# SUCCESSORS OF SINGULAR CARDINALS AND COLORING THEOREMS I

TODD EISWORTH AND SAHARON SHELAH

ABSTRACT. We investigate the existence of strong colorings on successors of singular cardinals. This work continues Section 2 of [1], but now our emphasis is on finding colorings of pairs of ordinals, rather than colorings of finite sets of ordinals.

## 1. Introduction

The theme of this paper is that strong coloring theorems hold at successors of singular cardinals of uncountable cofinality, except possibly in the case where the singular cardinal is a limit of regular cardinals that are Jonsson in a strong sense.

Our general framework is that $\lambda = \mu^+$, where $\mu$ is singular of uncountable cofinality. We will be searching for colorings of pairs of ordinals $< \lambda$ that exhibit quite complicated behaviour. The following definition (taken from [2]) explains what "complicated" means in the previous sentence.

**Definition 1.1.** Let $\lambda$ be an infinite cardinal, and suppose $\kappa+\theta \leq \mu \leq \lambda$. $\mathrm{Pr}_1(\lambda, \mu, \kappa, \theta)$ means that there is a symmetric two–place function $c$ from $\lambda$ to $\kappa$ such that if $\xi < \theta$ and for $i < \mu$, $\langle \alpha_{i,\zeta} : \zeta < \xi \rangle$ is a strictly increasing sequence of ordinals $< \lambda$ with all $\alpha_{i,\zeta}$'s distinct, then for every $\gamma < \kappa$ there are $i < j < \mu$ such that

$$\zeta_1 < \xi \text{ and } \zeta_2 < \xi \Longrightarrow c(\alpha_{i,\zeta_1}, \alpha_{i,\zeta_2}) = \gamma. \tag{1.1}$$

Just as in [1], one of our main tools is a game that measures how "Jonsson" a given cardinal is.

Recall that a cardinal $\lambda$ is a Jonsson cardinal if for every $c : [\lambda]^{<\omega} \to \lambda$, we can find a subset $I \subseteq \lambda$ of cardinality $\lambda$ such that the range of $c \restriction I$ is a proper subset of $\lambda$. A reader seeking more background should investigate [4] and [3] in [5].

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**Definition 1.2.** Assume $\mu \leq \lambda$ are cardinals, $\gamma$ is an ordinal, $n \leq \omega$, and $J$ is an ideal on $\lambda$. We define the game $\mathrm{Gm}^n_J[\lambda, \mu, \gamma]$ as follows:

A play lasts $\gamma$ moves.

In the $\alpha^{\text{th}}$ move, the first player chooses a function $F_\alpha : [\lambda]^{<n} \to \mu$, and the second player responds by choosing (if possible) a subset $A_\alpha \subseteq \lambda$ such that

- $A_\alpha \subseteq \bigcap_{\beta<\alpha} A_\beta$
- $A_\alpha \in J^+$
- $\mathrm{ran}(F_\alpha \restriction [A_\alpha]^{<n})$ is a proper subset of $\mu$.

The second player loses if he has no legal move for some $\alpha < \gamma$, and he wins otherwise.

In the previous definition, if $J = J^{\mathrm{bd}}_\lambda$ then we may omit it. Note that it causes no harm if we use a set $E$ of cardinality $\lambda$ instead of $\lambda$ itself; in this case, we write $\mathrm{Gm}^n_J[E, \mu, \gamma]$.

Note that $\lambda$ is a Jonsson cardinal if and only if Player I does not have a winning strategy in the game $\mathrm{Gm}^\omega[\lambda, \lambda, 1]$. One may view the lack of a winning strategy for Player I in games of longer length as a strong version of Jonsson-ness or a weak version of measurability — if $\lambda$ is measurable, then Player II can make sure her moves are elements of some $\lambda$–complete ultrafilter.

The following claim investigates how the existence of winning strategies is affected by modifications to the game; the proof is left to the reader.

**Claim 1.3.**

(1) If $\mu' \leq \mu$ and the first player has a winning strategy in $\mathrm{Gm}^n_J[\lambda, \mu, \gamma]$, then she has a winning strategy in $\mathrm{Gm}^n_J[\lambda, \mu', \gamma]$.

(2) Suppose we weaken the demand on the second player to

$$\text{“}(\exists \zeta < \lambda)[\mathrm{ran}(F_\alpha \restriction [A_\alpha \setminus \zeta]^{<n}) \text{ is a proper subset of } \mu].\text{”} \tag{1.2}$$

If $\mathrm{cf}(\lambda) \geq \gamma$ and $J \supseteq J^{\mathrm{bd}}_\lambda$, then the first player has a winning strategy in the revised game if and only if she has a winning strategy in the original game.

(3) If $J$ is $\gamma$–complete, then the same applies to the case where we weaken the demand on the second player to

$$\text{“}(\exists Y \in J)[\mathrm{ran}(F_\alpha \restriction [A_\alpha \setminus Y]^{<n}) \text{ is a proper subset of } \mu].\text{”} \tag{1.3}$$

(4) We can allow the second player to pass, i.e., to let $A_\alpha = \bigcap_{\beta<\alpha} A_\beta$ (even if this is not a legal move) as long as we declare that the

second player loses if the order–type of the set of moves where he did not pass is $< \gamma$.

(5) If Player I has a winning strategy in $\mathrm{Gm}^n_J[\lambda, \mu, \gamma]$ for every $\mu < \mu^*$ where $\mu^*$ is singular and $\gamma > \mathrm{cf}(\mu^*)$ is regular, then Player I has a winning strategy in $\mathrm{Gm}^n_J[\lambda, \mu^*, \gamma]$. We can weaken the requirement that $\gamma$ is regular and instead require that $\mathrm{cf}(\gamma) > \mathrm{cf}(\mu^*)$ and $\omega^\gamma = \gamma$.

In Section 2 of [1], the existence of winning strategies for Player I in variants of the game is investigated. We will prove one such result here; the reader should look in [1] for others.

**Claim 1.4.** If $2^\chi < \lambda < \beth_{(2^\chi)^+}(\chi)$ then Player I has a winning strategy in $\mathrm{Gm}^\omega[\lambda, \chi, (2^\chi)^+]$.

*Proof.* At a stage $i$, Player I will select a function $F_i : [\lambda]^{<\omega} \to \chi$ coding the Skolem functions of some model $M_i$.

For the initial move, we let the model $M_0$ have universe $\lambda$, and include in our language all relations on $\lambda$ and all functions from $\lambda$ to $\lambda$ of any finite arity that are first order definable in the structure $\langle H(\lambda^+), \in, <^*_{\lambda^+}\rangle$ with the parameters $\chi$ and $\lambda$.

For subsequent moves, $M_i$ is an expansion of $M_0$ with universe $\lambda$ that has all relations on $\lambda$ and all functions from $\lambda$ to $\lambda$ of any finite arity that are first order definable in the structure $\langle H(\lambda^+), \in, <^*_{\lambda^+}\rangle$ from the parameters $\chi$, $\lambda$, $M_0$, and $\langle A_j : j < i\rangle$.

To obtain the function $F_i$, we let $\langle F^i_n : n < \omega\rangle$ list the Skolem functions of $M_i$ in such a way that $F^i_n$ has $m_i(n) \leq n$ places. Let $h : \omega \to \omega$ be such that for all $n$, $h(n) \leq n$ and $h^{-1}(\{n\})$ is infinite. We then define

$$F_i(u) = \begin{cases} F^i_{h(|u|)}(\{\alpha \in u : |u \cap \alpha| < m_i(n)\}) & \text{if this is } < \chi \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

The point of doing this is that whenever Player II chooses $A_i$, we know that $\mathrm{ran}(F_i \restriction [A_i]^{<\omega})$ will look like the result of intersecting an elementary submodel of $M_i$ with $\chi$; in particular, this range will be closed under the functions from $M_i$.

Note that $M_0$ (and all expansions of it) has definable Skolem functions and so for any $i$ and $A \subseteq \lambda$, the Skolem hull of $A$ in $M_i$ (denoted by $\mathrm{Sk}_{M_i}(A)$) is well–defined.

Let $\langle (F_i, A_i) : i < (2^\chi)^+\rangle$ be a play of the game in which Player I uses this strategy (with $M_i$ the model corresponding to $F_i$). For each $i$, define

$$\alpha_i = \min\{\alpha : |\mathrm{Sk}_{M_0}(A_i) \cap \beth_\alpha(\chi)| > \chi\}. \tag{1.5}$$

By the choice of $M_0$ and $M_i$, clearly $\alpha(i)$ is a successor ordinal or a limit ordinal of cofinality $\chi^+$, and

$$|\operatorname{Sk}_{M_0}(A_i) \cap \beth_{\alpha_i}(\chi)| \leq 2^\chi. \tag{1.6}$$

Since $A_i \subseteq A_j$ for $i > j$, we know the sequence $\langle \alpha_i : i < (2^\chi)^+ \rangle$ is non–decreasing. Furthermore, for each $i$ we know

$$\alpha_i < \min\{\beta : \lambda \leq \beth_\beta(\chi)\} < (2^\chi)^+. \tag{1.7}$$

This means that the sequence $\langle \alpha_i : i < (2^\chi)^+ \rangle$ is eventually constant, say with value $\alpha^*$. Let $i^*$ be the least ordinal $< (2^\chi)^+$ such that $\alpha_i = \alpha^*$ for $i \geq i^*$.

**Proposition 1.5.** If $i^* \leq i < (2^\chi)^+$, then $\operatorname{Sk}_{M_0}(A_{i+1}) \cap \beth_{\alpha^*}(\chi)$ is a proper subset of $\operatorname{Sk}_{M_0}(A_i) \cap \beth_{\alpha^*}(\chi)$.

