense as well to talk about one $N_0[\bar{G}]$–spine refining another, as $\bar{G}$ has enough information to determine if this is going to happen in the generic extension. Now we come to yet another version of the Focus Lemma, this time formulated in terms of names.

Proposition 6.5. Suppose $\dot{\vec{U}}$ is an $N_0[\bar{G}]$–spine in N_1, and let $\dot{f} \in N_0$ be a P–name for a promise. If we are given $\ell < k$, then $\dot{\vec{U}}$ can be refined to an $N_0[\bar{G}]$ spine $\dot{\vec{V}} = \langle (\dot{V}_j, \dot{B}_j) : j < \omega \rangle$ such that $\dot{\vec{V}} \in N_1$, and

$$G^\ell \Vdash \{x \in \mathrm{dom}\, \dot{f} : \bigcup_{j<\omega} \dot{V}_j \cap \dot{B}_j \subseteq \dot{f}(x)\} \text{ is large.} \tag{6.2}$$

Proof. Let G be any generic subset of P with $G^\ell \subseteq G$. In the model $V[G]$, all of our names get interpreted as concrete objects. In particular, $\dot{\vec{U}}[G]$ is an $N_0[G]$–spine lying in the model $N_1[G]$ and $\dot{f}$ is interpreted as a promise $f := \dot{f}[G]$ in $N_0[G]$. We apply Proposition 5.10 inside the model $N_1[G]$ to obtain an $N_0[G]$–spine $\vec{V} = \langle (V_j, B_j) : j < \omega \rangle$ in $N_1[G]$ refining $\dot{\vec{U}}[G]$ such that

$$\{x \in \mathrm{dom} : \bigcup_{j<\omega} V_j \cap B_j \subseteq f(x)\} \text{ is large.} \tag{6.3}$$

This is almost what we need — what remains is to take the $N_0[G]$–spine $\vec{V}$ (which lives only in the generic extension $V[G]$) and replace it by an equivalent $N_0[\bar{G}]$–spine $\dot{\vec{V}}$ (which lives in the ground model). We can do this because the notion of forcing P is totally proper, as the following argument shows.

Given $j < \omega$, in the generic extension we know that both V_j and B_j are the evaluations of P–names from N_0 using the generic filter G. In the model $N_1[G]$, we

[3]This is technically different from evaluating a forcing name using a generic object, but no harm is done if the reader mentally identifies the two processes.

can choose P–names $\dot{V}_j$ and $\dot{B}_j$ witnessing this. Since forcing with P adds no new ω–sequences to the ground model, it follows that the sequence $\langle(\dot{V}_j, \dot{B}_j) : j < \omega\rangle$ must actually lie in $N_1[G] \cap V = N_1$, and we are done as G^ℓ will still force this sequence to have the required properties. □

Iterating the above argument gives us a refinement of $\dot{\vec{U}}$ that works uniformly for *all* $\ell < k$.

Corollary 6.6. Suppose $\dot{\vec{U}}$ is an $N_0[\bar{G}]$–spine in N_1, and let $\dot{f} \in N_0$ be a P–name for a promise. Then $\dot{\vec{U}}$ can be refined to an $N_0[\bar{G}]$ spine $\dot{\vec{V}} = \langle(\dot{V}_j, \dot{B}_j) : j < \omega\rangle$ such that $\dot{\vec{V}} \in N_1$, and *for every* $\ell < k$,

$$G^\ell \Vdash \{x \in \operatorname{dom} \dot{f} : \bigcup_{j<\omega} \dot{V}_j \cap \dot{B}_j \subseteq \dot{f}(x)\} \text{ is large.} \tag{6.4}$$

Proof. The proof consists of iterating Proposition 6.5 k times. □

Now at last we can show the validity of Hypothesis 3 of Theorem 4.1 for our iteration.

