Lecture 1: Introduction to D-spaces

**Definition 1.** A space $X$ is a $D$–space if for every neighborhood assignment $x \mapsto U_x$ (so $U_x$ is an open neighborhood of $x$) there is a closed discrete $D \subseteq X$ such that

$$X \subseteq \bigcup_{d \in D} U_d.$$  

Clearly every compact space is a $D$–space.

**Theorem 1.** Metrizable spaces are $D$–spaces.

Let $X$ be a metrizable space. Our proof of this theorem does not use the full strength of metrizability; rather, we make use of the following property of metrizable spaces:

**Claim 2.** Let $\tau$ be the topology on $X$. Then there is a function $F$ with domain $\tau \times \omega$ such that

1. $F(U, n)$ is a closed subset of $U$
2. $U = \bigcup_{n \in \omega} F(U, n)$
3. $F(U, n) \subseteq F(U, n + 1)$
4. If $U \subseteq V$, then $F(U, n) \subseteq F(U, n + 1)$.

**Proof.** Given $U$ an open subset of $X$ and $n < \omega$, define

$$F(U, n) := \{x \in X : B(x, 1/n) \subseteq U\},$$

where $B(x, 1/n)$ is the open ball of radius $1/n$ centered at $x$. It is easy to verify that $F$ has all the required properties. \qed

Given a neighborhood assignment $x \mapsto U_x$ for $X$, one can think of the function $F$ as a way to organize our space into “layers”:

For $n < \omega$, define

$$X_n := \{x \in X : x \in F(U_x, n)\}.$$  

The “layer” analogy holds, because it is clear that $X_n \subseteq X_{n+1}$ and $\bigcup_{n<\omega} X_n = X$.

Why are these layers helpful? They give us a way of making sure that points stay “spread apart” which is quite important given that we are trying to build closed discrete subsets. The idea is that if $x \in F(U, n)$ and $y \notin U_x$, then $d(x, y) \geq 1/n$; this is immediate from the definitions involved.

Now suppose we are given $n < \omega$. We show that there is a closed discrete $D_n \subseteq X_n$ such that $X_n \subseteq \bigcup_{x \in D_n} U_x$. Once we do this, we show how to improve this to show that $X$ is a $D$–space.

We define a sequence $\langle x_\alpha : \alpha < \gamma \rangle$ (for some $\gamma$) as follows:

**Stage 0:**

Ask if $X_\gamma = \emptyset$. If so, we set $\gamma = 0$ and stop the construction as there is nothing to be done. If not, then let $x_0 \in X_\gamma$ be arbitrary.

**Stage $\alpha > 0$:**
Given \( \langle x_\beta : \beta < \alpha \rangle \), we ask if \( X_n \) is covered by \( \bigcup_{\beta < \alpha} U_{x_\beta} \). If so, we set \( \gamma = \alpha \) and stop the construction. If not, we let \( x_\alpha \) be some element of \( X_n \setminus \bigcup_{\beta < \alpha} U_{x_\beta} \), and the induction continues.

Clearly the above construction terminates at some \( \gamma \leq |X_n| \), and we let \( D_n = \{ x_\alpha : \alpha < \gamma \} \) be the sequence produced.

Let us define
\[
U := \bigcup_{\alpha < \gamma} U_{x_\alpha}.
\]
Our construction guarantees that \( X_n \subseteq U \), as we keep the induction going until this is accomplished. So why is \( D_n = \{ x_\alpha : \alpha < \gamma \} \) closed and discrete?

This is trivial if we are willing to use the full power of metrizability – \( D_n \) is uniformly discrete. However, we are committed to getting a proof to go through using only the properties of \( F \) enumerated in Claim 2, so we must work a little harder.

**Claim 3.** Given \( z \in X \), there is an open neighborhood \( V \) of \( z \) such that \( |V \cap D_n| \leq 1 \).

**Proof.** Let \( z \in X \) be given. We start by noting that since \( x_\alpha \in F(U_{x_\alpha}, n) \) for each \( \alpha < \gamma \), we have
\[
D \subseteq F\left( \bigcup_{\alpha < \gamma} U_{x_\alpha}, n \right) = F(U, n).
\]
This means that if \( z \notin F(U, n) \), then \( X \setminus F(U, n) \) is an open set containing \( z \) that is disjoint to \( D_n \). Thus, we may assume that \( z \in F(U, n) \). In particular, \( z \in U \) and so there is a least \( \alpha < \gamma \) such that \( z \in U_{x_\alpha} \).

By construction,
\[
\{ x_\beta : \beta < \alpha \} \subseteq F\left( \bigcup_{\beta < \alpha} U_{x_\beta}, n \right).
\]
This latter set is closed, hence \( X \setminus F(\bigcup_{\beta < \alpha} U_{x_\beta}, n) \) is an open neighborhood of \( z \) disjoint to \( \{ x_\beta : \beta < \alpha \} \).

For \( \beta > \alpha \), we have made sure that \( x_\beta \notin U_{x_\alpha} \), and so \( U_{x_\alpha} \) is an open neighborhood of \( z \) disjoint to \( \{ x_\beta : \beta > \alpha \} \). Thus
\[
V := U_{x_\alpha} \cap \left( X \setminus F\left( \bigcup_{\beta < \alpha} U_{x_\beta}, n \right) \right)
\]
is an open neighborhood of \( z \) meeting \( D_n \) at most one point (namely \( x_\alpha \)). This implies immediately that \( D_n \) is closed and discrete. (Check this!) We are almost done; the first of the questions below asks about taking the above argument and “iterating it” to finish our proof.

**Question:** Can you finish the proof that metrizable spaces are \( D \)–spaces? Verify that all we used are the properties of \( F(U, n) \) outlined in Claim 2.

**Question:** Is the Sorgenfrey line a \( D \)–space? Is the Sorgenfrey line “semi-stratifiable”? (i.e., can you define a function \( F \) as in Claim 2 for the Sorgenfrey line?)

**Question:** Suppose \( Y \subseteq X \) and \( X \) is a \( D \)–space. Is \( Y \) a \( D \)–space? What if \( Y \) is closed? open? \( G_\delta \), \( F_\sigma \)?

**Question:** Suppose \( f : X \to Y \) is a closed mapping. If \( X \) is a \( D \)–space, is \( Y \)? If \( Y \) is a \( D \)–space, is \( X \)? If not, can you come up with some additional conditions on \( f \) that will give you a theorem?