*Proof.* Note that $i^*$, $\alpha^*$, and $\beth_{\alpha^*}(\chi)$ are all elements of $M_{i+1}$ as they are definable in $\langle H(\lambda^+), \in, <_{\lambda^+} \rangle$ from the parameters $M_0$ and $\langle A_j : j \leq i \rangle$. Furthermore,

$$\gamma^* := \min\{\gamma < \lambda : |\operatorname{Sk}_{M_0}(A_i) \cap \gamma| = \chi\} \tag{1.8}$$

is also definable in $M_{i+1}$ (and $< (2^\chi)^+$). Thus the language of $M_{i+1}$ includes a bijection between $\operatorname{Sk}_{M_0}(A_i) \cap \gamma^*$ and $\chi$.

If Player I has not won the game at this stage, after Player I selects $A_{i+1}$ we will be able to find an ordinal $\beta < \chi$ such that $\beta \notin \operatorname{ran}(F_{i+1} \restriction [A_{i+1}]^{<\omega})$. By definition of $h$, we know $\beta' := h^{-1}(\beta)$ is an element of $\operatorname{Sk}_{M_0}(A_i) \cap \beth_{\alpha^*}(\chi)$. However, $\beta'$ is not an element of $\operatorname{Sk}_{M_{i+1}}(A_{i+1})$ – since $F_{i+1}$ codes the Skolem functions of $M_{i+1}$, the range of $F_{i+1} \restriction [A_{i+1}]^{<\omega}$ is $\operatorname{Sk}_{M_{i+1}}(A_{i+1}) \cap \chi$. Since $\operatorname{Sk}_{M_{i+1}}(A_{i+1})$ is closed under $h$, this contradicts our choice of $\beta$. Since $\operatorname{Sk}_{M_0}(A_{i+1}) \subseteq \operatorname{Sk}_{M_{i+1}}(A_{i+1})$, we have established the proposition. □

Note that the preceding proposition finishes the proof of the claim — if play of the game continues for all $(2^\chi)^+$ steps, then $\langle \operatorname{Sk}_{M_0}(A_i) \cap \beth_{\alpha^*}(\chi) : i < (2^\chi)^+ \rangle$ is a strictly decreasing family of subsets of $\operatorname{Sk}_{M_0}(A_{i^*})$, contradicting (1.6). □

## 2. Club–guessing technology

In this section, we prove that if $\lambda = \mu^+$, where $\mu$ is singular, then under certain circumstances we can find a complicated "library" of colorings of smaller cardinals. In the next section, we will use this library of colorings to get a complicated coloring of $\lambda$.

The basics of club–guessing are explained in [4], but we will take a few minutes to recall some of the definitions.

Let us recall that if $S$ is a stationary subset of $\lambda$, then an $S$–club system is a sequence $\bar{C} = \langle C_\delta : \delta \in S\rangle$ such that for (limit) $\delta \in S$, $C_\delta$ is closed unbounded in $\delta$.

In this section, we will be concerned with the case where $\lambda$ is the successor of a singular cardinal, i.e., $\lambda = \mu^+$ where $\mathrm{cf}(\mu) < \mu$. In this context, if $\bar{C}$ is an $S$–club system, then for $\delta \in S$ we define an ideal $J^{b[\mu]}_\delta$ on $C_\delta$ by $A \in J^{b[\mu]}_\delta$ if and only if $A \subseteq C_\delta$, and for some $\theta < \mu$ and $\gamma < \delta$,

$$\beta \in A \cap \mathrm{nacc}(C_\delta) \Rightarrow [\beta < \gamma \text{ or } \mathrm{cf}(\beta) < \theta]\,.$$

Note that it is a bit easier to understand the definition of $J^{b[\mu]}_\delta$ by looking at the contrapositive — a subset $A$ of $C_\delta$ is "large", i.e., not in $J^{b[\mu]}_\delta$, if and only if $A \cap \mathrm{nacc}(C_\delta)$ is cofinal in $\delta$, and the cofinalities of members of any end segment of $A \cap \mathrm{nacc}(C_\delta)$ are unbounded below $\mu$.

**Claim 2.1.** Let $\lambda = \mu^+$, where $\mu$ is a singular cardinal of cofinality $\kappa < \mu$. Let $S \subseteq \lambda$ be stationary, and assume that $\sup\{\mathrm{cf}(\delta) : \delta \in S\} = \mu^* < \mu$. Let $\bar{C}$ be an $S$–club system, and for each $\delta \in S$, let $J_\delta$ be the ideal $J^{b[\mu]}_\delta$. Let $\langle \kappa_i : i < \kappa\rangle$ be a non–decreasing sequence of cardinals such that

$$\kappa^* = \sum_{i<\kappa} \kappa_i \leq \mu, \tag{2.1}$$

and let $\gamma^* < \mu$.

Assume we are given a $\lambda$–club system $\bar{e}$ and a sequence of ideals $\bar{I} = \langle I_\alpha : \alpha < \lambda\rangle$ such that

(1) $I_\alpha$ is an ideal on $e_\alpha$ extending $J^{\mathrm{bd}}_{e_\alpha}$

(2) if $\delta \in S$, then for each $i < \kappa$,

$$\{\alpha \in \mathrm{nacc}(C_\delta) : \text{ Player I wins } \mathrm{Gm}^\omega_{I_\alpha}[e_\alpha, \kappa_i, \gamma^*]\} = \mathrm{nacc}(C_\delta) \mod J_\delta$$

(3) for any club $E \subseteq \lambda$, for stationarily many $\delta \in S$,

$$\{\alpha \in \mathrm{nacc}(C_\delta) : B_0[E, e_\alpha] \notin I_\alpha\} \notin J_\delta,$$

where

$$B_0[E, e^*_\alpha] = \{\beta \in \mathrm{nacc}(e_\alpha) : E \text{ meets the interval } (\sup(\beta \cap e_\alpha), \beta)\}.$$

Then there is a function $h : \lambda \to (\kappa + 1)$ and a sequence

$$\bar{F} = \langle F_\delta : \delta < \lambda,\ \delta \text{ a limit } \rangle$$

such that

$\circledast_1$ $F_\delta : [e_\delta]^{<\omega} \longrightarrow \kappa_{h(\delta)}$ (where $\kappa^* := \kappa_\kappa$ )

and

$\circledast_2$ for every club $E \subseteq \lambda$, for each $i < \kappa$ there are stationarily many $\delta \in S$ such that the set of $\beta \in \mathrm{nacc}(C_\delta)$ satisfying the following

- $h(\beta) \geq i$
- $B_0[E, e_\beta] \notin I_\beta$
- for all $\gamma < \beta$, $\kappa_{h(\beta)} \subseteq \mathrm{ran}(F_\beta \restriction \left[B_0[E, e_\beta] \setminus \gamma\right]^{<\omega})$

is *not* in $J_\delta$.

Now admittedly the previous claim is quite a lot to digest, so we will take a little time to illuminate the basic situation we have in mind.

**Claim 2.2.** The assumptions of Claim 2.1 are satisfied if

(1) $\lambda = \mu^+$ where $\kappa = \mathrm{cf}(\mu) < \mu$
(2) $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$
(3) $\delta \in S \rightarrow |\delta| = \mu$ (i.e., $S \subseteq \lambda \setminus \mu$)
(4) $\bar{C}$ is an $S$–club system
(5) $\bar{J} = \langle J_\delta : \delta \in S \rangle$ where $J_\delta = J^{b[\mu]}_{C_\delta}$
(6) $\mathrm{id}_p(\bar{C}, \bar{J})$ is a proper ideal
(7) $\langle \kappa_i : i < \kappa \rangle$ is a non–decreasing sequence of cardinals with supremum $\kappa^* \leq \mu$
(8) $\gamma^* < \mu$, and for each $i < \kappa$, Player I wins the game $\mathrm{Gm}^\omega[\theta, \kappa_i, \gamma^*]$ for all large enough regular $\theta < \mu$
(9) $\bar{e}$ is a $\lambda$–club system such that $|e_\beta| < \mu$
(10) for $\alpha < \lambda$, $I_\alpha = J^{\mathrm{bd}}_{e_\alpha}$

*Proof of Claim 2.2.* We need only check items (2) and (3) in the statement of Claim 2.1 — everything else is trivially satisfied. Concerning (2), given $\delta \in S$ and $i < \kappa$, we need to show

$$\{\alpha \in \mathrm{nacc}(C_\delta) : \text{ Player I wins } \mathrm{Gm}^\omega[e_\alpha, \kappa_i, \gamma^*]\} = \mathrm{nacc}(C_\delta) \mod J_\delta.$$

Let $A$ consist of those $\alpha \in \mathrm{nacc}(C_\delta)$ for which Player I does not win the game $\mathrm{Gm}^\omega[e_\alpha, \kappa_i, \gamma^*]$. By our assumptions, there is a $\theta < \mu$ such that $|e_\alpha| < \theta$ for all $\alpha \in A$, and therefore $A$ is in the ideal $J^{b[\mu]}_{C_\delta} = J_\delta$ and we have what we need.

Concerning (3), given $E \subseteq \lambda$ club, we must find stationarily many $\delta \in S$ such that

$$\{\alpha \in \mathrm{nacc}(C_\delta) : B_0[E, e_\alpha] \notin I_\alpha\} \notin J_\delta.$$

Let $E' = \{\xi \in E : \mathrm{otp}(E \cap \xi) = \xi \text{and } \mu \text{ divides } \xi\}$. Clearly $E'$ is a closed unbounded subset of $E$, and since $\mathrm{id}_p(\bar{C}, \bar{J})$ is a proper ideal, the set

$$S^* := \{\delta \in S \cap E' : E' \cap \mathrm{nacc}(C_\delta) \notin J_\delta\}$$

is stationary.

Fix $\delta \in S^*$, and suppose we are given $\theta < \mu$ and $\xi < \delta$. Since $E' \cap \mathrm{nacc}(C_\delta) \notin J_\delta$, we can find $\alpha \in E' \cap \mathrm{nacc}(C_\delta)$ such that $\alpha > \max\{\xi, \mu\}$ and $\mathrm{cf}(\alpha) > \theta$. Since the order–type of $E \cap \alpha$ is $\alpha \geq \mu > |e_\alpha|$, we know that $B_0[E, e_\alpha]$ is unbounded in $e_\alpha$ hence a member of $I_\alpha$. This shows that the set of such $\alpha$ is in $J^+_\delta$, as required. □

Now we return to the proof of Claim 2.1.