Proposition 6.7. For each α, if N_0, N_1, $\dot{q}$, $\bar{G}$, and $\langle G^\ell : \ell < k\rangle$ satisfy

- N_0 and N_1 are countable elementary submodels of $H(\chi)$
- $N_0 \in N_1$
- $\{\mathbb{P}, \alpha, \dot{q}\} \in N_0$
- $\Vdash_{P_\alpha} \dot{q} \in \dot{Q}_\alpha$
- $\bar{G} \in \operatorname{Gen}^+(N_0, P_\alpha, p) \cap N_1$
- for $\ell < k$, $G^\ell \in \operatorname{Gen}(N_1, P_\alpha)$
- for $\ell < k$, $\bar{G} \subseteq G^\ell$

then there is a sequence $\langle \dot{q}_n : n \in \omega\rangle$ in $N_1 \cap \operatorname{Gen}(N_0[\bar{G}], \dot{Q}, \dot{q})$ such that for all $\ell < k$,

$$G^\ell \Vdash \langle \dot{q}_n : n \in \omega\rangle \text{ has a lower bound in } \dot{Q}.$$

Proof. This involves repeating the proof that our notion of forcing is totally proper. The new twist is that the sequence $\langle q_n : n < \omega\rangle$ consists of names forced by $\bar{G}$ to form a generic sequence, instead of actual conditions in an extant notion of forcing. To ensure that each filter G^ℓ forces the sequence to have a lower bound, we use Corollary 6.6 to "take care of" each (name for a) promise that appears along the way. □

7. On the $\aleph_2$–chain condition

We need one last ingredient before we can construct our model. So far, we have demonstrated that the notions of forcing that we will be using can be iterated without adding new reals, but we will also need that the limit of our iteration sequence satisfies the $\aleph_2$–chain condition. Our tool for showing this is a standard one — the $\aleph_2$–properness isomorphism condition (abbreviated $\aleph_2$–p.i.c.) of Shelah.

Definition 7.1. P satisfies the $\aleph_2$–p.i.c. provided the following holds (for χ a large enough regular cardinal):
If

(1) $i < j < \aleph_2$
(2) N_i and N_j are countable elementary submodels of $H(\chi)$

(3) $i \in N_i$, $j \in N_j$
(4) $N_i \cap \omega_2 \subseteq j$
(5) $N_i \cap i = N_j \cap j$
(6) h is an isomorphism from N_i onto N_j
(7) $h(i) = j$
(8) h is the identity map on $N_i \cap N_j$
(9) $P \in N_i \cap N_j$
(10) $p \in N_i \cap P$

then (letting $\dot{G}$ be the P–name for the generic set) there is a $q \in P$ such that:

(11) $q \Vdash$ "$(\forall r \in N_i \cap P)[r \in \dot{G} \iff h(r) \in \dot{G}]$"
(12) $q \Vdash$ "$p \in \dot{G}$"
(13) q is (N_i, P)–generic.

Notice that if N_i and N_j are as in the above definition, then $N_i \cap \omega_1 = N_j \cap \omega_1$. It also does not matter if we require that the models under consideration contain a fixed parameter $x \in H(\chi)$. In particular, if we assume the Continuum Hypothesis then without loss of generality there is an enumeration of $[\omega_1]^{\aleph_0}$ in order–type ω_1 that is an element of both N_i and N_j. From this (and the fact that $N_i \cap \omega_1 = N_j \cap \omega_1$) we conclude that N_i and N_j contain exactly the same countable sequences of countable ordinals.

The only role that the $\aleph_2$–p.i.c. plays in our theorem is the result of Shelah that the limit of a countable support iteration of length at most ω_2 in which each iterand is proper and $\aleph_2$–p.i.c. will satisfy the (weaker) $\aleph_2$–chain condition.

Theorem 7.2. *Assume the Continuum Hypothesis, and suppose $(X, \mathcal{F})$ is a pair of objects as in Section 2 with $|X| = \aleph_1$. Then P, the notion of forcing from Section 3 associated with $(X, \mathcal{F})$, satisfies the $\aleph_2$–p.i.c..*

Proof. Let N_i and N_j be models as in Definition 7.1. Without loss of generality, X has ω_1 as its underlying set, and so $N_i \cap X = N_j \cap X$. Since N_i and N_j contain exactly the same countable subsets of ω_1, it follows that N_i and N_j contain exactly the same separable closed subsets of X. Since $\mathcal{F}$ is generated by separable sets, it follows that

$$\mathrm{Tr}(N_i) = \mathrm{Tr}(N_j). \tag{7.1}$$

Now suppose that $r = (\sigma, A, \Phi)$ is a condition in $N_i \cap P$. We may assume that A is separable, because the set of such conditions is dense in P. Then $h(r)$ is a condition in $N_j \cap h(P) = P$, and since N_j contains the same countable subsets of ω_1 and the same separable closed subsets as N_i, it follows that $h(r)$ is of the form $(\sigma, A, h(\Phi))$. From the definition of P, we see that

$$r \oplus h(r) := (\sigma, A, \Phi \cup h(\Phi)) \tag{7.2}$$

is a condition in P that extends both r and $h(r)$.[4]