*Proof of Claim 2.1.* Let $\sigma = \mathrm{cf}(\sigma)$ be a regular cardinal $< \mu$ that is greater than $\mu^*$ and $\gamma^*$. For each limit $\beta < \lambda$, if there is an $i \leq \kappa$ such that Player I wins the version of $\mathrm{Gm}^\omega_{I_\beta}[e_\beta, \kappa_i, \sigma^+]$ where we allow Player II to pass, then we let $h(\beta)$ be the maximal such $i$ — note that $i$ exists by (5) of Claim 1.3 — and let $\mathrm{Str}_\beta$ be a strategy that witnesses this.

Note that since $\gamma^* < \sigma^+$ and $J_\delta = J^{b[\mu]}_\delta$ for $\delta \in S$, we have that for $\delta \in S$ and $i < \kappa$ that

$$\{\beta \in \mathrm{nacc}(C_\delta) : \mathrm{Str}_\beta \text{ is defined and } i \leq h(\beta)\} = \mathrm{nacc}(C_\delta) \mod J_\delta.$$

We will make $\sigma^+$ attempts to build $\bar{F}$ witnessing the conclusion. In stage $\zeta < \sigma^+$, we assume that our prior work has furnished us with a decreasing sequence $\langle E_\xi : \xi < \zeta\rangle$ of clubs in $\lambda$, and, for each $\beta < \lambda$ where $\mathrm{Str}_\beta$ is defined, an initial segment $\langle F^\xi_\beta, A^\xi_\beta : \xi < \zeta\rangle$ of a play of $\mathrm{Gm}^\omega_{I_\beta}[e_\beta, \kappa^*_{h(\beta)}, \sigma^+]$ in which Player I uses $\mathrm{Str}_\beta$. (Note that our convention is that if Player II chooses to pass at a stage, we let $A^\xi_\beta$ be undefined.)

For each such $\beta$, let $F^\zeta_\beta : [e_\beta]^{<\omega} \to \kappa_{h(\beta)}$ be given by $\mathrm{Str}_\beta$, and for those $\beta$ for which $\mathrm{Str}_\beta$ is undefined, we let $F^\beta_\zeta$ be any such function. Now if $\langle F^\zeta_\beta : \beta < \lambda\rangle := \bar{F}^\zeta$ is as required then we are done. Otherwise, there is a club $E' \subseteq \lambda$ and $i_\zeta < \kappa$ exemplifying the failure of $\bar{F}^\zeta$, and without loss of generality,

$$(\forall \delta \in S)\big[B_{i_\zeta}[E'_\zeta, C_\delta, \bar{I}, \bar{e}, \bar{F}^\zeta]\big] \in J_\delta. \tag{2.2}$$

Now let $E_\zeta = \mathrm{acc}(E'_\zeta \cap \bigcap_{\xi<\zeta} E_\xi)$. For each $\beta$ where $\mathrm{Str}_\beta$ is defined, we let Player II respond to $F^\zeta_\beta$ by playing the set $B_0[E_\zeta, e_\beta]$ if it is a legal move, otherwise we let him pass. We then proceed to stage $\zeta + 1$.

Assuming that this construction continues for all $\sigma^+$ stages, we will arrive at a contradiction. Let $E = \bigcap_{\zeta<\sigma^+} E_\zeta$. By assumption (3) there is a $\delta(*) \in S$ for which

$$A_1 := \{\beta \in \mathrm{nacc}(C_{\delta(*)}) : B_0[E, e_\beta] \notin I_\beta\} \notin J_{\delta(*)}.$$

By assumption (2), we have

$$A_2 := \{\beta \in A_1 : \mathrm{Str}_\beta \text{ is defined }\} \notin J_{\delta(*)}.$$

For $\beta \in A_2$, look at the play $\langle F^\zeta_\beta, A^\zeta_\beta : \zeta < \sigma^+\rangle$. Since Player I wins, there is a $\zeta_\beta < \sigma^+$ such that Player II passed at stage $\zeta$ for all $\zeta \geq \zeta_\beta$. Since $\sigma > \mu^*$ and $J_{\delta(*)}$ is $\mu^*$–based, for some $\zeta^* < \sigma^+$,

$$A_3 = \{\beta \in A_1 : \mathrm{Str}_\beta \text{ is defined and } \zeta_\beta \leq \zeta^*\} \notin J_{\delta(*)}.$$

Now $E_{\zeta^*}$ was defined so that for some $i_{\zeta^*}$, for all $\delta \in S$,

$$B_{i_{\zeta^*}}[E_{\zeta^*}, C_\delta, \bar{I}, \bar{e}, \bar{F}^{\zeta^*}] \in J_\delta, \tag{2.3}$$

but (again by assumption (2))

$$A_4 = \{\beta \in A_1 : \mathrm{Str}_\beta \text{ is defined, } \zeta_\beta \leq \zeta^*, \text{ and } i_{\zeta^*} \leq h(\beta)\} \notin J_{\delta(*)}.$$

For $\beta \in A_4$, we know that at stage $\zeta^*$ of our play of $\mathrm{Gm}^\omega_{I_\beta}[e_\beta, \kappa_{h(\beta)}, \sigma^+]$ the set $B_0[E_{\zeta^*}, e_\beta]$ was not a legal move. Since our sequence of clubs is decreasing, we know that $B_0[E_{\zeta^*}, e_\beta]$ is a subset of $B_0[E_\xi, e_\beta]$ for all $\xi < \zeta^*]$, so we have

$$B_0[E_{\zeta^*}, e_\beta] \subseteq \bigcap_{\xi < \zeta^*} A^\xi_\beta.$$

Since $\beta \in A_1$, we know that $B_0[E_{\zeta^*}, e_\beta] \notin I_\beta$. Thus the reason for $B_0[E_{\zeta^*}, e_\beta]$ being an illegal move must be that for all $\gamma < \beta$,

$$\kappa^*_{h(\beta)} \subseteq \mathrm{ran}(F^{\zeta^*}_\beta \restriction [B_0[E_{\zeta^*}, e_\beta] \setminus \gamma]^{<\omega}).$$

All of these facts combine to tells us that $\beta \in B_{i_{\zeta^*}}[E_{\zeta^*}, C_\delta, \bar{I}, \bar{e}, \bar{F}^{\zeta^*}]$, and thus

$$A_4 \subseteq B_{i_{\zeta^*}}[E_{\zeta^*}, C_\delta, \bar{I}, \bar{e}^*, \bar{F}^{\zeta^*}] \notin J_{\delta(*)},$$

contradicting (2.3). □

The proofs in this section (and the next) can be considerably simplified if we are willing to restrict ourselves to the case $\kappa^* < \mu$, as we can dispense with the sequence $\langle \kappa_i : i < \kappa\rangle$.

## 3. Building the Coloring

We now come to the main point of this paper; we dedicate this section and the next to proving the following theorem.

**Theorem 1.** *Assume $\lambda = \mu^+$, where $\mu$ is a singular cardinal of uncountable cofinality, say $\aleph_0 < \kappa = \mathrm{cf}(\mu) < \mu$. Assume $\langle \kappa_i : i < \kappa \rangle$ is non–decreasing with supremum $\kappa^* \leq \mu$, and there is a $\gamma^* < \mu$ such that for each $i$, for every large enough regular $\theta < \mu$, Player I has a winning strategy in the game $\mathrm{Gm}^\omega[\theta, \kappa_i, \gamma^*]$. Then $\mathrm{Pr}_1(\lambda, \lambda, \kappa^*, \kappa)$ holds.*

Let $\langle S_i : i < \kappa \rangle$ be a sequence of pairwise disjoint stationary subsets of $\{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$. For $i < \kappa$, let $\bar{C}^i$ be an $S_i$–club system such that

- $\lambda \notin \mathrm{id}_p(\bar{C}^i, \bar{J}^i)$, where $\bar{J}^i = \langle J^{b[\mu]}_{C^i_\delta} : \delta \in S_i \rangle$
- for $\delta \in S_i$, $\mathrm{otp}(C^i_\delta) = \mathrm{cf}(\delta) = \kappa = \mathrm{cf}(\mu)$

Such ladder systems can be found by Claim 2.6 (and Remark 2.6A (6)) of [2] — for the second statement to hold, we need that $\mu$ has uncountable cofinality.

**Claim 3.1.** There is a $\lambda$–club system $\bar{e}$ such that $|e_\beta| \leq \mathrm{cf}(\beta) + \mathrm{cf}(\mu)$, and $\bar{e}$ "swallows" each $\bar{C}^i$, i.e., if $\delta \in S_i \cap (e_\beta \cup \{\beta\})$, then $C^i_\delta \subseteq e_\beta$.

*Proof.* Let $S = \cup_{i<\kappa} S_i$, and let $\beta < \lambda$ be a limit ordinal. Let $e^0_\beta$ be a closed cofinal subset of $\beta$ of order–type $\mathrm{cf}(\beta)$. We will construct the required ladder $e_\beta$ in $\omega$–stages, with $e^n_\beta$ denoting the result of the first $n$ stages of our procedure. The construction is straightforward, but it is worthwhile to note that we need to use the fact that each member of $S$ has uncountable cofinality.

Given $e^n_\beta$, let us define

$$B_n = S \cap (e^n_\beta \cup \{\beta\}). \tag{3.1}$$

Now we let $e^{n+1}_\beta$ be the closure in $\beta$ of

$$e^n_\beta \cup \bigcup \{C_\delta : \delta \in B_n\}. \tag{3.2}$$

Note that $|e^{n+1}| \leq \mathrm{cf}(\mu) + \mathrm{cf}(\beta)$ as $|C_\delta| = \mathrm{cf}(\mu) = \kappa$ for each $\delta \in S$. Finally, we let $e_\beta$ be the closure of $\cup_{n<\omega} e^n_\beta$ in $\beta$.

Clearly $|e_\beta| \leq \mathrm{cf}(\mu) + \mathrm{cf}(\beta)$. Also, since each element of $S$ has uncountable cofinality, if $\delta \in S \cap e_\beta$, then there is an $n$ such that $\delta \in e^n_\beta$, and therefore

$$C_\delta \subseteq e^{n+1}_\beta \subseteq e_\beta, \tag{3.3}$$

as required. ☐

For each $i < \kappa$, there are $h_i$ and $\bar{F}^i = \langle F^i_\delta : \delta < \lambda,\ \delta \text{ limit } \rangle$ as in the conclusion of Claim 2.1 applied to $\bar{C}^i$ and $\bar{e}$; note that we satisfy the assumptions of Claim 2.1 by way of Claim 2.2.

Let $\langle \lambda_i : i < \kappa \rangle$ be a strictly increasing sequence of regular cardinals $> \kappa$ and cofinal in $\mu$ such that

$$\lambda = \mathrm{tcf}\big(\prod_{i<\kappa} \lambda_i / J^{\mathrm{bd}}_\kappa\big), \tag{3.4}$$

and let $\langle f_\alpha : \alpha < \lambda \rangle$ exemplify this. Finally, let $h^*_0 : \kappa \to \omega$ and $h^*_1 : \kappa \to \kappa$ be such that

$$(\forall n)(\forall i < \kappa)(\exists^\kappa j < \kappa)[h^*_0(j) = n \text{ and } h^*_1(j) = i]. \tag{3.5}$$

Before we can define our coloring, we must recall some of the terminology of [2].

**Definition 3.2.** Let $0 < \alpha < \beta < \lambda$, and define

$$\gamma(\alpha, \beta) = \min\{\gamma \in e_\beta : \gamma \geq \alpha\}.$$

We also define (by induction on $\ell$)

$$\gamma_0(\alpha, \beta) = \beta,$$

$$\gamma_{\ell+1}(\alpha, \beta) = \gamma(\alpha, \gamma_\ell(\alpha, \beta)) \text{ (if defined)}.$$

We let $k(\alpha, \beta)$ be the first $\ell$ for which $\gamma_\ell(\alpha, \beta) = \alpha$. The sequence $\langle \gamma_i(\alpha, \beta) : i \leq k(\alpha, \beta) \rangle$ will be referred to as the *walk from $\beta$ to $\alpha$ along the ladder system $\bar{e}$.*

We now define the coloring $c$ that will witness $\mathrm{Pr}_1(\lambda, \lambda, \kappa^*, \kappa)$. Recall that $c$ must be a symmetric two–place function from $\lambda$ to $\kappa^*$.

Given $\alpha < \beta$, we let $i = i(\alpha, \beta)$ be the maximal $j < \kappa$ such that $f_\beta(j) < f_\alpha(j)$ (if such an $j$ exists). Next, we walk from $\beta$ down to $\alpha$ along $\bar{e}$ until we reach an ordinal $\nu(\alpha, \beta)$ such that

$$f_\alpha(i) < f_{\nu(\alpha,\beta)}(i),$$

(again, if such an ordinal exists.) After this, we walk along $\bar{e}$ from $\alpha$ toward the ordinal $\max(\alpha \cap e_{\nu(\alpha,\beta)})$ until we reach an ordinal $\eta(\alpha, \beta)$ for which

$$f_{\nu(\alpha,\beta)}(i) < f_{\eta(\alpha,\beta)}(i).$$

The idea now is to look at how the ladders $e_{\nu(\alpha,\beta)}$ and $e_{\eta(\alpha,\beta)}$ intertwine. Let us make a temporary definition by calling an ordinal $\xi \in e_{\nu(\alpha,\beta)}$ *relevant* if $e_{\eta(\alpha,\beta)}$ meets the interval $(\sup(\xi \cap e_{\nu(\alpha,\beta)}), \xi)$.

If it makes sense, we let $w(\alpha, \beta) \subseteq e_{\nu(\alpha,\beta)}$ be the last $h^*_0(i(\alpha, \beta))$ relevant ordinals in $e_{\nu(\alpha,\beta)}$ (so we need that the relevant ordinals have order–type $\gamma + h^*_0(i(\alpha, \beta))$ for some $\gamma$).

Finally, we define our coloring by

$$c(\alpha,\beta) = F^{h_1^*(i(\alpha,\beta))}_{\nu(\alpha,\beta)}(w(\alpha,\beta)). \tag{3.6}$$

If the attempt to define $c(\alpha,\beta)$ breaks down at some point for some specific $\alpha<\beta$, then we set $c(\alpha,\beta)=0$.

We now prove that this coloring works, so suppose $\langle t_\alpha : \alpha<\lambda\rangle$ are pairwise disjoint subsets of $\lambda$ such that $|t_\alpha| = \theta_1 < \kappa$ and $j^* < \kappa^*$, and without loss of generality $\alpha < \min t_\alpha$ and $\theta_1 \geq \omega$. We need to find $\delta_0$ and $\delta_1$ such that

$$\alpha \in t_{\delta_0} \text{ and } \beta \in t_{\delta_1} \Rightarrow \alpha < \beta \text{ and } c(\alpha,\beta) = j^*. \tag{3.7}$$

Let $j_1$ be the least $j$ such that $j^* < \kappa_j$, and let $S$, $\bar{C}$, and $\bar{F}$ denote $S_{j_1}$, $\bar{C}^{j_1}$, and $\bar{F}^{j_1}$ respectively.

Given $\delta<\lambda$, we define the *envelope of* $t_\delta$ (denoted $\operatorname{env}(t_\delta)$) by the formula

$$\operatorname{env}(t_\delta) = \bigcup_{\zeta\in t_\delta}\{\gamma_\ell(\delta,\zeta) : \ell \leq k(\delta,\zeta)\}. \tag{3.8}$$

The envelope of $t_\delta$ is the set of all ordinals obtained by walking down to $\delta$ from some $\zeta \in t_\delta$ using the ladder system $\bar{e}$. This makes sense as we have arranged that $\delta < \min t_\delta$. Note also that $|\operatorname{env}(t_\delta)| \leq |t_\delta| = \theta_1$.

Next we define functions $g^{\min}_\delta$ and $g^{\max}_\delta$ in $\prod_{i<\kappa}\lambda_i$ by

$$g^{\min}_\delta(i) = \min\{f_\gamma(i) : \gamma \in \operatorname{env}(t_\delta)\}, \tag{3.9}$$

and

$$g^{\max}_\delta(i) = \sup\{f_\gamma(i)+1 : \gamma \in \operatorname{env}(t_\delta)\}. \tag{3.10}$$

Note that $g^{\max}_\delta$ is well–defined as we assume that $\kappa < \min\{\lambda_i : i<\kappa\}$.

The following claim is quite easy, and the proof is left to the reader.

**Claim 3.3.**

1. $f_\delta =_{J^{\text{bd}}_\kappa} g^{\min}_\delta$
2. $g^{\min}_\delta(i) \leq g^{\max}_\delta(i)$ for all $i<\kappa$
3. There is a $\delta' > \delta$ such that $g^{\max}_\delta \leq_{J^{\text{bd}}_\kappa} g^{\min}_{\delta'}$.

Now let $\chi^* = (2^\lambda)^+$, and let $\langle M_\alpha : \alpha<\lambda\rangle$ be a sequence of elementary submodels of $\langle H(\chi^*), \in, <^*_{\chi^*}\rangle$ that is increasing and continuous in $\alpha$ and such that each $M_\alpha \cap \lambda$ is an ordinal, $\langle M_\beta : \beta \leq \alpha\rangle \in M_{\alpha+1}$, and $\langle f_\alpha : \alpha < \lambda\rangle$, $g$, $c$, $\bar{e}$, $S$, $\bar{C}$, $\langle t_\alpha : \alpha<\lambda\rangle$ all belong to $M_0$. Note that $\mu+1 \subseteq M_0$.

The set $E = \{\alpha<\lambda : M_\alpha\cap\lambda = \alpha\}$ is closed unbounded in $\lambda$, and furthermore,

$$\alpha < \delta \in E \Rightarrow \sup t_\alpha < \delta. \tag{3.11}$$

By the choice of $\bar{C}$ and $\bar{F}$, for some $\delta \in S \cap E$ we have the set
(3.12)
$$A = \{\beta \in \mathrm{nacc}(C_\delta) : (\forall \gamma < \beta)\, \mathrm{ran}(F_\beta \restriction \left[B_0[E, e_\beta] \setminus \gamma\right]^{<\omega}) \supseteq \kappa_{j_1}\}$$
is not in $J^{b[\mu]}_{C_\delta}$.

Note that $A \subseteq \mathrm{acc}(E)$, as $B_0[E, e_\beta]$ is unbounded in $\beta$ for $\beta \in A$. For $\beta \in t_\delta$, if $\ell < k(\delta, \beta)$ then $e_{\gamma_\ell(\delta,\beta)} \cap \delta$ is bounded in $\delta$, and since it is closed it has a well–defined maximum. Since $|t_\delta| < \kappa = \mathrm{cf}(\delta)$, this means the ordinal
$$\gamma^\otimes := \sup\{\max[e_{\gamma_\ell(\delta,\beta)} \cap \delta] : \beta \in t_\delta \text{ and } \ell < k(\delta, \beta)\}$$
is strictly less than $\delta$.