Now suppose we are given $p \in N_i \cap P$, and let $\langle D_n : n < \omega \rangle$ enumerate the dense open subsets of P that are elements of N_i. To prove that P has the $\aleph_2$ –p.i.c., it suffices to construct a decreasing sequence $\langle p_n : n < \omega \rangle$ of conditions in $N_i \cap P$ such that

- $p_0 = p$,

[4]In fact, $r \oplus h(r)$ is the greatest lower bound of r and $h(r)$ in P.

- $p_{n+1} \in N \cap D_n$, and
- the sequence $\langle p_n \oplus h(p_n) : n < \omega\rangle$ has a lower bound q in P.

If we do this, then the condition q demonstrates that P has the $\aleph_2$–p.i.c..

The argument we use is a simple modification of that used to show that P is totally proper. We need yet another version of the Focus Lemma.

Lemma 7.3. Let $f \in N_i$ be a promise, and let U be an open set that meets $\mathrm{Tr}(N_i)$. Then there is a separable set $A \in N_i \cap \mathcal{F}$ and an open $V \subseteq U$ such that

- $V \cap \mathrm{Tr}(N_i) \neq \emptyset$,
- $\{x \in \mathrm{dom}(f) : V \cap A \subseteq f(x)\}$ is large, and
- $\{x \in \mathrm{dom}(h(f)) : V \cap A \subseteq h(f)(x)\}$ is large.

Proof. The proof is quite simple. From Lemma 3.14, we know there is a separable set $A^* \in N_i \cap \mathcal{F}$ and an open $V^* \subseteq U$ such that V^* meets $\mathrm{Tr}(N_i)$, and

$$\{x \in \mathrm{dom}(f) : V^* \cap A^* \subseteq f(x)\} \text{ is large.} \tag{7.3}$$

Since $\mathrm{Tr}(N_i) = \mathrm{Tr}(N_j)$, we can take the set V^* and apply Lemma 3.14 once more (this time to the model N_j and the promise $h(f)$) to get a separable $A \in N_j \cap \mathcal{F}$ and open $V \subseteq V^*$ such that

$$\{x \in \mathrm{dom}(h(f)) : V \cap A \subseteq h(f)(x)\} \text{ is large.} \tag{7.4}$$

Since $A^* \in N_j$ (as it is separable and closed), without loss of generality $A \subseteq A^*$. Since $V \subseteq V^*$, it follows that A and V are as required. □

The proof now goes just as in Section 3 — we construct the sequence $\langle p_n : n < \omega\rangle$ so that $p_{n+1} \in N_i \cap D_n$, and we use Lemma 7.3 instead of Lemma 3.14 to take care of each promise along the way. We leave the details to the reader. □

8. Constructing the model

Let us now turn to the problem of constructing a model of $\mathsf{CH}+\otimes$, where $\otimes$ abbreviates the following statement:

> A regular countably compact space that is hereditarily of countable π–character is either compact or contains an uncountable free sequence.

Lemma 8.1. Assume the Continuum Hypothesis. Then the full version of $\otimes$ follows from the weaker statement where we require the spaces to be of cardinality and weight at most $\aleph_1$.

Proof. We show that if $\otimes$ fails, then in the presence of the Continuum Hypothesis there is a counterexample to $\otimes$ of cardinality and weight $\aleph_1$. Thus, suppose that the Continuum Hypothesis holds, and we are given objects (X, τ) and $\mathcal{F}$ such that

- (X, τ) is a countably compact, non–compact regular space,
- $h\pi\chi(X) = \aleph_0$, and
- X does not contain an uncountable free sequence.

We observe that without loss of generality the space X is separable, i.e., if there is a counterexample to $\otimes$ then there is a separable one. To see this, let $\mathcal{F}$ be a maximal filter of closed subsets of X with $\bigcap \mathcal{F} = \emptyset$. Since X has no uncountable free sequences, we know that $\mathcal{F}$ is generated by separable sets.