For $\beta \in t_\delta$, let us define
$$A_\beta := \{\beta' \in A : (\exists \ell \leq k(\beta, \delta))[\mathrm{cf}(\beta') \leq |e_{\gamma_\ell(\delta,\beta)}|]\}. \tag{3.13}$$
Since the cardinality of each ladder in $\bar{e}$ is less than $\mu$, each set $A_\beta$ is an element of $J^{b[\mu]}_{C_\delta}$. The ideal $J^{b[\mu]}_{C_\delta}$ is $\kappa$–complete, so the fact that $|t_\delta| < \kappa$ and $k(\beta, \delta)$ is finite for each $\beta \in t_\delta$ together imply that
$$\bigcup_{\beta \in t_\delta} A_\beta \in J^{b[\mu]}_{C_\delta}. \tag{3.14}$$
By the definition of $A$ and our choice of $\delta$, this means it is possible to choose $\beta^* \in A \setminus (\gamma^\otimes + 1)$ that is not in any $A_\beta$, i.e.,
$$\beta \in t_\delta \text{ and } \ell < k(\delta, \beta) \implies \mathrm{cf}(\beta^*) > |e_{\gamma_\ell(\delta,\beta)}|. \tag{3.15}$$

**Claim 3.4.**

1. If $\epsilon \in t_\delta$, and $\ell = k(\delta, \epsilon) - 1$, then $\beta^* \in \mathrm{nacc}(e_{\gamma_\ell(\delta,\epsilon)})$.
2. If $\epsilon \in t_\delta$ and $\gamma^\otimes < \gamma' \leq \beta^*$, then
   - $\gamma_\ell(\delta, \epsilon) = \gamma_\ell(\gamma', \epsilon)$ for $\ell < k(\delta, \epsilon)$, and
   - $\gamma_{k(\delta,\epsilon)}(\gamma', \epsilon) = \beta^*$

*Proof.* For the first clause, note that $\delta$ is an element of $e_{\gamma_\ell(\delta,\epsilon)}$ and hence by our choice of $\bar{e}$, $C_\delta \subseteq e_{\gamma_\ell(\delta,\epsilon)}$. Thus $\beta^* \in e_{\gamma_\ell(\delta,\epsilon)}$, and since $\mathrm{cf}(\beta^*) > |e_{\gamma_\ell(\delta,\epsilon)}|$, we know that $\beta^*$ cannot be an accumulation point of $e_{\gamma_\ell(\delta,\epsilon)}$.

The first part of the second statement follows because of the definition of $\gamma^\otimes$. As far as the second part of the second statement goes, it is best visualized as follows:

We walk down the ladder system $\bar{e}$ from $\epsilon$ to $\gamma'$, we eventually hit a ladder that contains $\delta$ — this happens at stage $k(\delta, \epsilon) - 1$. Since $C_\delta$ is a subset of this ladder, the next step in our walk from $\epsilon$ to $\gamma'$ must be down to $\beta^*$ because $\gamma^\otimes < \gamma' < \beta^*$. □

We can visualize the preceding claim in the following manner: $\beta^*$ is chosen so that for all sufficiently large $\gamma' < \beta^*$, all the walks from some element of $t_\delta$ to $\gamma'$ are funnelled through $\beta^*$ — $\beta^*$ acts as a bottleneck. This will be key when want to prove that our coloring works.

Since $\beta^* \in A$, we can choose a finite increasing sequence $\xi_0 < \xi_1 < \cdots < \xi_n$ of ordinals in $\mathrm{acc}(E) \cap \mathrm{nacc}(e_{\beta^*}) \setminus (\gamma^\otimes + 1)$ such that $F^{j_1}_{\beta^*}(\{\xi_0, \ldots, \xi_n\}) = j^*$, the color we are aiming for.

For each $\ell \leq n$, we can find $\zeta_\ell \in E \setminus (\gamma^\otimes + 1)$ such that

$$\sup(e_{\beta^*} \cap \xi_\ell) < \zeta_\ell < \xi_\ell.$$

Now we let $\phi(x_0, y_0, x_1, y_1, \ldots, x_n, y_n, z_0, z_1)$ be the formula (with parameters $\gamma^\otimes$, $\bar{f}$, $\langle \lambda_i : i < \kappa \rangle$, $\bar{C}$, $\bar{e}$, $\langle t_\alpha : \alpha < \lambda \rangle$, $h$, $h_0$, $j^*$) that describes our current situation with $x_\ell$, $y_\ell$ standing for $\zeta_\ell$, $\xi_\ell$, and $z_0$, $z_1$ standing for $\beta^*$, $\delta$ , i.e., $\phi$ states

- $\gamma^\otimes < x_0 < y_0 < \cdots < x_n < y_n < z_0 < z_1$ are ordinals $< \lambda$
- $z_1 \in S$ and $z_0 \in \mathrm{nacc}(C_{z_1})$
- $\gamma^\otimes = \sup\{\max[e_{\gamma_\ell(z_1,\zeta)} \cap z_1] : \ell < k(z_1, \zeta) \text{ and } \zeta \in t_{z_1}\}$
- $z_0 \in \mathrm{nacc}(e_{\gamma_{k(z_1,\epsilon)}(z_1,\epsilon)})$ for all $\epsilon \in t_{z_1}$
- $F^{j_1}_{z_0}(\{y_0, \ldots, y_n\}) = j^*$

Now clearly we have

$$H(\chi) \models \phi[\zeta_0, \xi_0, \ldots, \zeta_n, \xi_n, \beta^*, \delta]. \tag{3.16}$$

Recall that all the parameters needed in $\phi$ are in $M_0$, except possibly for $\gamma^\otimes$, so the model $M_{\gamma^\otimes+1}$ contains all the parameters we need. Also, $\{\zeta_0, \xi_0, \ldots, \zeta_n, \xi_n\} \in M_{\beta^*}$, $\beta^* \in M_\delta \setminus M_{\beta^*}$, and since $\delta \in \lambda \setminus M_\delta$, we have (recalling that $\exists^* z < \lambda$ means "for unboundedly many $z < \lambda$)

$$M_\delta \models (\exists^* z_1 < \lambda)\phi(\zeta_0, \xi_0, \ldots, \zeta_n, \xi_n, \beta^*, z_1). \tag{3.17}$$

Therefore, this formula is true in $H(\chi)$ because of elementarity. Similarly, we have

$$H(\chi) \models (\exists^* z_0 < \lambda)(\exists^* z_1 < \lambda)\phi(\zeta_0, \xi_0, \ldots, \zeta_n, \xi_n, z_0, z_1).$$

Now each of the intervals $[\gamma^\otimes + 1, \zeta_0)$, $[\zeta_0, \xi_0)$, $\ldots$, contains a member of $E$, so (by the definition of $E$) similar considerations give us

$$H(\chi) \models (\exists^* x_0 < \lambda) \ldots (\exists^* y_n < \lambda)(\exists^* z_0 < \lambda)(\exists^* z_1 < \lambda)\phi(x_0, y_0, \ldots, z_0, z_1).$$

Now we can choose (in order)

$$\zeta^a_0 < \zeta^b_0 < \xi^a_0 < \zeta^a_1 < \xi^b_0 < \zeta^b_1 < \cdots < \zeta^a_n < \xi^b_{n-1} < \zeta^b_n < \xi^a_n \tag{3.18}$$

such that

$$(\exists^* z_0 < \lambda)(\exists^* z_1 < \lambda)[\phi(\zeta_0^a, \dots, \xi_{n-1}^a, \zeta_n^a, \xi_n^a, z_0, z_1)], \tag{3.19}$$

and

$$(\exists^* y_n < \lambda)(\exists^* z_0 < \lambda)(\exists^* z_1 < \lambda)[\phi(\zeta_0^b, \dots, \xi_{n-1}^b, \zeta_n^b, y_n, z_0, z_1)], \tag{3.20}$$

Our goal is to show that for all sufficiently large $i < \kappa$, it is possible to choose objects $\beta^a$, $\delta^a$, $\xi_n^b$, $\beta^b$, and $\delta^b$ such that

| |
|---|
| (1) $\zeta_n^b < \beta^a < \delta^a < \min(t_{\delta^a}) \leq \max(t_{\delta^a}) < \xi_n^b < \beta^b < \delta^b$ |
| (2) $\phi(\zeta_0^a, \dots, \xi_n^a, \beta^a, \delta^a)$ |
| (3) $\phi(\zeta_0^b, \dots, \xi_n^b, \beta^b, \delta^b)$ |
| (4) for all $\epsilon \in \mathrm{env}(t_{\delta^a})$, $g_{\delta^a}^{\min} \upharpoonright [i, \kappa) \leq f_\epsilon \upharpoonright [i, \kappa) \leq g_{\delta^a}^{\max} \upharpoonright [i, \kappa)$ |
| (5) for all $\epsilon \in \mathrm{env}(t_{\delta^b})$, $g_{\delta^b}^{\min} \upharpoonright [i, \kappa) \leq f_\epsilon \upharpoonright [i, \kappa) \leq g_{\delta^b}^{\max} \upharpoonright [i, \kappa)$ |
| (6) $g_{\delta^b}^{\max}(i) < g_{\delta^a}^{\min}(i) \leq g_{\delta^a}^{\max}(i) < f_{\beta^b}(i) < f_{\beta^a}(i)$ |
| (7) $g_{\delta^a}^{\max} \upharpoonright [i+1, \kappa) < g_{\delta^b}^{\min} \upharpoonright [i+1, \kappa)$ |

**Table 1**

**Claim 3.5.** If for all sufficiently large $i < \kappa$ it is possible to find objects satisfying the requirements of Table 1, then we can find $\delta^a < \delta^b$ such that $c(\epsilon^a, \epsilon^b) = j^*$ for all $\epsilon^a \in t_{\delta^a}$ and $\epsilon^b \in t_{\delta^b}$.

*Proof.* Let us choose $i^* < \kappa$ such that

- suitable objects (as above) can be found, and
- $h_1^*(i^*) = j_1$ and $h_0^*(i^*) = n$

Choose $\epsilon^a \in t_{\delta^a}$ and $\epsilon^b \in t_{\delta^b}$; we verify that $c(\epsilon^a, \epsilon^b) = j^*$.

*Subclaim* 1. $i(\epsilon^a, \epsilon^b) = i^*$.

*Proof.* Immediate by (4)-(7) in the table. □

*Subclaim* 2. $\nu(\epsilon^a, \epsilon^b) = \beta^b$.

*Proof.* Note that $\gamma^\otimes < \epsilon^a < \beta^b$. Clause (3) of the table implies that the assumptions of Claim 3.4 hold. Thus by Claim 3.4, for $\ell < k(\delta^b, \epsilon^b)$ we have

$$\gamma_\ell(\epsilon^a, \epsilon^b) = \gamma_\ell(\delta^b, \epsilon^b),$$

hence $\gamma_\ell(\epsilon^a,\epsilon^b)\in \operatorname{env}(t_{\delta^b})$ and (by (6) of the table and the definitions involved)

$$f_{\gamma_\ell(\epsilon^a,\epsilon^b)}(i^*)\le g^{\max}_{\delta^b}(i^*)<g^{\min}_{\delta^a}(i^*)\le f_{\epsilon^a}(i^*). \tag{3.21}$$

For $\ell=k(\delta^b,\epsilon^b)$, Claim 3.4 tells us

$$\gamma_\ell(\epsilon^a,\epsilon^b)=\beta^b,$$

and we have arranged that

$$f_{\epsilon^a}(i^*)\le g^{\max}_{\delta^a}(i^*)<f_{\beta^b}(i^*). \tag{3.22}$$

This establishes $\beta^b=\nu(\epsilon^a,\epsilon^b)$. □

*Subclaim* 3. $\eta(\epsilon^a,\epsilon^b)=\beta^a$.

*Proof.* Let $\alpha=\max(e_{\beta^b}\cap\epsilon^a)$. We have arranged that

$$\zeta^b_n<\beta^a<\delta^a<\epsilon^a<\xi^b_n$$

and $\gamma^\otimes<\max(e_{\beta^b}\cap\delta^a)$, hence $\gamma^\otimes<\alpha<\beta^a$. For $\ell<k(\delta^a,\epsilon^a)$, Claim 3.4 implies

$$\gamma_\ell(\alpha,\epsilon^a)=\gamma_\ell(\delta^a,\epsilon^a)\in\operatorname{env}(t_{\delta^a}).$$

By our choice of $i^*$, we have

$$f_{\gamma_\ell(\alpha,\epsilon^a)}(i^*)\le g^{\max}_{\delta^a}(i^*)<f_{\beta^b}(i^*). \tag{3.23}$$

For $\ell=k(\delta^a,\epsilon^a)$, Claim 3.4 implies $\gamma_\ell(\alpha,\epsilon^a)=\beta^a$, and we have ensured

$$f_{\beta^b}(i^*)<f_{\beta^a}(i^*). \tag{3.24}$$

Thus $\beta^a$ is the first ordinal $\eta$ in the walk from $\epsilon^a$ to $\max(e_{\beta^b}\cap\epsilon^a)$ for which $f_\eta(i^*)>f_{\beta^b}(i^*)$, and therefore $\eta(\epsilon^a,\epsilon^b)=\beta^a$. □

*Subclaim* 4. $w(\epsilon^a,\epsilon^b)=\{\xi^b_0,\dots\xi^b_n\}$.

*Proof.* Our previous subclaims imply that an ordinal $\xi\in e_{\beta^b}$ is relevant if and only if the ladder $e_{\beta^a}$ meets the interval $(\sup(e_{\beta^b}\cap\xi),\xi)$. Since $h^*_0(i^*)=n+1$, we know that $w(\epsilon^a,\epsilon^b)$ consists of the last $n+1$ relevant ordinals in $e_{\beta^b}$.

For $i\le n$, clearly $\xi^b_i\in e_{\beta^b}$ and $\sup(\xi^b_i\cap e_{\beta^b})\le\zeta^b_n$. We have made sure that $e_{\beta^a}\cap(\zeta^b_i,\xi^b_i)\ne\emptyset$ (for example, $\xi^a_i$ is an element in this intersection) and so each $\xi^b_i$ is relevant.

Since $\beta^a<\xi^b_n$, it is clear that there are no relevant ordinals larger than $\xi^b_n$.

Given $i<n$, if $\xi\in e_{\beta^b}\cap(\xi^b_i,\xi^b_{i+1})$, then

$$\xi^b_i\le\sup(\xi\cap e_{\beta^b})\le\xi\le\zeta^b_{i+1}.$$

Since $\zeta^a_{i+1} < \xi^b_i < \zeta^b_{i+1} < \xi^a_{i+1}$, it follows that

$$[\sup(\xi \cap e_{\beta^b}), \xi) \subseteq [\zeta^a_{i+1}, \xi^a_{i+1}),$$

and so $\xi$ is not relevant. Thus $\{\xi^b_0, \dots, \xi^b_n\}$ are the last $n+1$ relevant elements of $e_{\beta^b}$, as was required. $\square$

To finish the proof of Claim 3.5, we note that as $h^*_1(i^*) = j^*$, we have

$$c(\epsilon^a, \epsilon^b) = F^{j_1}_{\beta^b}(\{\xi^b_0, \dots, \xi^b_n\}) = j^*. \tag{3.25}$$

$\square$

## 4. Finding the required ordinals

The whole of this section will be occupied with showing that for all sufficiently large $i < \kappa$, it is possible to find objects satisfying the requirements of Table 1.

We begin with some notation intended to simplify the presentation.

- $\phi^a(z_0, z_1)$ abbreviates the formula $\phi(\zeta^a_0, \dots, \xi^a_n, z_0, z_1)$
- $\phi^b(y_n, z_0, z_1)$ abbreviates the formula $\phi(\zeta^b_0, \zeta^n_b, y_n, z_0, z_1)$
- For $i < \kappa$, $\psi(i, z_1)$ abbreviates the formula

$$(\forall \epsilon \in \operatorname{env}(t_{z_1}))[g^{\min}_{z_1} \upharpoonright [i, \kappa) \leq f_\epsilon \upharpoonright [i, \kappa) \leq g^{\max}_{z_1} \upharpoonright [i, \kappa)] \tag{4.1}$$

We have arranged things so that the sentence

$$\begin{aligned} &(\exists^* z^a_0 < \lambda)(\exists^* z^a_1 < \lambda)(\exists^* y^b_n < \lambda) \\ &\qquad (\exists^* z^b_0 < \lambda)(\exists^* z^b_1 < \lambda)[\phi^a(z^a_0, z^a_1) \wedge \phi^b(y^b_n, z^b_0, z^b_1)] \end{aligned} \tag{4.2}$$

holds.

There are far too many alternations of quantifiers in the above formula for most people to deal with comfortably; the best way to view them is as a single quantifier that asserts the existence of a tree of 5–tuples with the property that every node of the tree has $\lambda$ successors, and every branch through the tree gives us five objects satisfying $\phi^a \wedge \phi^b$.

Let $\Phi(i, z^a_0, \dots, z^b_1)$ abbreviate the formula

$$\begin{aligned} &\phi^a(z^a_0, z^a_1) \wedge \phi^b(y^b_n, z^b_0, z^b_1) \wedge \psi(i, z^a_1) \wedge \psi(i, z^b_1) \\ &\qquad \wedge \left( g^{\max}_{z^a_1} \upharpoonright [i+1, \kappa) < g^{\min}_{z^b_1} \upharpoonright [i+1, \kappa) \right). \end{aligned}$$

By pruning the tree so that every branch through it is a strictly increasing 5–tuple, we get

$$(\exists^* z_0^a < \lambda)(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)(\exists^* z_1^b < \lambda)(\forall^* i < \kappa)[\Phi(i, z_0^a, \ldots, z_1^b)]. \tag{4.3}$$

We now make a rather *ad hoc* definition of another quantifier in an attempt to make the arguments that follow a little bit clearer. Given $i < \kappa$, let the quantifier $\exists^{*,i} z_0^b < \lambda$ mean that not only are there unboundedly many $z_0^b$'s below $\lambda$ satisfying whatever property, but also that for each $\alpha < \lambda_i$, we can find unboundedly many suitable $z_0^b$'s for which $f_{z_0^b}(i)$ is greater than $\alpha$.

**Claim 4.1.** If we choose $\beta^a < \delta^a < \xi_n^b$ such that

$$(\exists^* z_0^b < \lambda)(\exists^* z_1^b < \lambda)(\forall^* i < \kappa)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)], \tag{4.4}$$

then

$$(\forall^* i < \kappa)(\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)]. \tag{4.5}$$

*Proof.* Suppose that we have $\beta^a < \delta^a < \xi_n^b$ such that (4.4) holds but (4.5) fails. Then there is an unbounded $I \subseteq \kappa$ such that for each $i \in I$,

$$\neg(\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)]. \tag{4.6}$$

In (4.4), we can move the quantifier "$\forall^* i < \kappa$" past the quantifiers to its left, i.e.,

$$(\forall^* i < \kappa)(\exists^* z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)], \tag{4.7}$$

so without loss of generality, for all $i \in I$,

$$(\exists^* z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)]. \tag{4.8}$$

Since (4.6) holds for all $i \in I$, it must be the case that for each $i \in I$, there is a value $g(i) < \lambda_i$ such that for all sufficiently large $\beta < \lambda$, if

$$(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, \beta, z_1^b)], \tag{4.9}$$

then

$$f_\beta(i) \leq g(i). \tag{4.10}$$

Since $\{f_\alpha : \alpha < \lambda\}$ witnesses that the true cofinality of $\prod_{i<\kappa} \lambda_i$ is $\lambda$, we know

$$(\forall^* x < \lambda)(\forall^* i \in I)[g(i) < f_x(i)]. \tag{4.11}$$

When we combine this with (4.4), we see that it is possible to choose $\beta^b < \lambda$ such that

$$(\forall^* i \in I)[g(i) < f_{\beta^b}(i)], \tag{4.12}$$

and

$$(\exists^* z_1^b < \lambda)(\forall^* j < \kappa)[\Phi(j, \beta^a, \delta^a, \xi_n^b, \beta^b, z_1^b)]. \tag{4.13}$$