Choose a separable $Y \in \mathcal{F}$. Since Y is a subspace of X, we know that Y is regular and $h\pi\chi(Y) = \aleph_0$. Since Y is closed, it follows that Y is countably compact and

Y cannot contain an uncountable free sequence. Finally, since $Y \in \mathcal{F}$ we know that Y is not compact. Thus, we may assume that X, our counterexample to $\otimes$, is separable.

We now claim that $w(X) = \aleph_1$. Since X is separable, it follows that there are most $2^{\aleph_0}$ open sets that are equal to the interior of their closures, i.e., X contains at most $\aleph_1$ regular open sets. Since X is regular, X has a base consisting of such sets, and so $w(X) \leq \aleph_1$. The space X is countably compact and non–compact, so $w(X) \neq \aleph_0$, and the result follows.

Let χ be a regular cardinal much larger than any other cardinals under discussion, and let M be an elementary submodel of $H(\chi)$ of size $\aleph_1$ containing the space (X, τ), and closed under ω–sequences. The existence of such a model follows from the Continuum Hypothesis.

We now utilize M in a standard fashion to define an approximation to X — we let X_M be the topological space whose underlying set of points is $M \cap X$, and whose topology is generated by the base whose elements are of the form $M \cap U$, where U is an open subset of X and $U \in M$. The space X_M is of size and weight at most $\aleph_1$, and we claim that it can serve as a counterexample to $\otimes$.

Standard arguments using elementary submodels (along the lines of Junqueira and Tall [11]) tell us that X_M is a regular, countably compact, non–compact space. Since $w(X) = \aleph_1$, it follows that M contains every member of a base for X. From this it follows that the topology on X_M defined above coincides with the topology on X_M obtained by considering it as a subspace of X. From this we conclude that $h\pi\chi(X_M) = \aleph_0$.

Finally, we show that X_M does not contain an uncountable free sequence. By way of contradiction, assume that this fails and let $\{x_\alpha : \alpha < \omega_1\}$ be a free sequence in X_M. For $\alpha < \omega_1$, let $A_\alpha = \{x_\beta : \beta < \alpha\}$ and $B_\alpha = \{x_\beta : \beta \geq \alpha\}$. Our assumption is that

$$\mathrm{cl}_{X_M}(A_\alpha) \cap \mathrm{cl}_{X_M}(B_\alpha) = \emptyset \text{ for all } \alpha < \omega_1, \tag{8.1}$$

and we will get a contradiction if we establish

$$\mathrm{cl}_X(A_\alpha) \cap \mathrm{cl}_X(B_\alpha) = \emptyset \text{ for all } \alpha < \omega_1, \tag{8.2}$$

and clearly this follows from the following lemma.

Lemma 8.2. Let A and B be subsets of X_M. Then

$$\mathrm{cl}_X(A) \cap \mathrm{cl}_X(B) \neq \emptyset \Longleftrightarrow \mathrm{cl}_{X_M}(A) \cap \mathrm{cl}_{X_M}(B) \neq \emptyset. \tag{8.3}$$

Proof. If $x \in \mathrm{cl}_{X_M}(A) \cap \mathrm{cl}_{X_M}(B)$ then x is certainly in $\mathrm{cl}_X(A) \cap \mathrm{cl}_X(B)$ because X_M has the subspace topology. For the other direction, assume that x is in both $\mathrm{cl}_X(A)$ and $\mathrm{cl}_X(B)$. We know that X is countably tight (this follows from $h\pi\chi(X) = \aleph_0$) and so there are countable sets $A_0 \subseteq A$ and $B_0 \subseteq B$ with

$$x \in \mathrm{cl}_X(A_0) \cap \mathrm{cl}_X(B_0). \tag{8.4}$$

Now M is closed under countable sequences, so both A_0 and B_0 are elements of M. Thus, both $\mathrm{cl}_X(A_0)$ and $\mathrm{cl}_X(B_0)$ are elements of M as well. By elementarity,

$$M \models \mathrm{cl}_X(A_0) \cap \mathrm{cl}_X(B_0) \neq \emptyset, \tag{8.5}$$