(Note that we have quietly used the fact that $|I| < \lambda = \mathrm{cf}(\lambda)$ to get a $\beta^b$ that is "large enough" so that (4.9) implies (4.10) for all $i \in I$ for this particular $\beta^b$.) This last equation implies

$$(\forall^* j < \kappa)(\exists^* z_1^b < \lambda)[\Phi(j, \beta^a, \delta^a, \xi_n^b, \beta^b, z_1^b)],$$

so it is possible to choose $i \in I$ large enough so that

$$g(i) < f_{\beta^b}(i)$$

and

$$(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, \beta^b, z_1^b)].$$

This is a contradiction, as (4.9) holds for our choice of $i$ and $\beta = \beta^b$, yet (4.10) fails. □

Notice that an immediate corollary of the preceding claim is

$$(\exists^* z_0^a < \lambda)(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)(\forall^* i < \kappa)$$
$$(\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)]. \tag{4.14}$$

**Claim 4.2.** If $\beta^a < \lambda$ is chosen so that

$$(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)(\forall^* i < \kappa)$$
$$(\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)], \tag{4.15}$$

then

$$(\forall^* i < \kappa)(\exists v < \lambda_i)(\exists^* z_1^a < \lambda)[\psi' \wedge \psi'']$$

where

$$\psi' := g^{\max}_{z_1^a}(i) < v,$$

and

$$\psi'' := (\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)\left[v < f_{z_0^b}(i) \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]\right].$$

*Proof.* In (4.15), we can move the quantifier "$(\forall^* i < \kappa)$" past the other quantifiers to its left, so

$$(\forall^* i < \kappa)(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)$$
$$(\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)] \tag{4.16}$$

holds. The claim will be established if we show that for each $i < \kappa$ for which

$$(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)$$
$$(\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)] \tag{4.17}$$

holds, it is possible to find $v < \lambda_i$ such that

$$(4.18)\quad (\exists^* z_1^a < \lambda)\Big[g^{\max}_{z_1^a}(i) < v \text{ and}$$
$$(\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)\Big[v < f_{z_0^b}(i) \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]\Big]\Big].$$

Despite the lengths of the formulas involved, this is not that hard to accomplish. Since $\lambda_i < \lambda = \mathrm{cf}(\lambda)$, we can find $v < \lambda_i$ such that

$$(\exists^* z_1^a < \lambda)\big[g^{\max}_{z_1^a}(i) < v \text{ and}$$
$$(\exists^* y_n^b < \lambda)(\exists^{*,i} z_0^b < \lambda)(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]\big],$$

and now the result follows from of the definition of "$\exists^{*,i} z_1^b < \lambda$". □

Thus there are unboundedly many $z_0^a < \lambda$ for which there is a function $g \in \prod_{i<\kappa} \lambda_i$ such that for all sufficiently large $i < \kappa$,

$$(4.19)\quad (\exists^* z_1^a < \lambda)\Big[g^{\max}_{z_1^a}(i) \leq g(i) \text{ and}$$
$$(\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)\big[g(i) < f_{z_0^b}(i)$$
$$\text{and } (\exists^* z_1^b < \lambda)[\Phi(i, z_0^a, z_1^a, y_n^b, z_0^b, z_1^b)]\big]\Big].$$

Now this is logically equivalent to the statement

$$(4.20)\quad (\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)$$
$$\big[g^{\max}_{z_1^a}(i) \leq g(i) < f_{z_0^b}(i) \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, z_0^a, z_1^a, y_n^b, z_0^b, z_1^b)]\big].$$

Suppose we are given a particular $z_0^a < \lambda$ for which a function $g$ as above can be found, and let us fix $i < \kappa$ "large enough" so that (4.19) holds. Also fix ordinals $\delta^a < \lambda$ and $\xi_n^b < \lambda$ that serve as suitable $z_1^a$ and $y_n^b$. Just to be clear, this means that for these choices we have

$$(\exists^* z_0^b < \lambda)\big[g^{\max}_{\delta^a}(i) \leq g(i) < f_{z_0^b}(i) \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)]\big].$$

Since $\lambda_i < \lambda = \mathrm{cf}(\lambda)$, there must be some value $w$ satisfying

$$(\exists^* z_0^b < \lambda)\big[g(i) < f_{z_0^b}(i) < w \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, \xi_n^b, z_0^b, z_1^b)]\big].$$

This implies for our particular $\beta^a$, $g$, $i$, $\delta^a$, and $\xi_n^b$ that

$$(4.21)\quad (\forall^* w < \lambda_i)(\exists^* z_0^b < \lambda)\big[g^{\max}_{\delta^a}(i) \leq g(i) < f_{z_1^b}(i) < w \text{ and}$$
$$(\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, \delta^a, y_n^b, z_0^b, z_1^b)]\big].$$

Since $\lambda_i < \lambda = \mathrm{cf}(\lambda)$, the quantifier $(\forall^* w < \lambda_i)$ can move to the left past the quantifiers $(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)$. This tells us that for our $\beta^a$ and $g$,

$$(4.22)\quad (\forall^* i < \kappa)(\forall^* w < \lambda_i)(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda) \big[g^{\max}_{z_1^a}(i) \leq g(i) < f_{z_0^b}(i) < w \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]\big].$$

When we put all this together, we end up with the statement

$$(4.23)\quad (\exists^* z_0^a < \lambda)(\forall^* i < \kappa)(\exists v < \lambda_i)(\forall^* w < \lambda_i)(\exists^* z_1^a < \lambda) (\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)\big[g^{\max}_{z_1^a}(i) \leq v < f_{z_0^b}(i) < w \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]\big].$$

Since both $\kappa$ and $\lambda_i$ are less than $\lambda = \mathrm{cf}(\lambda)$, we can move some quantifiers around and achieve

$$(4.24)\quad (\forall^* i < \kappa)(\forall^* w < \lambda_i)(\exists^* z_0^a < \lambda)(\exists v < \lambda_i)(\exists^* z_1^a < \lambda) (\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)\big[g^{\max}_{z_1^a}(i) \leq v < f_{z_0^b}(i) < w \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]\big].$$

Thus there is a function $h \in \prod_{i<\kappa} \lambda_i$ such that

$$(4.25)\quad (\forall^* i < \kappa)(\exists^* z_0^a < \lambda)(\exists v < \lambda_i)(\exists^* z_1^a < \lambda) (\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)\big[g^{\max}_{z_1^a}(i) \leq v < f_{z_0^b}(i) < h(i) \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i, \beta^a, z_1^a, y_n^b, z_0^b, z_1^b)]\big].$$

After all this work, it is finally time to prove that we can select objects $\beta^a < \delta^a < \xi_n^b < \beta^b < \delta^b$ that satisfy all of our requirements.

Clearly, for every unbounded $\Lambda \subseteq \lambda$,

$$(\exists i < \kappa)(\exists^* x \in \Lambda)(h \restriction [i, \kappa) < f_x \restriction [i, \kappa).$$

Thus we can choose $i^* < \kappa$ such that $h_1^*(i^*) = j_1$ and $h_0^*(i^*) = n$, and

$$(\exists^* z_0^a < \lambda)\Bigg[h \restriction [i^*, \kappa) < f^{z_0^a} \restriction [i^*, \kappa) \text{ and } (\exists v < \lambda_i)(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda) (\exists^* z_0^b < \lambda)\big[g^{\max}_{z_1^a}(i^*) \leq v < f_{z_0^b}(i^*) < h(i^*) \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i^*, z_0^a, \dots, z_1^b)]\big]\Bigg].$$

So now we choose $\beta^a$ such that $h(i^*) < f_{\beta^a}(i^*)$ and for some $\alpha < \lambda_{i^*}$,

$$(\exists^* z_1^a < \lambda)(\exists^* y_n^b < \lambda)(\exists^* z_0^b < \lambda)\big[g^{\max}_{z_1^a}(i^*) \leq \alpha < f_{z_0^b}(i^*) < h(i^*) \text{ and } (\exists^* z_1^b < \lambda)[\Phi(i^*, z_0^a, \ldots, z_1^b)]\big].$$

Now we choose $\delta^a$, $\xi_n^b$, $\beta^b$, and $\delta^b$ such that

- $\beta^a < \delta^a < \xi_n^b < \beta^b$
- $g^{\max}_{\delta^a}(i^*) \leq \alpha < f_{\beta^b}(i^*) < h(i^*) < f_{\beta^a}(i^*)$
- $\Phi(i^*, \beta^a, \delta^a, \xi_n^b, \beta^b, \delta^b)$

It is straightforward to check that these objects satisfy all the requirements listed in Table 1, so by Claim 3.5, we are done.

## 5. Conclusions

In this final section, we will deduce some conclusions in a few concrete cases.

**Theorem 2.** *If $\mu$ is a singular cardinal of uncountable cofinality that is not a limit of regular Jonsson cardinals, then* $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$ *holds.*

*Proof.* The proof of this theorem occurs in two stages—we first show that $\mathrm{Pr}_1(\mu^+, \mu^+, \mu, \mathrm{cf}(\mu))$ holds, and then we show that this result can be upgraded to obtain $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu)$.

Let $\mu$ be as hypothesized, and let us define $\lambda = \mu^+$ and $\kappa = \mathrm{cf}(\mu)$.

**Claim 5.1.** $\mathrm{Pr}_1(\lambda, \lambda, \mu, \kappa)$ holds.

*Proof.* Let $\langle \kappa_i : i < \kappa \rangle$ be a strictly increasing continuous sequence cofinal in $\mu$. Let $S \subseteq \{\delta \in [\mu, \lambda) : \mathrm{cf}(\delta) = \kappa\}$ be stationary. Standard club–guessing results tell us that there is an $S$–club system $\bar{C}$ such that $\mathrm{id}_p(\bar{C}, \bar{J})$ is a proper ideal, where $J_\delta$ is the ideal $J^{b[\mu]}_{C_\delta}$ for $\delta \in S$, and furthermore, satisfying $|C_\delta| = \kappa$. (Note that this last requires that $\kappa = \mathrm{cf}(\mu)$ is uncountable.)