and so

$$M \cap \mathrm{cl}_X(A_0) \cap \mathrm{cl}_X(B_0) \neq \emptyset. \tag{8.6}$$

From this we conclude

$$\mathrm{cl}_{X_M}(A_0) \cap \mathrm{cl}_{X_M}(B_0) \neq \emptyset, \tag{8.7}$$

and therefore

$$\mathrm{cl}_{X_M}(A) \cap \mathrm{cl}_{X_M}(B) \neq \emptyset, \tag{8.8}$$

as required. □

Armed with the preceding lemma, we see that a free sequence in X_M also forms a free sequence in X, and this gives us the required contradiction. □

In summary, we have seen that in a model where CH holds, if there is a counterexample to $\otimes$ then there is a counterexample of weight and cardinality $\aleph_1$.

Theorem 8.3. *The statement $\otimes$ is consistent with the Continuum Hypothesis.*

Proof. In order to prove the consistency of CH+ $\otimes$, we start with a model of GCH and force with a countable support iteration $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_2 \rangle$ where $\dot{Q}_\alpha$ is a P_α–name for a forcing as defined in the earlier sections of this paper. Let P_{ω_2} be the limit of the countable support iteration $\mathbb{P}$. Theorem 4.1 together with our work in sections 4 through 6 tells us that P_{ω_2} is totally proper, so that the resulting model satisfies the Continuum Hypothesis. Our work in Section 7 establishes that P_{ω_2} satisfies the $\aleph_2$–chain condition as well. Thus, any potential counterexample to $\otimes$ of cardinality and weight $\aleph_1$ in the model obtained upon forcing with P_{ω_2} must in fact appear in after some initial stage of our iteration. Thus, standard bookkeeping techniques allow us to "take care of" any such counterexamples to $\otimes$. In light of Lemma 8.1, we conclude that $\otimes$ holds in the model obtained by forcing with P_{ω_2}. □

9. Conclusions

In the previous section, we proved that the combinatorial principle $\otimes$ is consistent with the Continuum Hypothesis. In this section we derive some consequences of this that were discussed in the introductory section of the paper.

Theorem 9.1. CH $+ \otimes$ *implies that compact S–spaces are sequential.*

Proof. By way of contradiction, assume that X is a compact S–space that is not sequential. Since CH implies $2^{\aleph_0} < 2^{\aleph_1}$, a result of Ismail and Nyikos [9] implies that X contains a countably compact, non–compact subset Y. Since the Continuum Hypothesis holds, we may assume that Y has size $\aleph_1$. Since X is hereditarily separable, we see that Y has at most $\aleph_1^{\aleph_0} = \aleph_1$ closed subsets, hence $w(Y) = \aleph_1$. Clearly an S–space is countably tight, so Šapirovskiĭ's theorem [15] implies that $h\pi\chi(X) = \aleph_0$. An application of $\otimes$ tells us that Y contains an uncountable free sequence, contradicting the fact that X is hereditarily separable. □

Corollary 9.2. CH+ $\otimes$ implies that compact S–spaces have size $\aleph_1$.

Proof. Suppose X is a compact S–space. Let X_0 be a countable dense subset of X. The sequential closure of X_0 has size at most $\aleph_1$ because the Continuum Hypothesis holds. Since X is sequential, the sequential closure of X_0 is equal to the real closure of X_0, hence $|X| \leq \aleph_1$. □

We close the paper with some comments on the famous problem whether the Continuum Hypothesis is strong enough to construct a countably tight compact space that is not sequential. Said another way, is CH together with the following statement consistent?

(∗) Countably tight compacta are sequential.

We showed in [3] that a notion of forcing quite similar to that defined in Section 3 can be used to "kill off" a single counterexample to (∗) without adding new reals, but to this point we have been unable to show that the forcings can be iterated without adding new reals. The main obstacle is our inability to prove that the notions of forcing are weakly $< \omega_1$ proper in this more general context. In the proof presented in Section 4 of the current paper, the fact that the filter $\mathcal{F}$ is generated by separable sets is used in a crucial manner. Perhaps one can prove a version of Theorem 4.1 that will handle such forcing notions, or perhaps one could show that the forcing notions in question are indeed weakly $< \omega_1$–proper. In any case, there is much more work to be done here.

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