At this point, we have satisfied all of the assumptions of Claim 2.2 except possibly for clause (8). It suffices to show that for each $i < \kappa$, for all sufficiently large regular $\theta < \mu$, Player I has a winning strategy in the game $\mathrm{Gm}^\omega[\theta, \kappa_i, 1]$. Since $\mu$ is not a limit of regular Jonsson cardinals, it follows that for all sufficiently large regular $\theta < \mu$, Player I has a winning strategy in $\mathrm{Gm}^\omega[\theta, \theta, 1]$. This implies, by Lemma 1.3 (1), that for all sufficiently large regular $\theta$, Player I has a winning strategy in $\mathrm{Gm}^\omega[\theta, \kappa_i, 1]$, and so clause (8) of Claim 2.2 is satisfied. □

To finish the proof of Theorem 2, it remains to show that we can increase the number of colors from $\mu$ to $\lambda = \mu^+$ — we need $\Pr_1(\lambda, \lambda, \lambda, \kappa)$ instead of $\Pr_1(\lambda, \lambda, \mu, \kappa)$.

**Lemma 5.2.** There is a coloring $c_1 : [\lambda]^2 \to \lambda$ such that whenever we are given

- $\theta < \kappa$,
- $\langle t_\alpha : \alpha < \lambda \rangle$ a sequence of pairwise disjoint elements of $[\lambda]^\theta$,
- $\zeta_\alpha \in t_\alpha$ for $\alpha < \lambda$, and
- $\Upsilon < \lambda$,

we can find $\alpha < \beta$ such that $t_\alpha \subseteq \min(t_\beta)$ and

$$(\forall \zeta \in t_\alpha)[c_1(\zeta, \zeta_\beta) = \Upsilon]. \tag{5.1}$$

*Proof.* Let $c : [\lambda]^2 \to \mu$ be a coloring that witnesses $\Pr_1(\lambda, \lambda, \mu, \kappa)$. For each $\alpha < \lambda$, let $g_\alpha$ be a one–to–one function from $\alpha$ into $\mu$. We define

$$c_1(\alpha, \beta) = g_\beta^{-1}(c(\alpha, \beta)). \tag{5.2}$$

Suppose now that we are given objects $\theta$, $\langle t_\alpha : \alpha < \lambda \rangle$, $\langle \zeta_\alpha : \alpha < \lambda \rangle$, and $\Upsilon$ as in the statement of the lemma. Clearly we may assume that $\min(t_\alpha) > \alpha$.

For $i < \mu$, we define $X_i := \{\alpha \in [\gamma, \lambda) : g_{\zeta_\alpha}(\Upsilon) = i\}$. Since $\lambda$ is a regular cardinal, it is clear that there is $i^* < \mu$ for which $|X_{i^*}| = \lambda$. Since $c$ exemplifies $\Pr_1(\lambda, \lambda, \mu, \kappa)$, for some $\alpha < \beta$ in $X_{i^*}$ we have $t_\alpha \subseteq \min(t_\beta)$ and

$$(\forall \zeta \in t_\alpha)[c(\zeta, \zeta_\beta) = i^*]. \tag{5.3}$$

By definition, this means

$$(\forall \zeta \in t_\alpha)[c_1(\zeta, \zeta_\beta) = g^{-1}(c(\alpha, \beta)) = g^{-1}(i^*) = \Upsilon], \tag{5.4}$$

hence $\alpha$ and $\beta$ are as required.

□

To continue the proof of Theorem 2, we define a coloring $c_2 : [\lambda]^2 \to \lambda$ by

$$c_2(\alpha, \beta) = c_1(\alpha, \nu(\alpha, \beta)), \tag{5.5}$$

where $\nu(\alpha, \beta)$ is as in the proof of Theorem 1.

It remains to check that $c_2$ witnesses $\Pr_1(\lambda, \lambda, \lambda, \kappa)$. Toward this end, suppose we are given $\theta < \kappa$, $\langle t_\alpha : \alpha < \lambda \rangle$ a sequence of pairwise disjoint members of $[\lambda]^\theta$, and $\Upsilon < \lambda$. We need to find $\delta^a$ and $\delta^b$ less than $\lambda$ such that

$$\epsilon^a \in t_{\delta^a} \wedge \epsilon^b \in t_{\delta^b} \Longrightarrow c_2(\epsilon^a, \epsilon^b) = \Upsilon. \tag{5.6}$$

**Lemma 5.3.** There is a stationary set of $\gamma_1 < \lambda$ such that for some $\gamma_0 < \gamma_1$ and $\beta \in [\gamma_1, \lambda)$, if $\gamma_0 \leq \alpha < \gamma_1$, then the function $\nu$ is constant on $t_\alpha \times t_\beta$.

*Proof.* Let $E$ be an arbitrary closed unbounded subset of $\lambda$, and let $W$ be the set of ordinals $< \lambda$ satisfying the properties of $\gamma_1$. In the proof of Theorem 1, without loss of generality we can have $E \in M_0$. This means that the ordinal $\beta^*$ found in the course of that proof will be in $E$, so we finish by observing that $\beta^* \in W$. □

An application of Fodor's Lemma gives us a single ordinal $\gamma_0$ and a stationary $W' \subseteq W$ such that for all $\gamma \in W'$, there is a $\beta_\gamma \in [\gamma, \lambda)$ such that for all $\alpha \in [\gamma_0, \gamma)$, $\nu \restriction (t_\alpha \times t_\beta)$ is constant.

Using properties of the coloring $c_1$, we can find $\alpha$ and $\gamma$ such that

- $\gamma_0 \leq \alpha < \lambda$
- $\gamma \in W' \setminus (\sup(t_\alpha) + 1)$, and
- $\zeta \in t_\alpha \implies c_1(\zeta, \gamma) = \Upsilon$.

Now given $\epsilon^a \in t_\alpha$ and $\epsilon^b \in t_{\beta_\gamma}$, we find

$$c_2(\epsilon^a, \epsilon^b) = c_1(\epsilon^a, \gamma) = \Upsilon, \tag{5.7}$$

and therefore $c_2$ exemplifies $\mathrm{Pr}(\lambda, \lambda, \lambda, \kappa)$. □

Theorem 2 strengthens results in [1] as clearly $\mathrm{Pr}_1(\mu^+, \mu^+, \mu^+, \mathrm{cf}(\mu))$ implies that $\mu^+$ has a Jonsson algebra (i.e., $\mu^+$ is not a Jonsson cardinal). The question of whether the successor of a singular cardinal can be a Jonsson cardinal is a well–known open question.

We note that many of the results from Section 2 of [1] dealing with the existence of winning strategies for Player I in $\mathrm{Gm}^\omega[\lambda, \mu, \gamma]$ can be combined with Theorem 1 to give new results. For example, we have the following result from [1].

**Proposition 5.4.** If $\tau \leq 2^\kappa$ but $(\forall \theta < \kappa)[2^\theta < \tau]$, then Player I has a winning strategy in the game $\mathrm{Gm}^\omega(\tau, \kappa, \kappa^+)$.

*Proof.* See Claim 2.3(1) and Claim 2.4(1) of [1]. □

Armed with this, the following claim is straightforward.

**Claim 5.5.** Let $\mu$ be a singular cardinal of uncountable cofinality. Further assume that $\chi$ is a cardinal such that $2^{<\chi} \leq \mu < 2^\chi$. Then $\mathrm{Pr}_1(\mu^+, \mu^+, \chi, \mathrm{cf}(\mu))$ holds.

*Proof.* If $2^{<\chi} < \mu$, then Claims 2.3(1) and 2.4(1) of [1] imply that for every sufficiently large $\theta < \mu$, Player I has a winning strategy in the game $\mathrm{Gm}^\omega(\theta, \chi, \chi^+)$.

If $\mu = 2^{<\chi}$, then $\mathrm{cf}(\mu) = \mathrm{cf}(\chi)$. Let $\langle \kappa_i : i < \mathrm{cf}(\mu)\rangle$ be a strictly increasing continuous sequence of cardinals cofinal in $\chi$. Given $i < \mathrm{cf}(\mu)$, we claim that for all sufficiently large regular $\tau < \mu$, Player I has a winning strategy in $\mathrm{Gm}^\omega(\tau, \kappa_i, \chi)$. Once we have established this, $\mathrm{Pr}_1(\mu^+, \mu^+, \chi, \mathrm{cf}(\mu))$ follows by Theorem 1.

Given $\tau = \mathrm{cf}(\tau)$ satisfying $2^{\kappa_i} < \tau < \mu$, let $\eta$ be the least cardinal such that $\tau \leq 2^\eta$. Clearly $\kappa_i < \eta < \chi$. By Proposition 5.4, Player I wins the game $\mathrm{Gm}^\omega(\tau, \eta, \eta^+)$. This implies (since $\eta^+ < \chi$ and $\kappa_i < \eta$) that Player I wins the game $\mathrm{Gm}^\omega(\tau, \kappa_i, \chi)$ as required. □

We can also use Claim 1.4 to prove similar results. For example we have the following.

**Claim 5.6.** Let $\mu$ be a singular cardinal of uncountable cofinality. Further assume that $\chi < \mu$ satisfies $2^\chi < \mu < \beth_{(2^\chi)^+}(\chi)$. Then $\mathrm{Pr}_1(\mu^+, \mu^+, \chi, \mathrm{cf}(\mu))$ holds.

*Proof.* Again, the main point is that for all sufficiently large regular $\theta < \mu$, Player I has a winning strategy in the game $\mathrm{Gm}^\omega[\theta, \chi, (2^\chi)^+]$. This follows immediately from Claim 1.4. Since $(2^\chi)^+ < \mu$, Theorem 1 is applicable. □

In a sequel to this paper, we will address the situation where $\lambda$ is the successor of a singular cardinal of countable cofinality. Similar results hold, but the combinatorics involved are trickier.

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Department of Mathematics, University of Northern Iowa, Cedar Falls, IA, 50614

*E-mail address*: eisworth@uni.edu

Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel

*E-mail address*: shlhetal@math.huji.ac.il